EXPLICIT COMPUTATIONS RELATED TO "RATIONAL CONNECTIVITY . . ." BY GRABER, HARRIS, MAZUR AND STARR

Abstract. These are some explicit computations for hypersurfaces related to the paper "Rational connectivity and sections of families over curves" [2].

1. Introduction

One of the results proved in [2] is that over $\mathbb{C}$ there exists a surjective morphism $f : X \to C$ from a smooth, projective threefold to a smooth, projective curve whose general fiber is an Enriques surface and such that f admits no section. However the method of proof is not constructive, and several questions were unanswered: *intro*

1. For such a family, what is the least possible genus of C?
2. With respect to some invertible sheaf on the moduli space of Enriques surfaces, what is the least possible degree of C?
3. Does such a family exist in positive characteristic?
4. Does such a family exist over a number field?
5. Can one find $f : X \to \text{Spec } \mathbb{C}[[t]]$ a flat, projective morphism from a regular scheme whose geometric generic fiber is an Enriques surface, and such that the closed fiber X_0 is everywhere non-reduced?

We give some answers to the first three questions by explicit computations.

The methods of the paper are very elementary. We produce a family of Enriques surfaces in the boundary of a parameter space of Enriques surfaces where we can check that there is no section by a monodromy argument. Then by deformation theory we can conclude that a very general deformation of the family also has no section. The same method gives similar results for families of hypersurfaces. We go through the construction for hypersurfaces as a warm-up for the Enriques construction, and also because of Corollary 3.6 which completes the argument from [2, Section 7.3] showing the dependence of the family of curve $\mathcal{H}_d$ on d. At one place we need to use the result that for a general degree n morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ the Galois group of the extension of function fields acts as the full symmetric group S_n on the geometric generic fiber. In characteristic 0 this is quite easy, and undoubtedly the result is already known in every characteristic. To keep the paper self-contained, a proof is given in Section 2.

Let k be a fixed algebraically closed field (not necessarily of characteristic 0). All schemes will be k-schemes and all absolute fiber products will be fiber products over Spec k.

Acknowledgments/Warning These are personal notes intended mainly for myself and the other authors of [2]. Many of the results are already known, and I haven't (yet) tracked down the original proofs. I am grateful to Igor Dolgachev and Mike Stillman for clarifying the parameter space G of quartic Enriques surfaces which led to Section 4. I am grateful to Harry Tamvakis for pointing out that three general points on the Grassmannian $\mathbf{G}(3,6)$ lie on a twisted cubic curve in the Grassmannian. Mostly I am grateful to Tom Graber, Joe Harris and Barry Mazur (for whom I am writing these notes!). I was partially supported by NSF Grant DMS-0201423.

sec-covers

2. Covers of $\mathbb{P}^1$ by $\mathbb{P}^1$

In this section it is proved that for every integer n there is a degree n morphism $\mathbb{P}^1 \to \mathbb{P}^1$ such that the action of the Galois group of the cover on the geometric generic fiber is the full symmetric group of the fiber. Over $\mathbb{C}$ there is a very simple topological proof that any degree n morphism $\mathbb{P}^1 \to \mathbb{P}^1$ with only simple branching has this property. In positive characteristic this argument does not work directly, although perhaps it can be made to work using the language of [3]. Another concern is that in characteristic 2, no map has simple branching. Almost certainly this result is already known in positive characteristic, but we give an explicit proof in order to be self-contained.

Our proof is a "deformation-and-specialization" argument. Using [REFERENCE - Harris, Galois groups of enumerative problems], we reformulate the condition that the Galois group is the full symmetric group in terms of the connectedness of a certain associated scheme. We prove the existence of a degree n morphism $f : C \to B$ of connected, proper, at-worst-nodal curves of arithmetic genus 0 such that the associated scheme is connected. Then we use deformation theory to prove that there is a family of maps $\mathbb{P}^1 \to \mathbb{P}^1$ specializing to f and conclude that for the general member of this family, the associated scheme is connected.

defn-mild

Definition 2.1. A morphism of proper curves $f : C \to B$ is *mild* if it satisfies the following:

1. The morphism f is finite.
2. The morphism f is a local complete intersection morphism, i.e. for every closed point $x \in C$ the induced map $f_x^\sharp : \widehat{\mathcal{O}}_{B,f(x)} \to \widehat{\mathcal{O}}_{C,x}$ is equivalent to the inclusion of $\widehat{\mathcal{O}}_{B,f(x)}$ in $\widehat{\mathcal{O}}_{B,f(x)}[t_1,\dots,t_n]/\langle g_1,\dots,g_n\rangle$ for some n and $g_1,\dots,g_n$.
3. The morphism f is étale near every associated point of C.

rmk-mild

Remark 2.2. There are a couple of remarks on the previous definition

1. Together items (1) and (2) imply that f is flat.
2. Item (3) implies that f is generically étale and that no embedded point is also a ramification point.
3. This is not the most general definition; in particular admissible coverings are typically not mild because they do not satisfy item (2) over a node of the target curve. In the following lemma, item (2) is used to show that a certain subscheme is *locally unobstructed* in the sense of [6, Def I.2.11.1]. For admissible coverings, probably the subscheme is again locally unobstructed so that the following lemma is still true.

lem-mild

Lemma 2.3. *Let $\pi : \mathcal{B} \to S$ be a flat, proper family of curves, let $x \in S$ be a closed point and let $f : C \to \mathcal{B}_x$ be a morphism of proper curves. If f is mild, then there exists an étale morphism $g : (S', x') \to (S, x)$, a flat, proper family of curves $\rho : \mathcal{C} \to S'$, and a flat S'-morphism $\Phi : \mathcal{C} \to \mathcal{B} \times_S S'$ whose fiber over x' is the base-change of f. Moreover every fiber of Φ is a mild morphism.*

Proof. After an étale base change of (S, x), we may assume that π is projective. Let $\mathcal{L}$ be a relatively ample invertible sheaf on $\mathcal{B}$. In a standard way we may find a closed immersion $\iota : C \to (\mathbb{P}^1)^r$ for some r such that for each projection $\mathrm{pr}_j : (\mathbb{P}^1)^r \to \mathbb{P}^1$ we have $H^1(C, \iota^*\mathrm{pr}_j^*\mathcal{O}_{\mathbb{P}^1}(2))$ is zero. Consider the morphism $(f, \iota) : C \to \mathcal{B}_x \times (\mathbb{P}^1)^r$. This is a closed immersion since ι is a closed immersion. Let $\mathcal{I}$ denote the ideal sheaf of this closed immersion. Since f is a local complete intersection morphism, (f, ι) is a regular embedding, i.e. $\mathcal{I}/\mathcal{I}^2$ is a locally free $\mathcal{O}_C$-module. By [6, Lemma I.2.12], $C \subset \mathcal{B}_x \times (\mathbb{P}^1)^r$ is locally unobstructed.

Denote by N the locally free $\mathcal{O}_C$-module $\mathit{Hom}_{\mathcal{O}_C}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C)$. There is a tautological map of $\mathcal{O}_C$-modules $\alpha\colon \iota^*T_{(\mathbb{P}^1)^r} \to N$. Since $f : C \to B$ is étale near every associated point of C, α is an injective sheaf map and the cokernel is a torsion sheaf. Consider the short exact sequence:

$$0 \longrightarrow \iota^*T_{(\mathbb{P}^1)^r} \xrightarrow{\alpha} N \longrightarrow \mathrm{coker}(\alpha) \longrightarrow 0. \tag{1}$$

By the long exact sequence of cohomology associated to this short exact sequence, we conclude that $H^1(C, \iota^*T_{(\mathbb{P}^1)^r}) \to H^1(C, N)$ is surjective. But of course $\iota^*T_{(\mathbb{P}^1)^r} \cong \oplus_{j=1}^r \iota^*\mathrm{pr}_j^*\mathcal{O}_{\mathbb{P}^1}(2)$, so $H^1(C, \iota^*T_{(\mathbb{P}^1)^r})$ is zero.

Let $P(t)$ be the Hilbert polynomial of $C \subset \mathcal{B}_x \times (\mathbb{P}^1)^r$ with respect to the ample invertible sheaf $\mathcal{L} \otimes \mathcal{O}_{(\mathbb{P}^1)^r}(1, 1, \dots, 1)$. Consider the relative Hilbert scheme $u : \mathcal{H} \to S$ where $\mathcal{H} = \mathit{Hilb}^{P(t)}(\mathcal{B} \times (\mathbb{P}^1)^r/S)$. We have the point $[C] \in \mathcal{H}$. As proved above, this point is locally unobstructed and $H^1(C, N)$ is zero. So by [6, Thm I.2.10, Prop I.2.14], we have that u is smooth at $[C]$. By étale slicing, we may find an étale morphism $g : (S', x') \to (S, x)$ and a section of $\mathcal{H} \times_S S'$ passing through $(x', [C])$. The pullback by this section of the universal family over $\mathcal{H}$ gives the flat, proper family of curves $\rho : \mathcal{C} \to S'$. And the projection of the universal family to $\mathcal{B}$ gives the morphism $\Phi : \mathcal{C} \to \mathcal{B} \times_S S'$. Since the items (1)–(3) in Definition 2.1 are clearly open conditions on S', up to shrinking S' we may suppose that every fiber of Φ is a mild morphism. □

subsec-cover

2.1. **Construction of the special covering.** Let S be a smooth, connected curve over k and let $x_0 \in S$ be a closed point. Let $t_1, \dots, t_{n-1} \in \mathbb{P}^1$ be $n-1$ distinct points. Let $\pi : \mathcal{B} \to S$ be the blowing up of $S \times \mathbb{P}^1$ in the points $(x_0, t_1), \dots, (x_0, t_{n-1})$. Denote by $\widetilde{B}_0 \subset \mathcal{B}_0$ the strict transform of $x_0 \times \mathbb{P}^1$. Denote by $E_1, \dots, E_{n-1} \subset \mathcal{B}_0$ the exceptional divisors above $(x_0, t_1), \dots, (x_0, t_{n-1})$ respectively. The curve $\mathcal{B}_0$ is a connected, proper, at-worst-nodal curve of arithmetic genus 0 whose irreducible components are $\widetilde{B}_0, E_1, \dots, E_{n-1}$. Moreover, for each $i = 1, \dots, n-1$ we have that $\widetilde{B}_0 \cap E_i$ consists of a unique node p_i, and for $i \neq j$ we have $E_i \cap E_j = \emptyset$.

