A NOTE ON FANO MANIFOLDS WHOSE SECOND CHERN CHARACTER IS POSITIVE

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ABSTRACT. This note outlines some first steps in the classification of Fano manifolds for which $c_2^2 - 2c_2$ is positive or nef.

1. INTRODUCTION

This note lists the few known examples of Fano manifolds $X$ for which the second graded piece of the Chern character is positive, $c_2(T_X) = (C_1^2 - 2C_2)(T_X)/2$. There are also many non-examples. Presumably there are many more positive examples. They do not seem easy to find.

Theorem 1.1. In the following cases $X$ is Fano and $c_2(T_X)$ is ample, positive or nef.

1. For every projective and weighted projective space, $c_2(T_X)$ is ample.
2. For a Grassmannian $X = \text{Grass}(k, n)$ of $k$-dimensional subspaces of a fixed $n$-dimensional space with $2k \leq n$, $c_2(T_X)$ is ample if $k = 1$, positive if $n = 2k$ or $n = 2k + 1$, and nef if $n = 2k + 2$.
3. A complete intersection $Y = D_1 \cap \cdots \cap D_r$ in $X$ is Fano if $(C_1(T_X) - ([D_1] + \cdots + [D_r]))|_Y$ is ample. And $c_2(T_Y)$ is ample, resp. positive, weakly positive, nef, if $(c_2(T_X) - 1/2([D_1]^2 + \cdots + [D_r]))|_Y$ is ample, resp. positive, weakly positive, nef.
4. In particular, for a complete intersection of type $(d_1, \ldots, d_r)$ in a $n$-dimensional weighted projective space, $c_2(T_X)$ is ample, resp. nef, if $d_1^2 + \cdots + d_r^2 < n + 1$, resp. $\leq n + 1$.
5. A product $X \times Y$ of Fano manifolds $X$ and $Y$ is Fano, and $c_2(T_{X \times Y})$ is nef if $c_2(T_X)$ and $c_2(T_Y)$ are nef.
6. A projective bundle $Y = \mathbb{P}(E)$ over a Fano manifold $X$ associated to an extension $E$ of $\mathcal{O}_X$ by an invertible sheaf $L$, is Fano if $c_1(T_X) - c_1(L)$ is ample. And $c_2(T_Y)$ is nef if $c_2(T_X) + C_1(L)^2/2$ is nef.
7. In particular, let $(n, d, a)$ be integers such that $d \geq 1$, $n \geq (d^2 + d + 1)/2$, and $n - d \geq a \geq \lfloor \sqrt{\max(0, d^2 - n - 1)} \rfloor$. Let $X$ be a degree $d$ hypersurface in $\mathbb{P}^n$, and let $E = (\mathcal{O}_{\mathbb{P}^n}(-a) \oplus \mathcal{O}_{\mathbb{P}^n})|_X$. Then $Y = \mathbb{P}(E)$ is Fano and $c_2(T_Y)$ is nef.

Theorem 1.2. In the following cases $c_2(T_X)$ is not ample.

1. For a Grassmannian $X = \text{Grass}(k, n)$ with $2k \leq n$, $c_2(T_X)$ is not ample if $k > 1$, and it is not nef if $(n - 2)/2 > k > 1$.
2. For a product $X \times Y$ of positive-dimensional Fano manifolds, $c_2(T_{X \times Y})$ is not weakly positive.
(3) For a projective bundle $Y = \mathbb{P}(E)$ over a positive-dimensional Fano manifold $X$, $\text{ch}_2(T_Y)$ is not weakly positive. Moreover, if $rk(E) > 2$, then $Y$ is nef only if the restriction to every curve in $X$ is semistable.

(4) For a blowing up $Y$ of $\mathbb{P}^n$ in a nonempty, codimension 2 center, $\text{ch}_2(T_Y)$ is not nef.

Following are the definitions of nef, weakly positive, positive and ample for cycles of codimension greater than one.

**Notation 1.3.** Let $X$ be a projective variety over an algebraically closed field. For every integer $k \geq 0$, denote by $N^*_k(X)$ the finitely-generated free Abelian group of $k$-cycles modulo numerical equivalence, and denote by $N^k(X)$ the $k^{th}$ graded piece of the quotient algebra $A^*(X)/\text{Num}^*(X)$, cf. [Ful84, Example 19.3.9]. For every $\mathbb{Z}$-module $B$, denote $N_k(X)_B := N_k(X) \otimes B$, resp. $N^k(X)_B := N^k(X) \otimes B$. Denote by $NE_k(X) \subset N_k(X)$ the semigroup generated by nonzero, effective $k$-cycles. For $B$ a subring of $\mathbb{R}$, denote by $NE_k(X)_B$ the $B_{>0}$-semigroup in $N_k(X)_B$ generated by $NE_k(X)$.

**Definition 1.4.** A class in $N^k(X)_{\mathbb{R}}$ is **nef** if it pairs nonnegatively with every element in $N^k(X)$. The corresponding cone is denoted $Nef^k(X)$. A class is **weakly positive** if it pairs positively with every element in $NE_k(X)$.

**Remark 1.5.** There are obvious inclusions,

$$\text{Ample}^k(X) \subset \text{Pos}^k(X) \subset \text{WPos}^k(X) \subset Nef^k(X).$$

For $k = 1$, $\text{Ample}^1(X) = \text{Pos}^1(X)$ by definition. Moreover, by Kleiman’s criterion, this is the $\mathbb{R}_{>0}$-semigroup generated by first Chern classes of ample invertible sheaves. For $k > 1$, it can happen that $\text{Ample}^k(X) \neq \text{Pos}^k(X)$; for instance, because $(N^1(X))^{\otimes k} \to N^k(X)$ is not surjective. There are also examples where $\text{Pos}^k(X) \neq \text{WPos}^k(X)$ and $\text{WPos}^k(X) \neq Nef^k(X)$.