For each $i = 1, \dots, n-1$ choose an isomorphism $E_i \cong \mathbb{P}^1$ such that the node $p_i \in E_i$ is identified with $[1 : 0] \in \mathbb{P}^1$. For each $j = 1, \dots, n$ define $f_0^{(j)} : \widetilde{B}_0^{(j)} \to \widetilde{B}_0$ to be any isomorphism, and define $p_1^{(j)}, \dots, p_{n-1}^{(j)} \in \widetilde{B}_0^{(j)}$ to be the preimages of

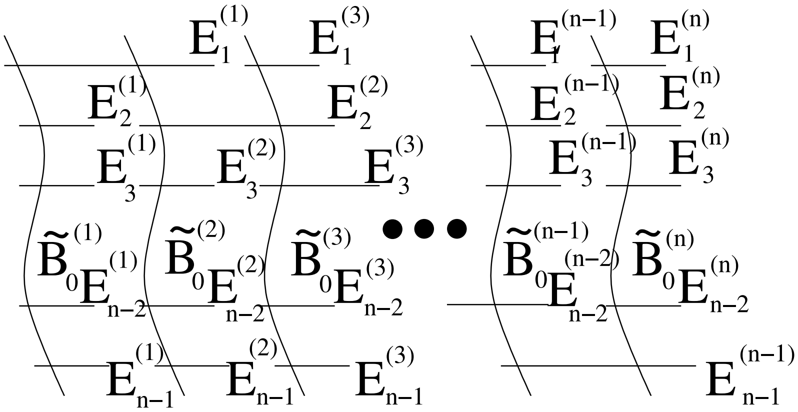

FIGURE 1. The reducible curve C

$p_1, \dots, p_{n-1} \in \widetilde{B}_0$. For $j = 1, \dots, n$ and for $i = 1, \dots, n-1$ satisfying $j \neq i, j \neq i+1$, define $f_i^{(j)} : E_i^{(j)} \to E_j$ to be any isomorphism, and define $q_i^{(j)} \in E_i^{(j)}$ to be the preimage of $p_i \in E_i$.

For $i = 1, \dots, n-1$ we define $f_i^{(i)} : E_i^{(i)} \to E_i$ to be the map $\mathbb{P}^1 \to \mathbb{P}^1$ given by $[X_0 : X_1] \mapsto [X_0^2 : X_1^2 - X_0X_1]$. The ramification divisor of this map is given by the equation $X_0(X_0 - 2X_1) = 0$ (so in characteristic 2 this is just $X_0^2 = 0$) and the branch divisor is given by the equation $X_0(X_0 + 4X_1)$ (so in characteristic 2 this is just $X_0^2 = 0$). Observe that the map is unramified over the point in the target $[1 : 0] \in \mathbb{P}^1$, i.e. $f_i^{(i)}$ is unramified over p_i. Also the map is Galois with Galois group cyclic of order 2. Let $\gamma_i : E_i^{(i)} \to E_i^{(i)}$ denote the nontrivial involution such that $f_i^{(i)} \circ \gamma_i = f_i^{(i)}$, i.e. $\gamma_i[X_0 : X_1] = [X_0 : X_0 - X_1]$. Choose a labelling of the preimage of p_i as $\{q_i^{(i)}, q_i^{(i+1)}\}$. For convenience, we define $f_i^{(i+1)} : E_i^{(i+1)} \to E_i$ to be the same as $f_i^{(i)}$.

We define C to be the connected, proper, at-worst-nodal curve of arithmetic genus 0 whose irreducible components are the n curves $\widetilde{B}_0^{(j)}, j = 1, \dots, n$, the $(n-1)(n-2)$ curves $E_j^{(i)}, i = 1, \dots, n-1, j = 1, \dots, n$ and $j \neq i, i+1$, and the $n-1$ curves $E_j^{(j)}, j = 1, \dots, n-1$ (see Figure 1). For each $i = 1, \dots, n-1, j = 1, \dots n$ such that $j \neq i+1$, we identify the point $p_i^{(j)} \in \widetilde{B}_0^{(j)}$ with $q_i^{(j)} \in E_i^{(j)}$. And for $i = 1, \dots, n-1$, we identify the point $p_i^{(i+1)} \in \widetilde{B}_0^{(i+1)}$ with $q_i^{(i+1)} \in E_i^{(i)}$. It is straightforward to check that there is a unique morphism $f : C \to B$ whose restriction to every irreducible component of C is the morphism $f_i^{(j)}$ defined above. Moreover, this morphism is étale away from the points $[1 : 0], [0 : 1] \in E_i, i = 1, \dots, n$. In particular, f is mild.

By Lemma 2.3, after making an étale base change of S, we can find a flat, proper family of curves $\rho : \mathcal{C} \to S$ and a flat morphism $\Phi : \mathcal{C} \to \mathcal{B}$ whose fiber over x_0 is just f and such that every fiber of Φ is mild. Shrinking S if necessary, we may suppose that every geometric fiber of ρ is a connected, at-worst-nodal curve of arithmetic genus 0 and that the ramification locus of Φ is contained in the smooth locus of ρ. For every $x \neq x_0$ in S, $\mathcal{C}_x$ is smooth at every ramification point of Φ_x. On the other hand, it is also smooth at every point where Φ_x is étale since $\mathcal{B}_x \cong \mathbb{P}^1$ is smooth. Thus $\mathcal{C}_x$ is smooth, i.e. $\rho : \mathcal{C} \to S$ is smooth away from x_0. So for $x \neq x_0$, $\Phi : \mathcal{C}_x \to \mathcal{B}_x$ is equivalent to a degree n map $\mathbb{P}^1 \to \mathbb{P}^1$.

subsec-Galois

2.2. **Computation of the Galois group.** Define $U \subset \mathcal{B}$ to be the largest open subset over which Φ is étale and define $W = \Phi^{-1}(U)$. For every positive integer a, denote by W^a_Φ the k-fold fiber product of W with itself over U. For each pair $1 \leq b < c \leq a$, there is a diagonal map $\Delta_{b,c} : W^{a-1}_\Phi \to W^a_\Phi$ defined so that $\mathrm{pr}_i \circ \Delta_{b,c} = \mathrm{pr}_i$ for $i < c$, $\mathrm{pr}_c \circ \Delta_{b,c} = \mathrm{pr}_b$, and $\mathrm{pr}_i \circ \Delta_{b,c} = \mathrm{pr}_{i-1}$ for $i > c$. The image of $\Delta_{b,c}$ is an open and closed subset of W^a_Φ. We define $\widetilde{W}^a_\Phi$ to be the complement of the union of the images of all maps $\Delta_{b,c}$. In particular, we denote $\mathcal{G} = \widetilde{W}^n_\Phi$. Loosely speaking, every fiber of $\mathcal{G} \to U$ parametrizes ordered n-tuples of distinct points in the fiber of Φ. Since Φ has degree n, any two ordered n-tuples of distinct points in a fiber of Φ are equivalent up to reordering.

The significance of $\mathcal{G}$ is the following: For $x \neq x_0$, the action of the Galois group of the field extension $K(\mathcal{B}_x) \to K(\mathcal{C}_x)$ on the geometric generic fiber of Φ_x is the full symmetric group of the fiber iff $\mathcal{G}_x$ is connected. Moreover, in this case the function field of $\mathcal{G}_x$ is a Galois closure of $K(\mathcal{C}_x)$.

Let $\overline{\mathcal{G}}$ be the normalization of $\mathcal{B}$ in the function field of $\mathcal{G}$, so the morphism $g : \overline{\mathcal{G}} \to \mathcal{B}$ is finite. The projection $\pi \circ g : \overline{\mathcal{G}} \to S$ is proper and flat. By a standard argument (e.g. [4, Exercise III.11.4]), to prove that every closed fiber of $\pi \circ g$ is connected, it suffices to prove that one closed fiber is a connected, reduced scheme.

lem-connd

Lemma 2.4. *The scheme $\mathcal{G}_0$ is connected and reduced. Therefore the Cartier divisor $\overline{\mathcal{G}}_0 \subset \overline{\mathcal{G}}$ is connected and reduced.*

Proof. Consider $g_0 : \mathcal{G}_0 \to U \cap \mathcal{B}_0$. This is a finite, étale morphism. Since $\mathcal{B}_0$ is reduced, also $\mathcal{G}_0$ is reduced. Since $\mathcal{B}_0$ is connected, every connected component of $\mathcal{G}_0$ surjects onto $\mathcal{B}_0$. Let $p \in \widetilde{B}_0$ be any closed point in the smooth locus of $\mathcal{B}_0$. The preimage $f^{-1}(p)$ is just $\left\{p^{(1)}, p^{(2)}, \ldots, p^{(n)}\right\}$ where $p^{(j)} \in \widetilde{B}_0^{(j)}$ is the unique mapping to p. The preimage $g_0^{-1}(p)$ is equivalent to the symmetric group S_n via the bijection

$$\sigma \in S_n \mapsto \widehat{\sigma} = \left((p^{(\sigma(1))}, p^{(\sigma(2))}, \ldots, p^{(\sigma(n))}\right). \tag{2}$$

Denote by $e \in S_n$ the identity permutation. To prove that $\mathcal{G}_0$ is connected, it suffices to prove that for every permutation $\sigma \in S_n$, the element $\widehat{\sigma}$ is in the same connected component of $\mathcal{G}_0$ as $\widehat{e}$.

Suppose $\sigma \in S_n$ is a permutation. The transpositions $(1,2), (2,3), \ldots, (n-1,n)$ generate the symmetric group S_n. Thus every permutation σ is equal to a product $\tau_N \cdot \tau_{N-1} \cdot \cdots \cdot \tau_1$ for some N and each $\tau_i \in \{(1,2), \ldots, (n-1,n)\}$. Let $N(\sigma)$ be the least N such that σ has such an expression, and set $N(e) = 0$. We will prove by induction on $N(\sigma)$ that $\widehat{\sigma}$ is in the same component as $\widehat{e}$.

If $N(\sigma) = 0$, then $\sigma = e$ and there is nothing to prove. Therefore suppose $N = N(\sigma) > 0$, and by way of induction suppose we have proved the result for all $\sigma' \in S_n$ with $N(\sigma') < N$. Let $\sigma = \tau_N \cdot \cdots \cdot \tau_1$ and define $\sigma' = \tau_N \cdot \sigma$ so that $N(\sigma') = N - 1$. Denote by i the unique integer such that $\tau_N = (i, i+1)$. Denote by $r : \widetilde{B}_0 \to \mathcal{G}_0$ the unique morphism such that for each $j = 1, \ldots, n$ we have $\mathrm{pr}_j \circ r = \left(f_0^{(\sigma'(j))}\right)^{-1}$. Since $\widetilde{B}_0$ is irreducible, the image of r is contained in one connected component of $\mathcal{G}_0$. Therefore $\widehat{(\sigma')} = r(p)$ is contained in the same connected component as $r(p_i) = \left(p_i^{(\sigma'(1))}, \ldots, p_i^{(\sigma'(n))}\right)$.

Denote by a and b the integers such that $\sigma'(a) = i$, $\sigma'(b) = i+1$. Let $R \subset E_i^{(i)}$ denote the ramification divisor of $f_i^{(i)}$. Denote by $s : E_i^{(i)} - R \to \mathcal{G}_0$ the unique morphism such that for $j \neq a, b$ we have

$$\mathrm{pr}_j \circ s = \left(f_i^{(\sigma'(j))}\right)^{-1} \circ f_i^{(i)}, \quad \mathrm{pr}_a \circ s = \mathrm{Id}_{E_i^{(i)}}, \quad \mathrm{pr}_b \circ s = \gamma_i. \tag{3}$$

Since $E_i^{(i)} - R$ is irreducible, the image of s is contained in one connected component of $\mathcal{G}_0$. By construction $r(p_i) = \left(p_i^{(\sigma'(1))}, \ldots, p_i^{(\sigma'(n))}\right)$ equals $s(q_i^{(i)})$ and $s(q_i^{(i+1)})$ equals $\left(p_i^{(\sigma(1))}, \ldots, p_i^{(\sigma(n))}\right)$. Therefore $r(p_i) = s(q_i^{(i)})$ is contained in the same connected component as $s(q_i^{(i+1)}) = \left(p_i^{(\sigma(1))}, \ldots, p_i^{(\sigma(n))}\right)$.

Finally, denote by $t : \widetilde{B}_0 \to \mathcal{G}_0$ the unique morphism such that for every $j = 1, \ldots, n$ we have $\mathrm{pr}_j \circ t = \left(f_0^{(\sigma(j))}\right)^{-1}$. Since $\widetilde{B}_0$ is irreducible, the image of t is contained in one connected component of $\mathcal{G}_0$. By construction $t(p_i) = \left(p_i^{(\sigma(1))}, \ldots, p_i^{(\sigma(n))}\right) = s(q_i^{(i+1)})$ and $t(p) = \widehat{(\sigma)}$. Therefore $s(q_i^{(i+1)})$ is contained in the same connected component as $\widehat{\sigma}$. Putting the pieces together, we have that $\widehat{\sigma'}$ is contained in the same connected component as $\widehat{\sigma}$. Since $N(\sigma') = N - 1$, by the induction assumption $\widehat{\sigma'}$ is in the same connected component as $\widehat{e}$. By transitivity, $\widehat{\sigma}$ is in the same connected component as $\widehat{e}$, and we conclude that $\mathcal{G}_0$ is connected by induction on N.

Since $\overline{\mathcal{G}}$ is normal, it satisfies Serre's condition (S_2). Since $\overline{\mathcal{G}}_0 \subset \overline{\mathcal{G}}$ is a Cartier divisor, $\overline{\mathcal{G}}_0$ satisfies Serre's condition (S_1), i.e. $\overline{\mathcal{G}}_0$ is Cohen-Macaulay. Since it is generically reduced and connected, it is reduced and connected. This finishes the proof of the lemma. $\square$

Combining the lemma with the arguments above, we conclude that for every $x \in S$, the fiber of $\pi \circ g : \overline{\mathcal{G}} \to S$ over x is connected. Of course for $x \neq x_0$ we have that $\mathcal{G}_x$ is smooth, so also we have that the fiber of $\overline{\mathcal{G}}$ over x is reduced, in particular $\pi \circ g$ is separably generated. Since $\overline{\mathcal{G}}$ is normal, by Serre's criterion it is regular in codimension 1. Putting this all together, we conclude that $\pi \circ g : \overline{\mathcal{G}} \to S$ is smooth with geometrically connected fibers over some dense open subset $S^o \subset S$. Therefore over S^o, the morphism $\mathcal{G} \to S^o$ has geometrically connected fibers. Therefore we conclude the following proposition.

prop-Galois

Proposition 2.5. *There exists a dense open subset $S^o \subset S$ such that for every closed point $x \in S^o$, the morphism $\Phi_x : \mathcal{C}_x \to \mathcal{B}_x$ is equivalent to a finite, separably-generated morphism of degree n, $\mathbb{P}^1 \to \mathbb{P}^1$, whose Galois group acts as the full symmetric group S_n on the geometric generic fiber of $\mathbb{P}^1 \to \mathbb{P}^1$. In particular, there exists a degree n morphism $\mathbb{P}^1 \to \mathbb{P}^1$ whose Galois group is the full symmetric group S_n.*

2.3. **Variant of the special covering.** For the Enriques construction, we need a slightly stronger result which is proved with the same technique. Let $f : C \to B$ be a degree n map from $\mathbb{P}^1$ to $\mathbb{P}^1$ and let $g : D \to C$ be a degree m map from $\mathbb{P}^1$ to $\mathbb{P}^1$. The Galois group of $K(D)/K(B)$ acts on the geometric generic fiber of $f \circ g : D \to B$. We can label the points on the geometric generic fiber of f as $c_1, \ldots, c_n$ and we can label the preimages of c_i under g as $(d_{i,1}, \ldots, d_{i,m})$. Then the Galois group acts on the set $\{d_{1,1}, \ldots, d_{1,m}, \ldots, d_{n,1}, \ldots, d_{n,m}\}$ by permutations which preserve the incidence relation that $d_{i,j}$ is related to $d_{i,k}$. The group of such permutations is the wreath product:

$$1 \longrightarrow (S_m)^n \longrightarrow W_{n,m} \longrightarrow S_n \longrightarrow 1 \tag{4}$$

Recall that $W_{n,m}$ is the semi-direct product of the product group $(S_m)^n$ and S_n where S_n acts on $(S_m)^n$ by permuting the factors.

The result we will prove is that there exists a pair $(g : D \to C, f : C \to B)$ such that the Galois group of $K(D)/K(B)$ acts as the full group $W_{n,m}$ on the geometric generic fiber of $f \circ g : D \to B$. By the same deformation theory as above, it suffices to construct covers of at-worst-nodal curves of arithmetic genus 0, $(g : D \to C, f : C \to B)$ such that the associated Galois scheme is irreducible. The details are mostly left to the reader: we only describe the covers which work.

Let S be a smooth, connected curve over k and let $x_0 \in S$ be a closed point. Let $t_1, \ldots, t_{n-1}, u_1, \ldots, u_{m-1} \in \mathbb{P}^1$ be $(n-1)+(m-1)$ distinct points. Let $\pi : \mathcal{B} \to S$ be the blowing up of $S \times \mathbb{P}^1$ in the points $(x_0, t_1), \ldots, (x_0, t_{n-1}), (x_0, u_1), \ldots, (x_0, u_{m-1})$ Denote by $\widetilde{B}_0 \subset \mathcal{B}_0$ the strict transform of $x_0 \times \mathbb{P}^1$. Denote by $E_1, \ldots, E_{n-1} \subset \mathcal{B}_0$ the exceptional divisors above $(x_0, t_1), \ldots (x_0, t_{n-1})$ respectively. And denote by $F_1, \ldots, F_{m-1} \subset \mathcal{B}_0$ the exceptional divisors above $(x_0, u_1), \ldots, (x_0, u_{m-1})$ respectively. For each $i = 0, \ldots, n-1$ let $p_i \in \widetilde{B}_0 \cap E_i$ be the unique node and for $i = 0, \ldots, m-1$ let $q_i \in \widetilde{B}_0 \cap F_i$ be the unique node.

First we describe the degree n map $f : C \to \mathcal{B}_0$. It is just as in Subsection 2.1. For each $j = 1, \ldots, n$ we have $f_0^{(j)} : \widetilde{B}_0^{(j)} \to \widetilde{B}_0^{(j)}$ and $f_{E,i}^{(j)} : E_i^{(j)} \to E_i$ just as before. And in addition, for each $i = 1, \ldots, m-1$ we define $f_{F,i}^{(j)} : F_i^{(j)} \to F_i$ to be an isomorphism. We define $q_i^{(j)} \in \widetilde{B}_0^{(j)}$ to be the preimage of $q_i \in \widetilde{B}_0$ and we define $q_i^{(j)} \in F_i^{(j)}$ to be the preimage of $q_i \in F_i$. We glue these irreducible components in the obvious way. In other words, the cover $f : C \to B$ is precisely as in Subsection 2.1 except that for the curve $\widetilde{B}_0$ and each of the curves $\widetilde{B}_0^{(j)}$ we have attached the $m-1$ rational curves F_i and $F_i^{(j)}$ respectively.

Now we describe the map $g : D \to C$. For each $j = 1, \ldots, n$ and each $k = 1, \ldots, m$, we define $g_0^{(j,k)} : \widetilde{B}_0^{(j,k)} \to \widetilde{B}_0^{(j)}$ and $g_{E,i}^{(j,k)} : E_i^{(j,k)} \to E_i^{(j)}$ to be isomorphisms. We define $p_i^{(j,k)} \in \widetilde{B}_0^{(j,k)}$ and $q_i^{(j,k)} \in \widetilde{B}_0^{(j,k)}$ to be the preimages under $f_0^{(j,k)}$ of $p_i^{(j)} \in \widetilde{B}_0^{(j)}$ and $q_i^{(j)} \in \widetilde{B}_0^{(j)}$. Similarly, we define $p_i^{(j,k)} \in E_i^{(j,k)}$ to be the preimage under $f_{E,i}^{(j,k)}$ of $p_i^{(j)} \in E_i^{(j)}$.

For each (j,k) such that $j > 1$ we define $g_{F,i}^{(j,k)} : F_i^{(j,k)} \to F_i^{(j)}$ to be an isomorphism and we define $q_i^{(j,k)} \in F_i^{(j,k)}$ to be the preimage under $g_{F,i}^{(j,k)}$ of $q_i^{(j)} \in F_i^{(j)}$. For each $(1,k)$ and i such that $k \neq i, i+1$, we define $g_{F,i}^{(j,k)} : F_i^{(1,k)} \to F_i^{(1)}$ to be an isomorphism and we define $q_i^{(1,k)} \in F_i^{(1,k)}$ to be the preimage under $g_{F,i}^{(1,k)}$ of $q_i^{(1)} \in F_i^{(1)}$.

Finally, for each $i = 1, \dots, m-1$ we identify $F_i^{(1)}$ with $\mathbb{P}^1$ where the node $q_i^{(1)}$ corresponds to $[1:1] \in \mathbb{P}^1$, and we define $g_{F,i}^{(1,i)} : F_i^{(1,i)} \to F_i^{(i)}$ to be the map $\mathbb{P}^1 \to \mathbb{P}^1$ given by $[X_0 : X_1] \mapsto [X_0^2 : X_1^2 - X_0X_1]$. Observe that $g_{F,i}^{(1,i)}$ is unramified over $[1:0]$, i.e. it is unramified over $q_i^{(1)}$. Also the map is Galois with Galois group cyclic of order 2. Let $\gamma_i : F_i^{(1,i)} \to F_i^{(1,i)}$ denote the generator of this Galois group, i.e. $\gamma_i[X_0 : X_1] = [X_0 : X_0 - X_1]$. Choose a labelling of the preimage of $q_i^{(1)}$ as $\{q_i^{(1,i)}, q_i^{(1,i+1)}\}$. For convenience, we define $g_{F,i}^{(1,i+1)} : F_i^{(1,i+1)} \to F_i^{(1)}$ to be the same as $g_{F,i}^{(1,i+1)}$.