2. **Projective spaces, Grassmannians, products and complete intersections**

2.1. **Projective spaces.** Let $X$ be a projective variety of dimension $n \geq 2$. Denote by $h \in N^1(\mathbb{P}^n)$ the first Chern class of $\mathcal{O}_{\mathbb{P}^n}(1)$. Using the Euler sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus (n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0,$$

the Chern character of $T_{\mathbb{P}^n}$ is $(n+1)! - 1$. In particular, $\text{ch}_k(T_X) = (n+1)! - 1$ for every $k = 1, \ldots, n$. So $\text{ch}_k(T_X)$ is ample for $k = 1, \ldots, n$.

Weighted projective spaces work the same way provided we consider the space as a smooth Deligne-Mumford stack.

2.2. **Grassmannians.** Let $X$ be the Grassmannian $\text{Grass}(k,n)$ of $k$-dimensional subspaces of a fixed $n$-dimensional space. Since $\text{Grass}(k,n) \cong \text{Grass}(n-k,n)$, assume $2k \leq n$ without loss of generality. Denote by $\mathcal{O}_X^{\oplus n} \to S_k^v$ the universal rank $k$ quotient. There is an analogue of the Euler sequence,

$$0 \longrightarrow \text{Hom}_{\mathcal{O}_X}(S_k^v, S_k^v) \longrightarrow (S_k^v)^{\oplus n} \longrightarrow T_X \longrightarrow 0.$$
The Chern classes of $S^r_Y$ are the Schubert classes,

$$C_m(S^r_Y) = \sigma_1^m = \sigma_{1, \ldots, 1}.$$  

Therefore, by standard Chern class computations

$$\text{ch}(T_X) = (n-k)k + n\sigma_1 + \left[\frac{n+2-2k}{2}\sigma_2 - \frac{n-2-2k}{2}\sigma_{1,1}\right].$$  

In particular, if $n > 2k+2$, then $\text{ch}_2(T_X)$ has negative intersection with the effective Schubert cycle dual to $\sigma_{1,1}$. If $n = 2k + 2$, $\text{ch}_3(T_X)$ has intersection number 0. If $n = 2k$ or $n = 2k + 1$, $\text{ch}_2(T_X)$ has positive intersection number with every irreducible surface in $X$. But it is not a multiple of $\sigma_1^2$, thus it is not ample.

2.3. **Products.** For a product $X \times Y$, there is an isomorphism

$$T_{X \times Y} \cong \text{pr}_X^* T_X \oplus \text{pr}_Y^* T_Y.$$  

Therefore there is an equation

$$\text{ch}(T_{X \times Y}) = \text{pr}_X^* \text{ch}(T_X) + \text{pr}_Y^* \text{ch}(T_Y).$$  

In particular $C_1(T_{X \times Y}) = \text{pr}_X^* C_1(T_X) + \text{pr}_Y^* C_1(T_Y)$ is ample if $C_1(T_X)$ and $C_1(T_Y)$ are ample. Similarly, $\text{ch}_2(T_{X \times Y})$ is nef if $\text{ch}_2(T_X)$ and $\text{ch}_2(T_Y)$ are nef. However, for every curve $A \subseteq X$ and every curve $B \subseteq Y$, the intersection number of $\text{ch}_2(T_{X \times Y})$ with $A \times B$ is 0. Therefore $\text{ch}_2(T_{X \times Y})$ is not weakly positive.

2.4. **Complete intersections.** Let $Y$ be a smooth complete intersection of divisors $D_1, \ldots, D_r$ in $X$. There is an exact sequence

$$0 \rightarrow T_Y \rightarrow T_X|_Y \rightarrow \bigoplus_{i=1}^r O_X(D_i)|_Y \rightarrow 0.$$  

Therefore there is an equation

$$\text{ch}(T_Y) = \left[\text{ch}(T_X) - \sum_{i=1}^r e^{[D_i]}\right]|_Y.$$  

In other words, for every integer $m$,

$$\text{ch}_m(T_Y) = \left[\text{ch}_m(T_X) - \frac{1}{m!} \sum_{i=1}^r [D_i]^m\right]|_Y.$$  

Therefore $Y$ is Fano if $(C_1(T_X) - ([D_1] + \cdots + [D_r])|_Y$ is ample. And $\text{ch}_2(T_Y)$ is ample, resp. positive, weakly positive, nef, if $(\text{ch}_2(T_X) - 1/2([D_1]^2 + \cdots + [D_r])|_Y$ is ample, resp. positive, weakly positive, nef.

In particular, taking $X$ to be an $n$-dimensional weighted projective spaces, and taking $[D_i] = d_i h$ for each $i = 1, \ldots, r$, the Chern character of $T_Y$ is $(n+1)e^h - 1 - \sum_{i=1}^r e^{d_i h}$. Thus $\text{ch}_k(T_Y) = 1/k!(n+1-(d_1^k + \cdots + d_r^k))h^k$ for $k = 1, \ldots, n-r$. In particular, if $d_1^k + \cdots + d_r^k < n+1$, resp. $\leq n+1$, then $\text{ch}_2(T_Y)$ is ample, resp. nef.

3. **Projective bundles**

One way to produce new examples of Fano manifolds is to form the projective bundle of a vector bundle of “low degree” over a given Fano manifold.