We define D to be the connected, proper, at-worst-nodal curve of arithmetic genus 0 whose irreducible components are the mn curves $\widetilde{B}_0^{(j,k)}$, the $mn(n-1)$ curves $E_i^{(j,k)}$, the $m(n-1)(m-1)$ curves $F_i^{(j,k)}$ with $j > 1$, the $(m-1)(m-2)$ curves $F_i^{(1,k)}$ with $k \neq i, i+1$, and the $m-1$ curves $F_i^{(1,i)}$. For each i, j, k we identify the point $p_i^{(j,k)}$ in $\widetilde{B}_0^{(j,k)}$ with the point $p_i^{(j,k)}$ in $E_i^{(j,k)}$. For each i, j, k with either $j > 1$ or $k \neq i, i+1$, we identify the point $q_i^{(j,k)}$ in $\widetilde{B}_0^{(j,k)}$ with the point $q_i^{(j,k)}$ in $F_i^{(j,k)}$. Finally, for each $i = 1, \dots, m-1$, we identify the point $q_i^{(1,i)}$ in $\widetilde{B}_0^{(1,i)}$ with the point $q_i^{(1,i)}$ in $F_i^{(1,i)}$ and we identify the point $q_{i+1}^{(1,i)}$ in $\widetilde{B}_0^{(1,i)}$ with the point $q_{i+1}^{(1,i)}$ in $F_i^{(1,i)}$.

It is straightforward to check that there is a unique morphism $g : D \to C$ whose restriction to every irreducible component of D is the morphism $g_0^{(j,k)}$, resp. $g_{E,i}^{(j,k)}$, $g_{F,i}^{(j,k)}$ constructed above. Of course f and g are mild, so by Lemma 2.3 after making an étale basechange of S we can suppose that we have morphisms $\Phi : \mathcal{C} \to \mathcal{B}$ and $\Gamma : \mathcal{D} \to \mathcal{C}$ extending f and g respectively. And we can assume that for every geometric point $x \in S$ with $x \neq x_0$, the fibers $\mathcal{C}_x$ and $\mathcal{D}_x$ are smooth, geometrically connected curves of genus 0.

To prove that the Galois group of a general map $\Phi_x \circ \Gamma_x : \mathcal{D}_x \to \mathcal{B}_x$ acts as the full symmetric group $W_{n,m}$, we perform a similar computation as in Subsection 2.2. We define $U \subset \mathcal{B}$ to be the locus over which $\Phi \circ \Gamma$ is étale and we define $W = (\Phi \circ \Gamma)^{-1}(U)$. The one difference is in the definition of $\mathcal{G} \to U$. Define L to be the reduced, 0-dimensional scheme over Spec k with mn closed points labelled $(l^{(j,k)})$ for $j = 1, \dots, n$ and $k = 1, \dots, m$. We define $\mathcal{G} \to U$ to be the open and closed subscheme of the relative Hom scheme $Hom_U(L \times U, W)$ (c.f. [6, Def. I.1.9]) whose fiber over a geometric point $u \in U$ is the set of bijections $\lambda : L \times \text{Spec } \kappa(u) \to W_u$ such that for every $j = 1, \dots, n$ and every k, k' we have that $\lambda(l^{(j,k)})$ and $\lambda(l^{(j,k')})$ are contained in the same fiber of Γ.

The significance of $\mathcal{G}$ is the following: for $x \neq x_0$, the action of the Galois group of $\Phi_x \circ \Gamma_x$ is the full group $W_{n,m}$ iff $\mathcal{G}_x$ is connected. As in Subsection 2.2, we can reduce this to the case x_0 which can be checked in the same way as in Subsection 2.2. We conclude the following proposition.

prop-Wnm

Proposition 2.6. *There exists a degree n morphism $f_x : \mathbb{P}^1 \to \mathbb{P}^1$ and a degree m morphism $g_x : \mathbb{P}^1 \to \mathbb{P}^1$ such that the Galois group of $f_x \circ g_x : \mathbb{P}^1 \to \mathbb{P}^1$ acts as the full group $W_{n,m}$ on the geometric generic fiber of $f_x \circ g_x$.*

3. The basic construction

sec-constr

Let V be a vector space of dimension $n+1$ over k and let d be an integer. The degree d hypersurfaces in $\mathbb{P}(V)$ are parametrized by the projective space $\mathbb{P}(\mathrm{Sym}^d V^\vee)$. Let B be a curve isomorphic to $\mathbb{P}^1$. By Proposition 2.5, we can find a curve C isomorphic to $\mathbb{P}^1$ and a finite, separably-generated morphism $f : C \to B$ of degree d whose Galois group acts as the full symmetric group S_d. Let $g : C \to \mathbb{P}(V^\vee)$ be a closed immersion whose image is a rational normal curve of degree n.

Consider the pullback of the tautological surjection, $V \otimes_k \mathcal{O}_C \to g^*\mathcal{O}(1)$. By adjointness, there is an induced map $\beta : V \otimes_k \mathcal{O}_B \to f_*(g^*\mathcal{O}(1))$. Now for any locally free sheaf $\mathcal{E}$ on C, there is a locally free sheaf on B

$$\mathrm{Nm}_f(\mathcal{E}) = \mathit{Hom}_{\mathcal{O}_B}(\bigwedge^d (f_*\mathcal{O}_B), \bigwedge^d (f_*\mathcal{E})) \tag{5}$$

called the *norm sheaf* of $\mathcal{E}$, and there is a map of $\mathcal{O}_B$-modules,

$$\alpha'_{\mathcal{E}} : \bigotimes^d (f_*\mathcal{E}) \to \mathrm{Nm}_f(\mathcal{E}) \tag{6}$$

, called the *norm map* of $\mathcal{E}$ which locally over B maps a tensor $e_1 \otimes \cdots \otimes e_d \in \bigotimes^d (f_*\mathcal{E})$ to the map $\phi \in \mathrm{Nm}_f(\mathcal{E})$ which sends an element $b_1 \wedge \cdots \wedge b_d \in \bigwedge^d (f_*\mathcal{O}_B)$ to $(b_1 \cdot e_1) \wedge \cdots \wedge (b_d \cdot e_d)$ where the products $b \cdot e$ are induced by the $\mathcal{O}_C$-module structure on $\mathcal{E}$. We shall only need the restriction of $\alpha'_{\mathcal{E}}$ to the subsheaf of symmetric tensors:

$$\alpha_{\mathcal{E}} : \mathrm{Syt}^d(f_*\mathcal{E}) \to \mathrm{Nm}_f(\mathcal{E}). \tag{7}$$

In particular, when $\mathcal{E} = \mathcal{O}_C$ then $\mathrm{Nm}_f(\mathcal{E}) = \mathcal{O}_B$ and the linear functional $\alpha_{\mathcal{O}_C} : \mathrm{Syt}^d(f_*\mathcal{O}_C) \to \mathcal{O}_B$ is equivalent to the polynomial $\alpha_{\mathcal{O}_C} \in \mathrm{Sym}^d(f_*\mathcal{O}_C)^\vee$ which is the usual norm (where we are using the canonical perfect pairing between $\mathrm{Syt}^d(\mathcal{F})$ and $\mathrm{Sym}^d(\mathcal{F}^\vee)$ for any locally free sheaf $\mathcal{F}$).

Denote by γ the composition of $\mathrm{Syt}^d(\beta) : \mathrm{Syt}^d V \otimes_k \mathcal{O}_B \to \mathrm{Syt}^d(f_*g^*\mathcal{O}(1))$ with the norm map $\alpha_{g^*\mathcal{O}(1)} : \mathrm{Syt}^d(f_*g^*\mathcal{O}(1)) \to \mathrm{Nm}_f(g^*\mathcal{O}(1))$. The claim is that γ is surjective. To see this, let $b \in B$ be a closed point and let $c_1, \dots, c_r$ be the closed points of C lying above b. For each $i = 1, \dots, r$ we have that the induced k-linear map $V \to g^*\mathcal{O}(1) \otimes_{\mathcal{O}_C} \kappa(c_i)$ is surjective. The kernel is a codimension one linear subspace. Since k is infinite, there exists an element $v \in V$ which is in none of these kernels. Let $U \subset C$ be the open subscheme where the induced map $\mathcal{O}_C \xrightarrow{v} V \otimes_k \mathcal{O}_C \to g^*\mathcal{O}(1)$ is surjective and let $D = C - U$. Then D is a closed subscheme and $f(D) \subset B$ does not contain b. So $W = B - f(D)$ is an open subscheme of B containing b and over W we have that $\mathcal{O}_C \xrightarrow{v} g^*\mathcal{O}(1)$ is an isomorphism. Via the induced isomorphism $\mathrm{Nm}_f(g^*\mathcal{O}(1))|_W \cong \mathrm{Nm}_f(\mathcal{O}_C)|_W$, the element $\gamma(v \otimes v \otimes \cdots \otimes v) \in \mathrm{Nm}_f(g^*\mathcal{O}(1))$ corresponds to $1 \in \mathcal{O}_B = \mathrm{Nm}_f(\mathcal{O}_C)$. So we conclude that γ is surjective. Therefore there is an induced morphism $h : B \to \mathbb{P}(\mathrm{Sym}^d(V^\vee))$ such that the pullback of the tautologial surjection on $\mathbb{P}(\mathrm{Sym}^d(V^\vee))$ equals $\gamma : \mathrm{Syt}^d V \otimes_k \mathcal{O}_B \to \mathrm{Nm}_f(g^*\mathcal{O}(1))$. Speaking loosely, if $b \in B$ is a closed point and $c_1, \dots, c_d$ are distinct closed points lying above b which map to elements

$[\Phi_1], \dots, [\Phi_d] \in \mathbb{P}(V^\vee)$ under g, then $h(b)$ is the element $[\Phi_1 \cdot \Phi_2 \cdot \dots \cdot \Phi_d] \in \mathbb{P}(\mathrm{Sym}^d(V^\vee))$. Observe that the degree of the invertible sheaf $\mathrm{Nm}_f(g^*\mathcal{O}(1))$ on B equals the degree of the invertible sheaf $g^*\mathcal{O}(1)$ on C, namely n.

Let $\mathcal{X}_h \subset \mathbb{P}(V) \times B$ denote the pullback by h of the universal hypersurfaces in $\mathbb{P}(V) \times \mathbb{P}(\mathrm{Sym}^d V^\vee)$. Let $\pi : \mathcal{X}_h \to B$ denote the projection morphism. Let $U \subset B$ denote the largest open subset over which f is finite étale and define $W = f^{-1}(U)$. Every geometric fiber of $\mathcal{X}_h \times_B U \to U$ is a simple normal crossings variety in $\mathbb{P}(V)$ which is a union of d hyperplanes. There is a stratification of $\mathcal{X}_h \times_B U$ into reduced, locally closed subschemes

$$\mathcal{X}_h \times_B U = \mathcal{X}_h^1 \sqcup \mathcal{X}_h^2 \sqcup \dots \sqcup \mathcal{X}_h^n \tag{8}$$

such that for every geometric point $b \in U$ the set $(\mathcal{X}_h^k)_b$ is the set of points in $(\mathcal{X}_h)_b$ which lie in the intersection of precisely k of the irreducible components of $(\mathcal{X}_h)_b$. Define $m = \min(n, d)$. If $k > m$, then $\mathcal{X}_h^k$ is empty and for $k \le m$, $\mathcal{X}_h^k$ is nonempty.