**Lemma 3.1.** Let $E$ be a vector bundle on $X$ of rank $r$. Denote by $\pi : \mathbb{P}E \rightarrow X$ the associated projective bundle. The graded pieces of the Chern character of $T_{\mathbb{P}E}$ are,

$$c_1(T_{\mathbb{P}E}) = r\zeta + \pi^*(c_1(T_X) + c_1(E))$$  

and $\text{ch}_2(T_{\mathbb{P}E}) = r\zeta^2/2 + \pi^*(c_1(E)\zeta + \pi^*(\text{ch}_2(T_X) + \text{ch}_2(E)))$, where $\zeta$ equals $c_1(O_{\mathbb{P}E}(1))$.
Proposition 3.2. Let $\pi : X \to F$ be a vector bundle on a smooth Fano manifold $F$. The projective bundle $\mathbb{P}E$ is Fano if and only if there exists $\epsilon > 0$ such that for every finite morphism $g : B \to \mathbb{P}E$ of a smooth, connected curve to $F$ for which $\pi \circ g$ is also finite. Using the universal property of $\mathbb{P}E$, this holds iff for every finite morphism $f : B \to X$ and every invertible quotient $f^*E \to L$, $
abla$ is ample.

Proof. By hypothesis, $\omega_X$ is ample. By Lemma 5.3, $\omega_{\mathbb{P}E}$ is ample iff there exists a real number $\epsilon > 0$ such that

$$\deg_B(g^*\omega_{\mathbb{P}E}) \geq \epsilon \deg_B(g^*\pi^*\omega_X),$$

for every finite morphism $g : B \to \mathbb{P}E$ of a smooth, connected curve to $F$ for which $\pi \circ g$ is also finite. Using the universal property of $\mathbb{P}E$, this holds iff for every finite morphism $f : B \to X$ and every invertible quotient $f^*E \to L$, $
abla$ is ample.

So, finally, $\omega_{\mathbb{P}E}$ is ample iff there exists $\epsilon > 0$ such that for every finite morphism $f : B \to X$ and every invertible quotient $f^*E \to L$, $
abla$ is ample.

Taking the supremum over covers of $B$ and invertible quotients of the pullback of $E$, this is,

$$\mu_B(L) - \mu_B(f^*E) \leq (1 - \epsilon)\deg_B(f^*c_1(T_X))/r.$$

Since every finite morphism $f : B \to X$ factors through its image, it suffices to consider only irreducible curves $B$ in $X$. 

For $r = 2$, there is a necessary and sufficient condition for $\chi_2(T_{\mathbb{P}E})$ to be nef.

Proposition 3.3. Let $E$ be a vector bundle on $X$ of rank 2. Denoting by $\pi : \mathbb{P}E \to X$ the projection, $\chi_2(T_{\mathbb{P}E}) = \pi^*(\chi_2(T_X) + 1/2(c_1^2 - 4c_2)(E))$. Therefore $\chi_2(T_{\mathbb{P}E})$ is nef iff $\chi_2(T_X) + 1/2(c_1^2 - 4c_2)(E)$ is nef. If $\dim(X) > 0$, $\chi_2(T_{\mathbb{P}E})$ is not weakly positive.
Proof. By Lemma 3.1, \( \text{ch}_2(T_{PE}) \) equals \( \zeta^2 + \pi^*c_1(E)\zeta + \pi^*(\text{ch}_2(T_X) + \text{ch}_2(E)) \). By definition of the Chern classes of \( E \), \( \zeta^2 + \pi^*c_1(E)\zeta + \pi^*c_2(E) \) equals 0. So the class above is \( -\pi^*c_2(E) + \pi^*(\text{ch}_2(T_X) + \text{ch}_2(E)) \). Finally, \( \text{ch}_2(E) - c_2(E) \) equals 
\[
1/2(c_1^2 - 2c_2(E)) - c_2(E) = 1/2(c_1^2 - 4c_2(E)).
\]
\[\square\]

Applying Proposition 3.2 and Proposition 3.3 to the vector bundle \( E = L^r \oplus \mathcal{O}_X \) gives Theorem 1.1(6).

Finally, for \( r > 2 \), there is a necessary condition for \( \text{ch}_2(T_{PE}) \) to be nef.

**Proposition 3.4.** Let \( E \) be a vector bundle of rank \( r > 2 \) on \( X \). If \( \text{ch}_2(T_{PE}) \) is nef, then the pullback of \( E \) to every smooth, projective, connected curve is semistable. Also, \( \text{ch}_2(T_{PE}) \) is not weakly positive if \( \dim(X) > 0 \) and if the pullback of \( E \) to some curve is strictly semistable, e.g., if \( X \) contains a rational curve.

**Proof.** If the pullback of \( E \) to some smooth, projective, connected curve is not semistable, then by Corollary 5.1 there exists a smooth, projective, connected curve \( B \), a morphism \( f : B \to X \), and a rank 2 locally free subsheaf \( F \) of \( f^*E \) such that \( f^*E/F \) is locally free and \( \mu_B(F) > \mu_B(E) \). There is an induced morphism \( g : \mathbb{P}F \to \mathbb{P}E \) such that \( \pi \circ g = f \circ \pi \). By Lemma 3.1 \( g^*\text{ch}_2(T_{PE}) \) equals \( r\xi^2/2 + \pi^*f^*c_1(E)\xi + \pi^*f^*(\text{ch}_2(T_X) + \text{ch}_2(E)) \), where \( \xi \) equals \( c_1(\mathcal{O}_{PF}(1)) \). Since \( B \) is a curve, \( f^*(\text{ch}_2(T_X) + \text{ch}_2(E)) \) equals 0. Also, by definition of the Chern classes of \( F \), \( \xi^2 + \pi^*c_1(F)\xi = 0 \). Substituting in,
\[
g^*\text{ch}_2(T_{PE}) = 1/2\pi^*(2c_1(f^*E) - r c_1(F))\xi.
\]
In particular, \( \deg_B(g^*\text{ch}_2(T_{PE})) \) equals \( 1/2(\deg_B(c_1(f^*E)) - r\deg_B(F)) \). This equals \( r(\mu_B(f^*E) - \mu_B(F)) \), which is negative by construction. Therefore \( \text{ch}_2(T_{PE}) \) is not nef. \[\square\]

**Remark 3.5.** A vector bundle on a product of projective spaces whose restriction to every curve is semistable is of the form \( L^{2r} \), where \( L \) is an invertible sheaf. [OSS80] Thm. 3.2.1. In this case, \( \mathbb{P}E \) is also a product of projective spaces.

**Corollary 3.6.** Let \( X \) be a Fano manifold. For every vector bundle \( E \) on \( X \) of rank \( r > 1 \), \( \text{ch}_2(T_{PE}) \) is not weakly positive.

### 4. Blowings up

Let \( X \) be a smooth, connected, projective variety, let \( i : Y \hookrightarrow X \) be the closed immersion of a smooth, connected subvariety of \( X \) of codimension \( c \). Denote by \( \nu : \tilde{X} \to X \) the blowing up of \( X \) along \( Y \). Denote by \( \pi : E \to \tilde{X} \) the exceptional divisor. Denote by \( j : E \to \tilde{X} \) the obvious inclusion. Then \( E = \mathbb{P}N_{Y/X} \) and \( i^*\mathcal{O}_{\mathbb{P}X}(E) \) is canonically isomorphic to \( \mathcal{O}_{\mathbb{P}X}(-1) \).

**Lemma 4.1.** The graded pieces of the Chern character of \( \tilde{X} \) are, \( c_1(T_{\tilde{X}}) = \nu^*c_1(T_X) - (c - 1)[E] \) and \( \text{ch}_2(T_{\tilde{X}}) = \nu^*\text{ch}_2(T_X) + (c + 1)[E]^2/2 - \nu^*c_1(N_{Y/X}) \)

**Proof.** Using the short exact sequence,
\[
0 \longrightarrow \nu^*\Omega_X \longrightarrow \Omega_{\tilde{X}} \longrightarrow j_*\Omega_{\pi} \longrightarrow 0,
\]
\( \text{ch}(\Omega_{\tilde{X}}) \) equals \( \nu^*\text{ch}(\Omega_X) + \text{ch}(j_*\Omega_{\pi}) \). Grothendieck-Riemann-Roch for the morphism \( j \) gives,
\[
\text{ch}(Rj_*a) = j_*(\text{ch}(a))(1 - e^{-[E]}/[E]).
\]
Using the Euler sequence for $\Omega_\pi$,

\[ 0 \longrightarrow \Omega_\pi \longrightarrow \pi^*N^\vee_{Y/X} \otimes O_{\mathbb{P}N}(-1) \longrightarrow O_E \longrightarrow 0, \]

$\text{ch}(\Omega_\pi)$ equals $\pi^*\text{ch}(N^\vee_{Y/X})i^*(1 + e[^E]) - 1$. Putting the pieces together gives the lemma. \hfill \square

When is $\tilde{X}$ Fano? Denote by $C_1$ the collection of finite morphisms $g : B \to X$ from a smooth, connected curve to $X$ whose image is not contained in $Y$. Denote by $C_2$ the collection of finite morphisms $g : B \to Y$ from a smooth, connected curve to $Y$. The following result is well-known.

**Proposition 4.2.** Let $h$ be the first Chern class of an ample invertible sheaf on $X$, e.g., $h = c_1(T_X)$ if $X$ is Fano. The blowing up $\tilde{X}$ is Fano iff there exists $\epsilon > 0$ such that,

(i) for every $g : B \to X$ in $C_1$,

\[ \deg_B(g^{-1}Y) \leq \frac{1}{c-1}(\deg_B(g^*c_1(T_X)) - \epsilon \deg_B(g^*h)), \]

and

(ii) for every $g : B \to Y$ in $C_2$,

\[ \mu_B^1(g^*N_{Y/X}) \leq \frac{1}{c-1}(\deg_B(g^*c_1(T_X)) - \epsilon \deg_B(g^*h)). \]

The proof is similar to the proof of Proposition 3.2. Using an analogue of Proposition 3.3 no blowing-up of $\mathbb{P}^n$ is a Fano manifold with $\text{ch}_2$ nef.

5. Theorems about Vector Bundles on Curves

There are two theorems in this section. The first theorem goes back to Shou-Wu Zhang, though possibly it is older. A much more sophisticated arithmetic analogue was also proved by Shou-Wu Zhang in [Zha95, Theorem 1.10]. The second theorem in this section is a variation of the first theorem.

**Definition 5.1.** Let $B$ be a smooth, projective curve. A cover of $B$ is a finite, flat morphism $f : C \to B$ of constant, positive degree. A vector bundle on $B$ is a locally free $O_B$-module of constant rank.

**Definition 5.2.** Let $B$ be a smooth, projective curve. For every non-zero vector bundle $E$ on $B$, the slope is,

\[ \mu_B(E) = \deg(E)/\text{rank}(E) = \chi(B, E)/\text{rank}(E) - \chi(B, O_B). \]

For every cover $f : C \to B$ and every non-zero vector bundle $E$ on $C$, the $B$-slope is,

\[ \mu_B(f, E) := \deg(E)/(\deg(f)\text{rank}(E)) = \mu_B(f_*E) - \mu_B(f_*O_C). \]

When there is no chance of confusion, this is denoted simply $\mu_B(E)$.

For every cover $g : C' \to C$, $f \circ g : C' \to B$ is a cover and $\mu_B(f \circ g, g^*E)$ equals $\mu_B(f, E)$. 