For each $k = 1, \dots, m$, the morphism $\mathcal{X}_h^k \to B^o$ is smooth. For each $k = 1, \dots, m$ there exists a scheme $\overline{\mathcal{X}_h^k}$ containing $\mathcal{X}_h^k$ as a dense open subscheme along with a morphism $\overline{\mathcal{X}_h^k} \to \mathbb{P}(V) \times U$ which restricts to the inclusion map on $\mathcal{X}_h^k$ and such that the projection $\overline{\mathcal{X}_h^k} \to U$ is proper and smooth, and every irreducible component of every geometric fiber of $\overline{\mathcal{X}_h^k} \to U$ maps to a linear $(n-k)$-space in $\mathbb{P}(V)$. The Stein factorization of $\overline{\mathcal{X}_h^k} \to U$ is of the form $\overline{\mathcal{X}_h^k} \xrightarrow{\rho_k} \widetilde{W}_k \xrightarrow{\sigma_k} U$ where $\widetilde{W}_f^k$ is as in Subsection 2.2, where $\widetilde{W}_k$ is the quotient by the action of the symmetric group $\widetilde{W}_f^k / S_k$, where σ_k is the obvious induced map, and where ρ_k is a projective bundle.

lem-Wk

Lemma 3.1. *For each $k = 1, \dots, m$ the curve $\widetilde{W}_k$ is irreducible and the degree of $\widetilde{W}_k \to U$ is $\binom{d}{k}$.*

Proof. This follows from the fact that the Galois group of W/U is the full symmetric group S_d on Σ, and S_d acts k-transitively on Σ. □

Recall that a *multisection* of $\mathcal{X}_h \to B$ is an irreducible curve $Y \subset \mathcal{X}_h$ such that the induced map $Y \to B$ is finite. In this case the degree of $Y \to B$ is called the *degree* of the multisection.

prop-multi

Proposition 3.2. *Let $Y \subset \mathcal{X}_h$ be a multisection of degree e. Then there exists an integer $k = 1, \dots, m = min(d, n)$ such that e is divisible by $\binom{d}{k}$.*

Proof. The generic point of Y is contained in the strata $\mathcal{X}_h^k$ for some $k = 1, \dots, m$. Thus $Y \cap W \to U$ yields a multisection of $W_k \to U$. So the degree e of $Y \cap W \to U$ is divisible by the degree of $W_k \to U$, i.e. $\binom{d}{k}$. □

rmk-multi

Remark 3.3. Of course this tells us nothing about the degree e in case $d \le n$ so that $\binom{d}{d} = 1$ is one of the possible divisors of e. In fact by the Tsen-Lang theorem [7], there is a multisection of $\mathcal{X}_h \to U$ of degree 1 (i.e. a section) in this case. Let $M_{d,n} \subset \mathbb{Z}_{>0}$ denote the additive sub-semigroup generated by the elements $\binom{n}{k}$, $k = 1, \dots, \min(d, n)$. Proposition 3.2 implies that for any reducible curve $Y \subset \mathcal{X}_h$ which is a union of multisections $Y_1, \dots, Y_s$, the degree of $Y \to B$ is contained in $M_{d,n}$.

Let H_n denote the connected component of the Hom scheme $Hom(B, \mathbb{P}(\mathrm{Sym}^d(V^\vee)))$ parametrizing morphisms $f : B \to \mathbb{P}(\mathrm{Sym}^d(V^\vee))$ such that $f^*\mathcal{O}(1)$ has degree n. Identifying B with $\mathbb{P}^1$, any such morphism is of the form $f([X_0 : X_1]) = [f_0(X_0, X_1) : \cdots : f_N(X_0, X_1)]$ and so H_n is an open subset of the projective space associated to the vector space $\mathrm{Hom}_k(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)), \mathrm{Sym}^d(V^\vee))$. Let $F : B \times H_n \to \mathbb{P}(\mathrm{Sym}^d(V^\vee))$ denote the universal morphism. Let $\mathcal{X}_F \subset \mathbb{P}(V) \times B \times H_n$ denote the pullback by F of the universal hypersurface $\mathcal{X} \subset \mathbb{P}(V) \times \mathbb{P}(\mathrm{Sym}^d(V^\vee))$. The projection morphism $\mathrm{pr}_3 : \mathcal{X}_F \to H_n$ is projective, and the invertible sheaf $\mathcal{L} = \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}(V)}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_B(1)$ is pr_3-relatively ample. For each integer p and each pair of positive integers e, e' denote by $\mathcal{H}_{e,e',p} \to H_n$ the relative Hilbert scheme $Hilb(\mathcal{X}_F/H_n)$ parametrizing closed subschemes $Y \subset (\mathcal{X}_F)_h$ whose Hilbert polynomial with respect to $\mathcal{L}$ is $e't + 1 - p$ and whose Hilbert polynomial with respect to $\mathrm{pr}_2^*\mathcal{O}_B(1)$ is $e + 1 - p$. Observe that $\mathcal{H}_{e,e',p} \to H_n$ is a projective morphism.

thm-multi

Theorem 3.4. *If e is not in the sub-semigroup $M_{d,n}$, then the morphism $\mathcal{H}_{e,e',p} \to H_n$ is not surjective, and therefore it is not dominant.*

Proof. Suppose, on the contrary, that it is surjective. Then in particular there is some 1-dimensional scheme $Y \subset \mathcal{X}_h$ whose degree with respect to $\mathcal{O}_B(1)$ is e. Now let $Y_1, \dots, Y_s$ be the reduced, irreducible components of Y which dominate B and let the multiplicity of Y_i in Y be m_i. Let the degree of $Y_i \to B$ be e_i. Then the degree of Y with respect to $\mathcal{O}_B(1)$ is $m_1e_1 + \cdots + m_se_s$. Now by Proposition 3.2, every integer $e_i \in M_{d,n}$. Therefore $e = m_1e_1 + \cdots + m_se_s \in M_{d,n}$ as well.

Since $\mathcal{H}_{e,e',p} \to H_n$ is projective, it is dominant iff it is surjective. □

cor-multi

Corollary 3.5. *If k is an uncountable field, there exists a map $h' : \mathbb{P}^1 \to \mathbb{P}(Sym^d(V^\vee))$ with $(h')^*\mathcal{O}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ whose image is not contained in the discriminant divisor of $\mathbb{P}(Sym^d(V^\vee))$ and such that every multisection of $\mathcal{X}_{h'} \subset \mathbb{P}(V) \times \mathbb{P}^1$ has degree e contained in $M_{d,n}$. Whether or not k is uncountable, there exists a finitely generated field extension K/k with algebraic closure $\overline{K}$ and a map $h' : \mathbb{P}^1_{\overline{K}} \to \mathbb{P}(Sym^d(V^\vee))$ as above.*

Proof. We define L to be the function field of H_n. If k is uncountable, then there exists k-points $h' \in H_n$ such that for every $e \notin M_{d,n}$, for every p and for every e', h' is not contained in the image of $\mathcal{H}_{e,e',p} \to H_n$. Moreover, such points are Zariski dense in H_n, so we can choose h' so that the image of $h' : \mathbb{P}^1 \to \mathbb{P}(\mathrm{Sym}^d(V^\vee))$ is not contained in the discriminant divisor. □

3.1. **Dependence of $\mathcal{H}_d$ on d.** In this subsection we apply Corollary 3.5 to complete the argument from [2, Section 7.3]. Let k be an uncountable, algebraically closed field (not necessarily of characteristic 0). Consider the surface $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$. Suppose that D is a reduced, geometrically irreducible divisor on Σ (but not necessarily smooth) with $\mathcal{O}_\Sigma(D) \cong \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(b)$. The claim is that there exists an integer d such that the family $\mathcal{H}_d$ from [2] is not contained in the linear system $|D|$. More precisely, the claim is that there exists a family of hypersurfaces $X \subset \mathbb{P}^{d+1} \times \Sigma$ such that for every curve $E \subset \Sigma$ in the divisor class $|D|$, we have that $X_E \to E$ admits a section, and yet there exists a smooth curve $C \subset \Sigma$ such that $X_C \to C$ does not admit a section.

Suppose first that $\min(a, b) = 0$, say $b = 0$. The assumption that D is reduced and geometrically irreducible implies that $(a, b) = (1, 0)$. Let $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ be any irreducible curve such that $\mathrm{pr}_2 : Y \to \mathbb{P}^1$ is not purely inseparable. For example, choose coordinates $[T_0 : T_1]$ on the first factor, $[X_0 : X_1]$ on the second factor, and let Y be the zero locus of $X_0(T_1^2 - T_0T_1) - X_1T_0^2$. Identifying $\mathbb{P}^1 \times \Sigma$ with $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, let $X \subset \mathbb{P}^1 \times \Sigma$ be $Y \times \mathbb{P}^1$. It is clear that $X_E \to E$ has a section for every $E \in |D|$. But for any curve $C \subset \Sigma$ with $\mathcal{O}_\Sigma(C) \cong \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(0) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(1)$, the morphism $X_C \to C$ has no section.

Thus suppose that $a, b > 0$. Define $n = 2D\dot{D} = 4ab$ and set $d = n$ (it is more convenient to work with $n = d + 1$ than with d). Let $m \geq n$ be any integer. Let V be a k-vector space of dimension $n + 1$ and let $\mathcal{X} \subset \mathbb{P}(V) \times \mathbb{P}(\mathrm{Sym}^m(V^\vee))$ be the universal family of degree m hypersurfaces in $\mathbb{P}(V)$. By Corollary 3.5, we can find a map $h : \mathbb{P}^1 \to \mathbb{P}(\mathrm{Sym}^m(V^\vee))$ such that every multisection of $\mathcal{X}_h \to \mathbb{P}^1$ has degree in $M_{m,n}$. In particular the degree of a multisection is at least m. Moreover, we can choose h so that $h^*\mathcal{O}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ and so that the induced map

$$\alpha : [\mathrm{Sym}^m(V^\vee)]^\vee \to H^0(\mathbb{P}^1, i^*\mathcal{O}(1)) \tag{9}$$

is surjective.