Definition 5.3. Let $B$ be a smooth, projective curve and let $E$ be a vector bundle on $B$ of rank $r > 0$. For every integer $1 \leq k \leq r$, define $\mu^k_B(E)$ to be,

$$\sup\{-\mu_B(f,F^\vee)|f:C \to B \text{ a cover }, f^*E^\vee \to F^\vee \text{ a rank } k \text{ quotient}\}$$

$$= \sup\{\mu_B(f,F)|f:C \to B \text{ a cover }, F \subset f^*E \text{ a rank } k \text{ subbundle whose cokernel is locally free}\}.$$ 

Let $f: X \to Y$ be a morphism of projective varieties. Denote by $C_1$ the collection of all irreducible curves in $X$ not contained in a fiber of $f$. Denote by $C_2$ the collection of finite morphisms $g: C \to X$ occurring as the normalization of an irreducible curve in $X$ not contained in a fiber of $f$. Finally, denote by $C_3$ the collection of all finite morphisms from smooth, connected curves to $X$ whose image is not contained in a fiber of $f$.

Lemma 5.4. Let $f: X \to Y$ be a morphism of projective varieties and let $L$ be an ample invertible $O_Y$-module. An $f$-ample invertible $O_X$-module $M$ is ample iff there exists a real number $\epsilon > 0$ such that for every morphism $g : C \to X$ in $C_1$, resp. $C_2, C_3$, $\deg_C(g^*M) \geq \epsilon \deg_C(g^*f^*L)$.

Proof. Because $M$ is $f$-ample and $L$ is ample, there exists an integer $n > 0$ such that $M \otimes f^*L^\otimes n$ is ample. By Kleiman’s criterion, $M$ is ample iff there exists a real number $0 < \delta < 1$ such that for every irreducible curve $C$ in $X$,

$$\deg_C(M) \geq \delta \deg_C(M \otimes f^*L^\otimes n).$$

Simplifying, this is equivalent to,

$$\deg_C(M) \geq \frac{n \delta}{1 - \delta} \deg_C(f^*L).$$

As $M$ is $f$-ample, this holds if $C$ is contained in a fiber of $f$. So $M$ is ample iff the inequality holds for every curve in $C_1$. Setting $\epsilon = n \delta/(1 - \delta)$, $\delta = \epsilon/(n + \epsilon)$, gives the lemma.

Since $C_2 \subset C_3$, the condition for $C_3$ implies the condition for $C_2$. Since degrees on a curve can be computed after pulling back to the normalization, the condition for $C_2$ implies the condition for $C_1$. Finally, for every morphism $g : C \to X$ in $C_3$, $g(C)$ is in $C_1$. The inequality for $g(C)$ implies the inequality for $C$. Thus the condition for $C_1$ implies the condition for $C_3$. \qed

Lemma 5.5. Let $B$ be a smooth, connected, projective curve. A nonzero vector bundle $E$ on $B$ is ample iff there exists a positive real number $\delta$ such that for every cover $f : C \to B$ and every invertible quotient $f^*E \to L$, $\mu_B(L) \geq \delta$. In other words, $E$ is ample iff $\mu^1_B(L^\vee) < 0$.

Proof. Denote by $\pi : \mathbb{P}E^\vee \to B$ the projective bundle associated to $E^\vee$, and denote by $\pi^*E \to \mathcal{O}_{\mathbb{P}E^\vee}(1)$ the tautological invertible quotient. By definition, $E$ is ample iff $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is an ample invertible sheaf. Of course $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is $\pi$-relatively ample. Let $M$ be an invertible $\mathcal{O}_B$-module of degree 1. Then $M$ is ample. By Lemma 5.4 $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is ample iff there exists $\epsilon > 0$ such that for every smooth, connected curve $C$ and every finite morphism $g : C \to \mathbb{P}E^\vee$ such that $\pi \circ g$ is finite, $\deg_C(g^*\mathcal{O}_{\mathbb{P}E^\vee}(1)) \geq \epsilon \deg_C(g^*\pi^*M)$. Of course $\deg_C(g^*\pi^*M) = \deg(\pi \circ g)$. Using
the universal property of $\mathbb{P}E^\vee$, this holds iff for every cover $f : C \to B$ and every invertible quotient $f^*E \to L$, 
\[ \deg_C(L) \geq \epsilon \deg(f) \iff \mu_B(L) \geq \epsilon. \]

\[ \square \]

**Lemma 5.6.** For every ample vector bundle $E$ on $B$, there exists a cover $f : C \to B$, invertible $\mathcal{O}_C$-modules $L_1, \ldots, L_r$, and a morphism of $\mathcal{O}_C$-modules, $\phi : f^*E \to (L_1 \oplus \cdots \oplus L_r)$ such that,

(i) the support of $\operatorname{coker}(\phi)$ is a finite set,

(ii) for every $i = 1, \ldots, r$, the projection $f^*E \to \oplus_{j \neq i} L_j$ is surjective, and

(iii) for every $i = 1, \ldots, r$, $\mu_B(L_i) = \deg_B(E)$.

**Proof.** Denote $r = \operatorname{rank}(E)$. The claim is that for every $k = 1, \ldots, r$, there exists a cover $f_k : C_k \to B$, invertible $\mathcal{O}_{C_k}$-modules $L_{k,1}, \ldots, L_{k,k}$, and a morphism of $\mathcal{O}_{C_k}$-modules, $\phi_k : f_k^*E \to (L_{k,1} \oplus \cdots \oplus L_{k,k})$ satisfying (ii) and (iii) above and the following variant of (i): for $k < r$, $\phi_k$ is surjective and for $k = r$, the support of $\operatorname{coker}(\phi_r)$ is a finite set. The lemma is the case $k = r$. The claim is proved by induction on $k$.

The base case is $k = 1$. Denote by $\pi : \mathbb{P}E^\vee \to B$ the projective bundle associated to $E^\vee$, and denote by $\pi^*E \to \mathcal{O}_{\mathbb{P}E^\vee}(1)$ the tautological invertible quotient. By hypothesis, $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is ample. By Bertini’s theorem, for $d_1, \ldots, d_{r-1} \gg 0$, there exist effective Cartier divisors $D_1, \ldots, D_{r-1}$ such that the intersection $C_1 = D_1 \cap \cdots \cap D_{r-1}$ is smooth, connected curve, cf. [Jou83]. Denote by $f_1 : C_1 \to B$ the restriction of $\pi$. Denote by $\phi_1 : f_1^*E \to L_{1,1}$ the restriction of $\pi^*E \to \mathcal{O}_{\mathbb{P}E^\vee}(1)$. This satisfies (i) because $\pi^*E \to \mathcal{O}_{\mathbb{P}E^\vee}(1)$ is surjective. It satisfies (ii) trivially. Finally, $\deg(f)$ equals $d_1 \times \cdots \times d_{r-1}$, and $\deg(C_1(L_{1,1}))$ equals $d_1 \times \cdots \times d_{r-1} \times |C_1(\mathcal{O}_{\mathbb{P}E^\vee}(1))|^r$, i.e., $d_1 \times \cdots \times d_{r-1} \times \deg_B(E)$. Therefore $\mu_B(L_{1,1}) = \deg_B(E)$, i.e., this satisfies (iii).

By way of induction, assume the result is known for $k < r$, and consider the case $k + 1$. Since $\phi_k$ is surjective, there is an induced closed immersion $\mathbb{P}(L_{k,1} \oplus \cdots \oplus L_{k,k})^\vee \subseteq \mathbb{P}(f_k^*E)^\vee$. The image is irreducible and has codimension $r - k - 1$. For every $i = 1, \ldots, k$, the image of $\mathbb{P}(\oplus_{j \neq i} L_{k,j})^\vee$ is irreducible and has codimension $r - k - 1 \geq 2$. Associated to the finite morphism $f_k$, there is a finite morphism $\mathbb{P}(f_k^*E)^\vee \to \mathbb{P}E^\vee$. The pullback of an ample invertible sheaf by a finite morphism is ample; hence $\mathbb{P}(f_k^*(f_k^*E)^\vee)(1)$ is ample. By Bertini’s theorem, for $d_1, \ldots, d_{r-1} \gg 0$, there exist effective Cartier divisors $D_1, \ldots, D_{r-1}$ with $D_i \in |\mathcal{O}_{\mathbb{P}(f_k^*E)^\vee}(d_i)|$ such that the intersection $C_{k+1} = D_1 \cap \cdots \cap D_{r-1}$ is smooth, connected curve, disjoint from $\mathbb{P}(\oplus_{j \neq i} L_{k,j})^\vee$ for every $i = 1, \ldots, k$, and either disjoint from $\mathbb{P}(\oplus_i L_i)^\vee$ if $k < r - 1$, or else intersecting $\mathbb{P}(\oplus_i L_i)^\vee$ in finitely many points if $k = r - 1$. Define $g_{k+1} : C_{k+1} \to C_k$ to be the restriction of the projection. Define $f_{k+1} = f_k \circ g_{k+1}$, define $L_{k+1,i} = g_{k+1}^*L_{k,i}$ for $i = 1, \ldots, k$, and define $L_{k+1,k+1}$ to be the restriction of $\mathcal{O}_{\mathbb{P}(f_k^*E)^\vee}(1)$. Define $\phi_{k+1}$ to be the obvious morphism.

The cokernel of $\phi_{k+1}$ is supported on the intersection of $C_{k+1}$ with $\mathbb{P}(L_{k,1} \oplus \cdots \oplus L_{k,k})^\vee$. By construction, this is empty if $k < r - 1$, and is a finite set if $k = r - 1$. Thus $\phi_{k+1}$ satisfies (i). By the induction hypothesis, $f_{k+1}^*E \to (L_{k+1,1} \oplus \cdots \oplus L_{k+1,k})$, which is the pullback under $g_{k+1}$ of $\phi_k$, is surjective. For $i = 1, \ldots, k$, the cokernel of $f_{k+1}^*E \to \oplus_{j \neq i} L_{k+1,j}$ is supported on the intersection of $C_{k+1}$ with the
image of $\mathcal{P}(\oplus_{j \neq i} L_{k,j})^\vee$. By construction, this is empty, i.e., $f_{k+1}^* E \to \oplus_{j \neq i} L_{k+1,j}$ is surjective. Thus $\phi_{k+1}$ satisfies (ii). Finally, $\phi_{k+1}$ satisfies (iii) by the same argument as in the base case. The claim is proved by induction on $k$. \hfill \Box

**Theorem 5.7.** For every non-zero vector bundle $E$ on $B$, for every $\epsilon > 0$, there exists a cover $f : C \to B$ and an invertible quotient $f^* E \to L$ such that $\mu_B(L) < \mu_B(E) + \epsilon$. In other words, $\mu_B(E^\vee) \geq \mu_B(E^\vee)$.

**Proof.** Denote $r = \text{rank}(E)$. If $r = 1$, set $f = \text{Id}_B$ and $L = E$. Then $L$ is an invertible quotient of $f^* E$, and $\mu_B(L)$ equals $\mu_B(E)$ which is less than $\mu_B(E) + \epsilon$. Therefore assume $r > 1$.