Now choose a closed immersion $i : \mathbb{P}^1 \to \Sigma$ is a divisor in the linear system of $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(2b)$. Observe that the pullback by h^* of $\mathcal{O}(1)$ is isomorphic to the pullback by i^* of $\mathcal{L} := \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(2a - 1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(2b)$. Choose an isomorphism $i^*\mathcal{L} \cong h^*\mathcal{O}(1)$. Observe that $\mathcal{L} \otimes \mathcal{O}_\Sigma(-i(C))$ is just $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(2a-2)$ so that $H^1(\Sigma, \mathcal{L} \otimes \mathcal{O}_\Sigma(-i(C)))$ is zero. Therefore the induced map

$$\beta : H^0(\Sigma, \mathcal{L}) \to H^0(\mathbb{P}^1, i^*\mathcal{L}) \tag{10}$$

is surjective. So we can find a factorization

$$\gamma : [\mathrm{Sym}^m(V^\vee)]^\vee \to H^0(\Sigma, \mathcal{L}) \tag{11}$$

so that $\beta \circ \gamma = \alpha$. Moreover, since the dimension of $[\mathrm{Sym}^m(V^\vee)]^\vee$ is larger than the dimension of $H^0(\Sigma, \mathcal{L})$, we can choose γ to be surjective. Since $\mathcal{L}$ is generated by global sections, there is an induced morphism $j : \Sigma \to \mathbb{P}(\mathrm{Sym}^m(V^\vee))$ such that

(1) The pullback $j^*\mathcal{O}(1)$ is isomorphic to $\mathcal{L}$.
(2) The pullback map $[\mathrm{Sym}^m(V^\vee)]^\vee \to H^0(\Sigma, \mathcal{L})$ is the map γ.
(3) The composition $j \circ i$ equals h.
(4) The morphism j is a closed immersion.

Now consider the pullback $X \subset \mathbb{P}(V) \times \Sigma$ of $\mathcal{X}$ by j. By construction the restriction of X to $i(C)$ has no section. But for any $E \in |D|$, the degree of $j(E)$ in

$\mathbb{P}(\mathrm{Sym}^m(V^\vee))$ is $n-b \leq n-1$. So the span of $j(E)$ is a linear space of dimension at most $n-1$. The corresponding linear system of hypersurfaces in $\mathbb{P}^n$ has a nonempty base locus. Let $p \in \mathbb{P}(V)$ be a point in the base locus of this linear system. Then $\{p\} \times E \subset \mathbb{P}(V) \times \Sigma$ is a curve in $X \subset \mathbb{P}(V) \times \Sigma$ which is a section of the restriction $X_E \to E$.

cor-dependence

Corollary 3.6. *Let B be a normal, projective variety of dimension at least 2. Let $\mathcal{H}$ be a finite type variety and let $\mathcal{C} \subset B \times \mathcal{H}$ be a flat family of reduced, irreducible curves in B. There exists an integer d and a flat family of hypersurfaces $X \subset \mathbb{P}^{d+1} \times B$ such that*

(1) *For every geometric point $p \in \mathcal{H}$ parametrizing a curve $C_p \subset B \otimes_k \kappa(p)$, the restriction $X_{C_p} \to C_p$ of X over C_p has a section.*
(2) *There exists a reduced, irreducible curve $C' \subset B$ such that the restriction $X_{C'} \to C'$ of X over C' has no rational section.*

Proof. Since $\mathcal{H}$ is of finite type, it has only finitely many geometric generic points, $\eta_1, \dots, \eta_k$. By a standard argument, we can find a dominant rational transformation $f : B \to \Sigma$ whose indeterminacy locus $Z \subset B$ is a subscheme of codimension at least 2 and such that for every geometric generic point η_i the corresponding curve $C_i \subset B \otimes_k \kappa(\eta_i)$ does not intersect $Z \otimes_k \kappa(\eta_i)$. Define a' to be the maximum of the degrees $f^*\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(1)|_{C_i}$ and define b' to be the maximum of the degrees $f^*\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(1)|_{C_i}$. Define $a = a' + 1$ and $b = b' + 1$.

Let $C \subset \Sigma$ be a smooth curve in the linear system of $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(2b)$. Consider the subvariety $W \subset B$ which is the closure of $f^{-1}(B)$. We can find an irreducible curve $C' \subset W$ such that C' is not contained in Z and such that $f : C' \to C$ is dominant. Define e to be $\deg(C'/C)$. Let r be a positive integer such that $4abr > e$.

Choose a degree r morphism $g : \mathbb{P}^1 \to \mathbb{P}^1$ and compose the morphism $f : B \to \Sigma$ with the automorphism $g \times \mathrm{Id}_{\mathbb{P}^1} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. Define the composite to be $\widetilde{f} : B \to \Sigma$. If we define $\widetilde{a}$ and $\widetilde{b}$ analogously to a and b, then we have $\widetilde{a} = a$ and $\widetilde{b} = rb$. And the image of C under $g \times \mathrm{Id}_{\mathbb{P}^1}$ is a smooth curve $\widetilde{C} \subset \Sigma$ in the linear system of $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(2rb)$.

Set $d = 4abr$, let $\mathcal{L}$ be the invertible sheaf $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(2a-1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(2rb)$ on Σ, and let $h : \widetilde{C} \to \mathbb{P}(\mathrm{Sym}^d(V^\vee))$ be as in Corollary 3.5. As above, we can find a morphism $j : \Sigma \to \mathbb{P}(\mathrm{Sym}^d(V^\vee))$ such that

(1) The pullback $j^*\mathcal{O}(1)$ is isomorphic to $\mathcal{L}$.
(2) The pullback map $[\mathrm{Sym}^d(V^\vee)]^\vee \to H^0(\Sigma, \mathcal{L})$ is surjective.
(3) The restriction $j|_{\widetilde{C}}$ equals h.
(4) The morphism j is a closed immersion.

Let $\mathcal{X} \subset \mathbb{P}(V) \times \mathbb{P}(\mathrm{Sym}^d(V^\vee))$ denote the universal degree m hypersurface, and define $X \subset \mathbb{P}(V) \times B$ to be the closure of the pullback of $\mathcal{X}$ by $j \circ \widetilde{f}$. For each $i = 1, \dots, k$ we have that $j(\widetilde{f}(C_i))$ spans a linear space of dimension less than $4abr - br < d$. The corresponding linear system of degree d hypersurfaces in $\mathbb{P}(V)$ has a nonempty base locus. For any point $p \in \mathbb{P}(V)$ in the base locus, we have that $\{p\} \times C_i \subset X$ is a section of $X_{C_i} \to C_i$. So $X_{C_i} \to C_i$ has a section.

On the other hand, we will prove by contradiction that there is no rational section of $X_{C'} \to C'$. We have that $X_{C'} = X_{\widetilde{C}} \times_{\widetilde{C}} C'$, where $X_{\widetilde{C}} \to \widetilde{C}$ is the pullback of $\mathcal{X}$ by h. By way of contradiction, suppose there exists an irreducible curve $D \subset X_{C'}$ which maps birationally to C'. The image of D in $X_{\widetilde{C}}$ is an irreducible curve E such that $E \to C$ has degree at most $\deg(C'/C) = e$. But by the choice of h, every multisection of $X_{\widetilde{C}} \to \widetilde{C}$ has degree at least d, which is strictly greater than e (by the choice of r). This contradiction shows that D does not exist. So $X_{C'} \to C'$ has no rational section. □

sec-param

4. A parameter space of Enriques surfaces

The parameter space for Enriques surfaces we use is essentially the same one as in [2, Section 5], namely the Grassmannian parametrizing nets of quadrics in $\mathbb{P}^5$ invariant under an involution with eigenvalues $(+1, +1, +1, -1, -1, -1)$. This is actually a parameter space for the $K3$-covers of Enriques surfaces rather than the Enriques surfaces themselves. That the $K3$-cover of an Enriques surface is isomorphic to the base locus of a net of invariant quadrics apparently goes back to Enriques, although it was only more recently that Cossec and Verra independently proved this is true for every Enriques surface over a field k with $\mathrm{char}(k) \neq 2$ (c.f. [1, Thm. 4.9.2 and p. 284]).

This section is tangential to the rest of the article: we give a slightly different description of the parameter space of Enriques surfaces. The main difference between the description of the Enriques surfaces given in [2, Section 5] and that given here is that instead of constructing the family of $K3$-covers of Enriques surfaces and defining the Enriques surfaces to be the quotient by an involution, we directly construct the family of quotient Enriques surfaces. The answer is rather nice: if $\mathrm{char}(k) \neq 2$, then every Enriques surface is isomorphic the intersection in $\mathbb{P}^8$ of the rank-1 loci of two symmetric 3×3 matrices of linear forms. Alternatively, every Enriques surface is isomorphic to a complete intersection of three divisors $D_1 \cap D_2 \cap D_3$ in a certain $\mathbb{P}^1$-bundle over $\mathbb{P}^2 \times \mathbb{P}^2$ where $\mathcal{O}(D_1) = \mathcal{O}(D_2) = \mathcal{O}(D_3)$ is the invertible sheaf of the relative hyperplane class on the $\mathbb{P}^1$-bundle. The second description turns out to be simpler to work with, but we also discuss and use the first description.

The reason for constructing the family of Enriques surfaces directly is the case that $\mathrm{char}(k) = 2$. In this case the group scheme which acts on the $K3$-surfaces can specialize to the non-reduced scheme μ_2 (an Enriques surface in characteristic 2 with this property is called a *classical* Enriques surface, c.f. [1, Sec. 1.1]). Rather than deal with the technical problem of forming the (purely inseparable) quotient of a scheme by an action of non-reduced group scheme, instead we prefer to do a little extra computation and construct the Enriques surfaces directly. The extra effort is a bit superfluous in that the results of the next section do not apply to classical Enriques surfaces in characteristic 2.

We will come back to the family of $K3$-covers at the end of the section, and the $K3$-covers will be essential for the next section.

Let R be a commutative ring with 1. In this section all schemes will be R-schemes and all absolute products will be fiber products over $\mathrm{Spec}\, R$. Let V_+ and V_- be finitely generated projective R-modules of constant rank 3. Define $V_+^\vee = \mathrm{Hom}_R(V_+, R), V_-^\vee = \mathrm{Hom}_R(V_-, R)$. Define P_+ and P_- to be the projective

R-schemes $P_+ = \mathbb{P}(V_+) = \mathrm{ProjSym}^*_R(V_+^\vee)$ and $P_- = \mathbb{P}(V_-) = \mathrm{ProjSym}^*_R(V_+^\vee)$. Denote by $\mathrm{pr}_+ : P_+ \times P_- \to P_+$ and $\mathrm{pr}_- : P_+ \times P_- \to P_-$ the two projection morphisms. Consider the locally free sheaf $\mathcal{E} = \mathrm{pr}_+^*\mathcal{O}_{P_+}(-2) \oplus \mathrm{pr}_-^*\mathcal{O}_{P_-}(-2)$ on $P_+ \times P_-$. Form the relative projective bundle $\pi : Z \to P_+ \times P_-$ where $Z = \mathbb{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_{P_+\times P_-} \mathrm{Sym}^*(\mathcal{E}^\vee)$. Let $\alpha : \pi^*\mathcal{E}^\vee \to \mathcal{O}_Z(1)$ denote the tautological surjection. Then we have $H^0(Z, \mathcal{O}_Z(1))$ equals $H^0(P_+ \times P_-, \mathcal{E}^\vee) = H^0(P_+, \mathcal{O}_{P_+}(2)) \oplus H^0(P_-, \mathcal{O}_{P_-}(2))$ which equals $\mathrm{Sym}^2_R(V_+^\vee) \oplus \mathrm{Sym}^2_R(V_-^\vee)$.