Certainly an effective version of the following argument can be given, but a simpler argument is by contradiction. **Hypothesis 5.8.** For every cover $f : C \to B$ and every invertible quotient $f^* E \to L$, $\mu_B(L) \geq \mu_B(E) + \epsilon$, i.e., $\mu_B(E^\vee) < \mu_B(E^\vee) - \epsilon$.

By way of contradiction, assume Hypothesis 5.8. Let $f : C \to B$ be a connected, smooth cover of degree $d$. For every $a/d \in \mathbb{Q}$, there exists an invertible sheaf $M$ on $C$ of degree $a$, and thus $\mu_B(M) = a/d$. In particular, for $d$ sufficiently large, there exists an invertible quotient $M$ such that $0 < \mu_B(E) - \mu_B(M) < \epsilon/(r-1)$. Denote $\delta = \mu_B(E) - \mu_B(M)$. Denote $F = f^* E \otimes M^\vee$. Then $\mu_B(F)$ equals $\delta$, and $0 < \delta < \epsilon/(r-1)$.

Let $g : C' \to C$ be any cover and let $g^* F \to N$ be any invertible quotient. Then $f \circ g : C' \to B$ is a cover and $(f \circ g)^* E = g^* F \otimes g^* M \to N \otimes g^* M$ is an invertible quotient. By Hypothesis 5.8,

$$\mu_C(N) = \deg(f) \mu_B(N) = \deg(f)(\mu_B(N \otimes g^* M) - \mu_B(M)) \geq \deg(f)(\mu_B(E) + \epsilon)$$

By Lemma 5.5 $F$ is an ample vector bundle on $C$. By Lemma 5.6 there exists a cover $g : C' \to C$ and an invertible quotient $g^* F \to P$ such that $\mu_B(P) = r \mu_B(F) = r \delta$. Therefore $L := g^* M \otimes P$ is an invertible quotient of $g^* f^* E$ and,

$$\mu_B(L) = \mu_B(g^* M \otimes P) = \mu_B(M) + r \delta = \mu_B(E) + (r-1) \delta.$$ 

By hypothesis, $(r-1) \delta < \epsilon$. So $\mu_B(L) < \mu_B(E) + \epsilon$, contradicting Hypothesis 5.8. The proposition is proved by contradiction. \hfill \Box

**Corollary 5.9.** For every non-zero vector bundle $E$ on $B$, for every $\epsilon > 0$, there exists a cover $f : C \to B$ and a sequence of vector bundle quotients,

$$f^* E = E^r \to E^{r-1} \to \cdots \to E^1,$$

such that each $E^k$ is a vector bundle of rank $k$ and $\mu_B(E^k) < \mu_B(E) + \epsilon$.

**Proof.** The proof is by induction on the rank $r$ of $E$. If $\text{rank}(E) = 1$, defining $f = \text{Id}_B$ and $E^1 = E$, the result follows. Thus, assume $r > 1$ and the result is known for smaller values of $r$. By Theorem 5.7 there exists a cover $g : B' \to B$ and a rank 1 quotient $g^* E \to L$ such that $\mu_B(L) < \mu_B(E) + \epsilon$. Denote by $K$ the kernel of $g^* E \to L$. Then $\text{rank}(K) = r - 1$ and $\mu_B(K) = (r \mu_B(E) - \mu_B(L))/(r-1)$. By the induction hypothesis, there exists a cover $h : C \to B'$ and a sequence of vector bundle quotients,

$$h^* K = K^{r-1} \to \cdots \to K^1,$$
such that each $K^k$ is a vector bundle of rank $k$, and $\mu_{B'}(K^k) \leq \mu_{B'}(K) + \deg(g)\epsilon$. Of course $\mu_B(F) = \mu_{B'}(F)/\deg(g)$ for every $F$. Thus $\mu_B(K^k) \leq \mu_B(K) + \epsilon$.

Define $f = h \circ g$, define $E^i = h^*L$, and for every $k = 2, \ldots, r$, define $f^*E \to E^k$ to be the unique quotient whose kernel is contained in $h^*K$ and such that $h^*K \to E^k$ has image $K^{k-1}$. Then $\mu_B(E^1) = \mu_B(L) \leq \mu_B(E) + \epsilon$, and for $k = 2, \ldots, r$,

$$\mu_B(E^k) = 1/k(\mu_B(L) + (k - 1)\mu_B(K^{k-1})) < 1/k(\mu_B(L) + (k - 1)\mu_B(K) + (k - 1)\epsilon) = \frac{r(k-1)}{(r-1)k} \mu_B(E) + \frac{r-k}{(r-1)k} \mu_B(L) + \frac{(r-1)(k-1)}{(r-1)k} \epsilon < \mu_B(E) + \epsilon.$$ 

□

For semistable bundles in characteristic zero, there is a more precise result. An arithmetic analogue is also proved by Zhang in [Zha95, Theorem 1.10].

**Theorem 5.10** (Zhang). Let $B$ be a smooth, projective curve over an algebraically closed field of characteristic 0. Let $E$ be a semistable vector bundle on $B$. Let $\epsilon$ be a positive real number. There exists a cover $f : C \to B$, invertible sheaves $L_1, \ldots, L_r$ on $C$, and a morphism of $O_C$-modules, $\phi : f^*E \to (L_1 \oplus \cdots \oplus L_r)$ such that,

(i) the support of $\ker(\phi)$ is a finite set,

(ii) for every $i = 1, \ldots, r$, the projection $f^*E \to \oplus_{j \neq i} L_j$ is surjective,

(iii) for every $i = 1, \ldots, r$, $\mu_B(L_i) \leq \mu_B(E) + \epsilon$.