Denote by $\sigma_+ : P_+ \times P_- \to Z$ the section of π such that the pullback of the tautological surjection $\sigma_+^*\alpha$ is just the projection

$$\mathcal{E}^\vee \cong \mathrm{pr}_+^*\mathcal{O}_{P_+}(2) \oplus \mathrm{pr}_-^*\mathcal{O}_{P_-}(2) \to \mathrm{pr}_+^*\mathcal{O}_{P_+}(2). \tag{12}$$

Define $\sigma_- : P_+ \times P_- \to Z$ to be the section of π such that the pullback of the tautological surjection $\sigma_-^*\alpha$ is the other projection $\mathcal{E}^\vee \to \mathrm{pr}_-^*\mathcal{O}_{P_-}(2)$. Denote by D_+, D_- the Cartier divisors in Z which are the images of σ_+ and σ_- respectively. Since we have the natural isomorphism

$$\sigma_+^*\mathcal{O}_Z(D_+) \cong \mathit{Hom}_{\mathcal{O}_{P_+\times P_-}}(\sigma_+^*\mathcal{O}_Z(-1), \sigma_+^*(\pi^*\mathcal{E}/\mathcal{O}_Z(-1))) \tag{13}$$

we conclude that $\sigma_+^*\mathcal{O}_Z(D_+) \cong \mathrm{pr}_+^*\mathcal{O}_{P_+}(2) \otimes \mathrm{pr}_-^*\mathcal{O}_{P_-}(-2)$. Therefore we have that $\mathcal{O}_Z(D_+) \cong \pi^*\mathrm{pr}_-^*\mathcal{O}_{P_-}(-2) \otimes \mathcal{O}_Z(1)$. Similarly we have that $\mathcal{O}_Z(D_-) \cong \pi^*\mathrm{pr}_+^*\mathcal{O}_{P_+}(-2) \otimes \mathcal{O}_Z(1)$.

Denote by G the Grassmannian R-scheme

$$G = \mathrm{Grass}(3, H^0(Z, \mathcal{O}_Z(1))) = \mathrm{Grass}(3, \mathrm{Sym}^2_R(V_+^\vee) \oplus \mathrm{Sym}^2_R(V_-^\vee)). \tag{14}$$

Denote the tautological surjection onto a rank 3 locally free sheaf by

$$\beta^\dagger : \left[\mathrm{Sym}^2_R(V_+^\vee) \oplus \mathrm{Sym}^2_R(V_-^\vee)\right]^\vee \otimes_R \mathcal{O}_G \to S_3^\vee. \tag{15}$$

More often we are concerned with the transpose of this map which is an injective sheaf map β with domain S_3, that is why we denote the surjection in this way. Consider the product $Z \times G$ and denote by $\mathrm{pr}_Z : Z \times G \to Z$ and $\mathrm{pr}_G : Z \times G \to G$ the two projection morphisms. On $Z \times G$ we can compose the pullback

$$\mathrm{pr}_G^*\beta : \mathrm{pr}_G^* S_3 \to \left[\mathrm{Sym}^2_R(V_+^\vee) \oplus \mathrm{Sym}^2_R(V_-^\vee)\right] \otimes_R \mathcal{O}_{Z\times G} \tag{16}$$

with the pullback by pr_Z of the canonical map $H^0(Z, \mathcal{O}_Z(1)) \otimes_R \mathcal{O}_Z \to \mathcal{O}_Z(1)$. Twisting the composition by $\mathrm{pr}_Z^*\mathcal{O}_Z(-1)$ we get a map

$$\gamma : \mathrm{pr}_Z^*\mathcal{O}_Z(-1) \otimes \mathrm{pr}_G^* S_3 \to \mathcal{O}_{Z\times G}. \tag{17}$$

We define $\iota : \mathcal{Y} \to \mathcal{O}_{Z\times G}$ to be the closed subscheme whose ideal sheaf is the image of γ.

lem-E1

Lemma 4.1. *The map of sheaves $H^0(Z, \mathcal{O}_Z(1)) \otimes_R \mathcal{O}_Z \to \mathcal{O}_Z(1)$ is a surjective map of locally free sheaves with a kernel $\mathcal{K}$ which is locally free of rank 11. Moreover the morphism $pr_Z|_{\mathcal{Y}} : \mathcal{Y} \to Z$ is isomorphic to the Grassmannian bundle $Grass_Z(3, \mathcal{K})$.*

Proof. Clearly $\mathcal{O}_Z(1)$ is π-relatively generated by global sections. So to prove that $\mathcal{O}_Z(1)$ is generated by global sections it suffices to prove that $\pi_*\mathcal{O}_Z(1) = \mathcal{E}^\vee$ is generated by global sections. Since $\mathcal{E}^\vee = \mathrm{pr}_+^*\mathcal{O}_{P_+}(2) \oplus \mathrm{pr}_-^*\mathcal{O}_{P_-}(2)$, by Künneth it suffices to prove that each of $\mathcal{O}_{P_+}(2)$ and $\mathcal{O}_{P_-}(2)$ are generated by global sections, which is obvious.

By construction of $\mathcal{Y}$ the restriction of $\mathrm{pr}_G^*\beta$ to $\mathcal{Y}$ factors through the restriction of $\mathrm{pr}_Z^*\mathcal{K}$. This defines the morphism $\mathcal{Y} \to \mathrm{Grass}_Z(3,\mathcal{K})$. Conversely, there is an obvious morphism $\mathrm{Grass}_Z(3,\mathcal{K}) \to Z \times G$ induced by the sheaf map $\mathcal{K} \to H^0(Z,\mathcal{O}_Z(1)) \otimes_R \mathcal{O}_Z$. It is a matter of diagram chasing to see that this morphism factors through $\mathcal{Y}$ and that the two morphisms are inverse to one another. $\square$

It follows from Lemma 4.1 that $\mathrm{pr}_Z|_{\mathcal{Y}}$ is smooth of relative dimension 24 which implies that $\mathcal{Y}$ is smooth over Spec R of relative dimension 29.

Define $G_+ = \mathrm{Grass}(3, \mathrm{Sym}^2_R(V_+^\vee))$ and define $G_- = \mathrm{Grass}(3, \mathrm{Sym}^2_R(V_-^\vee))$. Let $U' \subset G$ denote the open subscheme over which the following two component maps are surjective:

$$\left[\mathrm{Sym}^2_R(V_+^\vee)\right]^\vee \otimes_R \mathcal{O}_G \to \left[\mathrm{Sym}^2_R(V_+^\vee) \oplus \mathrm{Sym}^2_R(V_-^\vee)\right]^\vee \otimes_R \mathcal{O}_G \to S_3^\vee$$
$$\left[\mathrm{Sym}^2_R(V_-^\vee)\right]^\vee \otimes_R \mathcal{O}_G \to \left[\mathrm{Sym}^2_R(V_+^\vee) \oplus \mathrm{Sym}^2_R(V_-^\vee)\right]^\vee \otimes_R \mathcal{O}_G \to S_3^\vee$$

There is an induced morphism $U' \to G_+ \times G_-$ such that the pullbacks of the two tautological surjections are the ones above. Define $U_+ \subset G_+$ and $U_- \subset G_-$ to be the open subschemes parametrizing *base-point-free* linear systems of conics on P_+ and P_- respectively. In other words $U_+ \subset G_+$ is the open subscheme where the morphism $\mathrm{Sym}^2(V_+^\vee) \otimes_R S_3 \to \mathrm{Sym}^4(V_+^\vee) \otimes_R \mathcal{O}_{G_+}$ is surjective, and similarly for $U_- \subset G_-$. We define $U \subset U'$ to be the preimage of $U_+ \times U_-$. Working with coordinates, it is clear that $U' \to G_+ \times G_-$ is surjective and that U_+ and U_- are dense open subschemes (in every fiber over a geometric point of R). Therefore also $U \subset G$ is dense in every fiber over a geometric point of R. The parameter space we will be considering is U (and occasionally also G) and the family of Enriques surfaces we will be considering is $\mathcal{Y} \cap \pi_G^{-1}(U)$.

Denote by $\pi_G|_U : \mathcal{Y} \cap \pi_G^{-1}(U) \to U$ the restriction of π_G to $\mathcal{Y} \cap \pi_G^{-1}(U)$. Denote by $\pi_+ : \mathcal{Y} \cap \pi_G^{-1}(U) \to P_+ \times U$ and $\pi_- : \mathcal{Y} \cap \pi_G^{-1}(U) \to P_- \times U$ the obvious morphisms.

lem-E2

Lemma 4.2. *The two morphisms π_+ and π_- are surjective, finite and flat of relative degree 4. Therefore also $\pi_G|_U$ is projective and flat of relative dimension 2. Moreover $\mathcal{Y} \cap \pi_G^{-1}(U)$ has empty intersection with $D_+ \times U$ and $D_- \times U$.*

Proof. Of course both π_+ and π_- are projective morphisms of schemes such that both the domain and target are smooth over R. Therefore to prove that π_+ and π_- are surjective, finite and flat of relative degree 4, it suffices to prove that for every geometric point of $P_+ \times U$ (resp. $P_- \times U$) the fiber of π_+ (resp. π_-) is a nonempty, zero-dimensional scheme of length 4.

Let $(p,g) \in P_+ \times U$ be a geometric point, i.e. $p \in P_+$ is a geometric point and g parametrizes a linear system in $H^0(Z,\mathcal{O}_Z(1)) \otimes_R \kappa(p)$. We can choose homogeneous coordinates $[T_+ : T_-]$ on the fibers of π so that elements of $H^0(Z,\mathcal{O}_Z(1))$ are elements $T_-F_+ - T_+F_-$ where $F_+ \in H^0(P_+,\mathcal{O}_{P_+}(2))$ and $F_- \in H^0(P_-,\mathcal{O}_{P_-}(2))$. Since g is in U, the component 3-dimensional linear subsystem of $H^0(P_+,\mathcal{O}_{P_+}(2))$

is base-point-free. Thus we can write the linear system parametrized by g as the span of

$$\begin{aligned} &T_- F_{+,1} - T_+ F_{-,1}, \quad F_{+,1}(p) = 0 \\ &T_- F_{+,2} - T_+ F_{-,2}, \quad F_{+,2}(p) = 0 \\ &T_- F_{+,3} - T_+ F_{-,3}, \quad F_{+,3}(p) \neq 0 \end{aligned}$$

Moreover, we also have that the linear subsystem of $H^0(P_-, \mathcal{O}_{P_-}(2))$ spanned by $F_{-,1}, F_{-,2}, F_{-,3}$ is also base-point-free.

First of all, consider the subscheme of P_- which is the zero locus of $F_{-,1}$ and $F_{-,2}$. By Bézout's theorem, this is nonempty and is zero-dimensional of length 4 if it contains no curve. But since there is no common zero of $F_{-,1}, F_{-,2}$ and $F-,3$, we must have that the zero locus of $F_{-,1}$ and $F_{-,2}$ is zero-dimensional. Restricted over this zero-dimensional subscheme we can consider the section of π defined by $[T_+, T_-] = [F_{+,3}(p), F_{-,3}]$. This gives a map into the fiber $\pi_+^{-1}(p, g)$. So we conclude that the fiber is nonempty and contains a zero-dimensional subscheme of length 4.