**Proof.** Denote $r = \text{rank}(E)$. If $r$ equals 1, the theorem is trivial. Thus assume $r > 1$. As in the proof of Theorem 5.7 there exists a cover $g : C' \to B$ and an invertible sheaf $M$ on $C'$ such that $0 < \mu_B(E) - \mu_B(M) < \epsilon/(r-1)$. Denote $\delta = \mu_B(E) - \mu_B(M)$ and denote $F = g^*E \otimes M^\vee$. Then $\mu_B(F)$ equals $\delta$, and $0 < \delta < \epsilon/(r-1)$.

Let $h : C \to C'$ be any cover and let $h^*F \to N$ be an invertible quotient. The composition $g \circ h : C \to B$ is a cover. By Kempf’s theorem, [Kemp92], which ultimately relies on the theorem that every stable vector bundle admits a Hermite-Einstein metric, $(goh)^*E$ is semistable. (Note, there are counterexamples in positive characteristic.) Therefore $h^*F$ is semistable. So $\mu_C(L) \geq \mu_C(h^*F)$, i.e., $\mu_C(L) \geq \mu_{C'}(F) = \delta$. Thus by Lemma 5.6 $F$ is an ample vector bundle on $C'$. Thus by Lemma 5.6 there exists a cover $h : C \to C'$, invertible $O_C$-modules $N_1, \ldots, N_r$, and a morphism of $O_C$-modules $\psi : h^*F \to (N_1 \oplus \cdots \oplus N_r)$ satisfying (i), (ii) and (iii) of Lemma 5.6. Define $f = g \circ h$, $L_i = N_i \otimes h^*M$ and $\phi$ is the twist of $\psi$ by $\text{Id}_{h^*M}$. Then $\phi$ satisfies (i) and (ii). And for every $i = 1, \ldots, r$,

$$\mu_B(L_i) = \mu_B(N_i) + \mu_B(M) = \mu_{C'}(N_i)/\deg(g) + \mu_B(E) - \delta = \mu_B(E) + r\delta/\deg(g) - \delta \leq \mu_B(E) + (r-1)\delta/\deg(g) < \mu_B(E) + \epsilon.$$ 

□

Of course, $\mu'_B(E)$ equals $\mu_B(E)$. The other values are more interesting.

**Corollary 5.11.** The slopes $\mu'^k_B(E)$ satisfy $\mu'^1_B(E) \geq \mu'^2_B(E) \geq \cdots \geq \mu'^r_B(E) = \mu_B(E)$. For each $1 \leq k < r$, $\mu'^k_B(E) = \mu_B(E)$ iff $f^*E$ is semistable for every cover $f : C \to B$. 


Proof. By Corollary 5.9, for every $\epsilon > 0$, there exists a cover $f : C \to B$ and a rank $k$ quotient $f^*E \to E^k$ such that $\mu_B(E^k) < \mu_B(E) + \epsilon$. Thus $\mu^k_B(E) \geq \mu_B(E)$. Applying the same reasoning to rank $k - 1$ quotients of rank $k$ quotients of $f^*E$, $\mu^{k-1}_B(E) \geq \mu^k_B(E)$.

If $f^*E$ is semistable for every cover $f : C \to B$, then every vector bundle quotient of $f^*E$ has slope $\geq \mu_C(f^*E)$, and thus has $B$-slope $\geq \mu_B(f^*E)$. Therefore $\mu^k_B(E) \leq \mu_B(E)$, i.e., $\mu^k_B(E) = \mu_B(E)$.

Conversely, suppose there is a cover $f : C \to B$ such that $f^*E$ is not semistable. Then there exists a vector bundle quotient $f^*E \to F$ such that $\mu_B(F) < \mu_B(E)$. Denote the rank by $l$. Suppose first that $l \geq k$, and define $\epsilon = \deg(f)(\mu_B(E) - \mu_B(F))$. Then by Corollary 5.9, there exists a cover $g : C' \to C$ and a rank $k$ quotient $g^*F \to G$ such that $\mu_C(G) < \mu_C(F) + \epsilon$. Therefore $g^*f^*E \to g^*F \to G$ is a rank $k$ quotient of $g^*f^*E$ and $\mu_B(G) < \mu_C(F) + (\mu_B(E) - \mu_B(F)) = \mu_B(E)$. Therefore $\mu^k_B(E) > \mu_B(E)$.

Next suppose that $l < k$. Denote by $K$ the kernel of $f^*E \to F$. Then $r\mu_B(E) = l\mu_B(F) + (r - l)\mu_B(K)$. Define,

$$\epsilon = \frac{(r - k)\deg(f)(\mu_B(E) - \mu_B(F))}{(r - l)(k - l)}.$$ 

By Corollary 5.9, there exists a cover $g : C' \to C$ and a rank $k - l$ quotient $g^*K \to G'$ such that $\mu_C(G') < \mu_C(K) + \epsilon$. Therefore $\mu_B(G') < \mu_B(K) + \epsilon/\deg(f)$. Define $g^*f^*E \to G$ to be the unique vector bundle whose kernel is contained in $g^*K$ and such that the image of $g^*K \to G$ equals $G'$. Then,

$$k\mu_B(G) = l\mu_B(F) + (k - l)\mu_B(G') < l\mu_B(F) + (k - l)\mu_B(K) + (k - l)\epsilon/\deg(f) = l\mu_B(F) + \frac{k - l}{r - l}(r\mu_B(E) - l\mu_B(F)) + \frac{k - l}{r - l}\epsilon = k\mu_B(E) - \frac{(r - k)l}{r - l}(\mu_B(E) - \mu_B(F)) + \frac{(r - k)l}{r - l}(\mu_B(E) - \mu_B(F)) = k\mu_B(E).$$

Thus $\mu_B(G) < \mu_B(E)$, and therefore $\mu^k_B(E) > \mu_B(E)$. \hfill \Box

References