But, on the other hand, it is easy to see that this zero-dimensional, length 4 subscheme is the whole fiber $\pi_+^{-1}(p, g)$. Indeed, from the first two elements of the linear system we conclude that either $T_+ = 0$ or else $F_{-,1} = F_{-,2} = 0$. But if $T_+ = 0$, then substituting into the third equation and using that $F_{+,3}(p) \neq 0$, we conclude that $T_- = 0$ as well. But $[T_+, T_-] = [0, 0]$ doesn't correspond to an actual point in a fiber of π. So we must have that T_- is nonzero and that $F_{-,1} = F_{-,2} = 0$. Thus we see that $\pi_+^{-1}(p, g)$ is exactly the subscheme from the last paragraph. So π_+ is a surjective, finite and flat morphism of degree 4. By symmetry also π_- is a surjective, finite and flat morphism of degree 4. Since the two projections $P_+ \times U \to U$ and $P_- \times U \to U$ are surjective, projective and flat morphisms of relative dimension 2, we conclude that $\pi_G|_U : \mathcal{Y} \cap \pi_G^{-1}(U) \to U$ is a surjective, projective and flat morphism of relative dimension 2.

Finally, the intersection of $D_- \times U$ with a fiber of π_+ is exactly the set where the coordinate T_- is zero. But as we saw, T_- is never zero. So $D_- \times U$ has empty intersection with $\mathcal{Y} \cap \pi_G^{-1}(U)$. By symmetry, also $D_+ \times U$ has empty intersection with $\mathcal{Y} \cap \pi_G^{-1}(U)$. □

Let us just remark on the fact that $\mathcal{Y} \cap \pi_G^{-1}(U)$ has empty intersection with $D_+ \times U$ and $D_- \times U$. This implies that the restrictions to $\mathcal{Y} \cap \pi_G^{-1}(U)$ of the invertible sheaves $\mathrm{pr}_Z^* \mathcal{O}_Z(D_+)$ and $\mathrm{pr}_Z^* \mathcal{O}_Z(D_-)$ are trivial. This will prove crucial to computing that the relative dualizing sheaf of $\pi_G|_U$ is numerically trivial on fibers of $\pi_G|_U$.

The next step is to prove that $\pi_G|_U : \mathcal{Y} \cap \pi_G^{-1}(U) \to U$ is smooth over an open subscheme of U which is dense in every fiber over a geometric point of R. Of course we prove this via Bertini's theorem using a projective embedding of $\mathcal{Y} \cap \pi_G^{-1}(U)$. Denote by $\mathbb{P}$ the projective R-scheme $\mathbb{P}\left[\mathrm{Sym}^2_R(V_+^\vee) \oplus \mathrm{Sym}^2_R(V_-^\vee)\right]^\vee$, i.e. $\mathrm{ProjSym}^*_R\left[\mathrm{Sym}^2_R(V_+^\vee) \oplus \mathrm{Sym}^2_R(V_-^\vee)\right]$. Denote by Λ_+ the closed subscheme of $\mathbb{P}$ which is the projective space $\mathbb{P}\left[\mathrm{Sym}^2_R(V_+^\vee)\right]^\vee$, i.e. $\mathrm{ProjSym}^*_R\left[\mathrm{Sym}^2_R(V_+^\vee)\right]$. Similarly let Λ_- be the closed subscheme of $\mathbb{P}$ with is the projective space $\mathbb{P}\left[\mathrm{Sym}^2_R(V_-^\vee)\right]^\vee$. Denote by $v_+ : P_+ \to \Lambda_+$ the 2-uple Veronese map, i.e. the morphism such that the pullback of the tautological surjection is the sheaf map

$$\mathrm{Sym}^2_R(V_+^\vee) \otimes_R \mathcal{O}_{P_+} = H^0(P_+, \mathcal{O}_{P_+}(2)) \otimes_R \mathcal{O}_{P_+} \to \mathcal{O}_{P_+}(2). \tag{18}$$

Similarly denote by $v_- : P_- \to \Lambda_-$ the 2-uple Veronese map. It is a standard result that v_+ and v_- are closed immersions.

Define $W = \mathbb{P} - (\Lambda_+ \cup \Lambda_-)$, i.e. W is the open subscheme of $\mathbb{P}$ over which the two compositions of the tautological surjection with the two component maps are both surjections:

$$\mathrm{Sym}^2_R(V_+^\vee) \otimes_R \mathcal{O}_\mathbb{P} \hookrightarrow \left[\mathrm{Sym}^2_R(V_+^\vee) \oplus \mathrm{Sym}^2_R(V_-^\vee)\right] \otimes_R \mathcal{O}_\mathbb{P} \to \mathcal{O}_{PP}(1)$$
$$\mathrm{Sym}^2_R(V_-^\vee) \otimes_R \mathcal{O}_\mathbb{P} \hookrightarrow \left[\mathrm{Sym}^2_R(V_+^\vee) \oplus \mathrm{Sym}^2_R(V_-^\vee)\right] \otimes_R \mathcal{O}_\mathbb{P} \to \mathcal{O}_{PP}(1)$$

There is an induced morphism $\rho : W \to \Lambda_+ \times \Lambda_-$ such that the pullbacks of the two tautological surjections on Λ_+ and Λ_- are the compositions from above.

Let $\psi : T \to \Lambda_+ \times \Lambda_-$ be the relative projective bundle associated to the locally free sheaf on $\Lambda_+ \times \Lambda_-$ $\mathrm{pr}_+^*\mathcal{O}_{\Lambda_+}(-1) \oplus \mathrm{pr}_-^*\mathcal{O}_{\Lambda_-}(-1)$, i.e. we have

$$T = \underline{\mathrm{Proj}}_{\Lambda_+\times\Lambda_-} \mathrm{Sym}^* \left[\mathrm{pr}_+^*\mathcal{O}_{\Lambda_+}(1) \oplus \mathrm{pr}_-^*\mathcal{O}_{\Lambda_-}(1)\right]. \tag{19}$$

Let $\tau_+ : \Lambda_+ \times \Lambda_- \to T$ and $\tau_- : \Lambda_+ \times \Lambda_- \to T$ denote the two sections of ψ such that the pullbacks of the tautological surjection are respectively the two projections

$$\mathrm{pr}_+^*\mathcal{O}_{\Lambda_+}(1) \oplus \mathrm{pr}_-^*\mathcal{O}_{\Lambda_-}(1) \to \mathrm{pr}_+^*\mathcal{O}_{\Lambda_+}(1)$$
$$\mathrm{pr}_+^*\mathcal{O}_{\Lambda_+}(1) \oplus \mathrm{pr}_-^*\mathcal{O}_{\Lambda_-}(1) \to \mathrm{pr}_-^*\mathcal{O}_{\Lambda_-}(1)$$

Let E_+ and E_- denote the Cartier divisors in T which are the images of τ_+ and τ_- respectively.

There is a unique morphism $\rho' : T \to \mathbb{P}$ such that the pullback of the tautological surjection is the canonical map

$$\left[\mathrm{Sym}^2_R(V_+^\vee) \oplus \mathrm{Sym}^2_R(V_-^\vee)\right] \otimes_R \mathcal{O}_T = H^0(T, \mathcal{O}_T(1)) \otimes_R \mathcal{O}_T \to \mathcal{O}_T(1). \tag{20}$$

It is straightforward to see that $(\rho')^{-1}(\Lambda_+) = E_+$ and $(\rho')^{-1}(\Lambda_-) = E_-$ and the induced map $\rho' : T - (E_+ \cup E_-) \to \mathbb{P} - (\Lambda_+ \cup \Lambda_-)$ is an isomorphism. So $\rho' : T - (E_+ \cup E_-) \to \mathbb{P}$ is an open immersion.

By construction $(v_+, v_-)^* \left[\mathrm{pr}_+^*\mathcal{O}_{\Lambda_+}(1) \oplus \mathrm{pr}_-^*\mathcal{O}_{\Lambda_-}(1)\right]$ on $P_+ \times P_-$ is the locally free sheaf $\mathrm{pr}_+^*\mathcal{O}_{P_+}(2) \oplus \mathrm{pr}_-^*\mathcal{O}_{P_-}(2)$. Therefore we see that there is a unique morphism $h : Z \to T$ such that $\psi \circ h = (v_+, v_-) \circ \pi$ and such that the pullback by h of the tautological surjection on T is just the tautological surjection on Z. Of course $h^*(E_+) = D_+$ and $h^*(E_-) = D_-$ as Cartier divisors. By chasing through universal properties, we conclude that the following diagram is Cartesian:

$$\begin{array}{ccc} Z - (D_+ \cup D_-) & \xrightarrow{h} & T - (E_+ \cup E_-) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ P_+ \times P_- & \xrightarrow{(v_+, v_-)} & \Lambda_+ \times \Lambda_- \end{array} \tag{21}$$

Since v_+ and v_- are closed immersions, so is (v_+, v_-). Therefore h is a closed immersion. And ρ' is an open immersion when restricted to $T - (E_+ \cup E_-)$. Therefore we have a locally closed immersion $\rho' \circ h : Z - (D_+ \cup D_-) \to \mathbb{P}$. Therefore the following product morphism is a locally closed immersion

$$(\rho' \circ h, \mathrm{Id}_G) : (Z - (D_+ \cup D_-)) \times G \to \mathbb{P} \times G. \tag{22}$$

We can interpret G as the Grassmannian parametrizing codimension 3 linear subspaces of $\mathbb{P}$. Let $\mathcal{S} \subset \mathbb{P} \times G$ denote the universal linear subspace. We can form the universal codimension 3 linear section of $(Z - (D_+ \cup D_-))$, i.e. we can form the intersection of $\mathcal{S}$ with the image of $(\rho' \circ h, \mathrm{Id}_G)$. Consider the projection of this intersection to G. Since $Z - (D_+ \cup D_-)$ is smooth and geometrically irreducible, it follows by Bertini's theorem [5, Thm. 6.3] that this projection is smooth with geometrically irreducible fibers over a dense open subset of G. So we conclude that the following subscheme is smooth over a dense open subset of U:

$$(\rho' \circ h, \mathrm{Id}_G)^{-1}(\mathcal{S}) \cap (Z - (D_+ \cup D_-)) \times U. \tag{23}$$

But a simple diagram chase shows that this subscheme is precisely $\mathcal{Y} \cap \pi_G^{-1}(U)$. We summarize in the following lemma:

lem-E3

Lemma 4.3. *The morphism $\rho' \circ h : Z - (D_+ \cup D_-) \to \mathbb{P}$ is a locally closed immersion. For $g \in U$ a point, $\mathcal{Y}_g \subset Z \otimes_R \kappa(g)$ equals $(\rho' \circ h)^{-1}(\mathcal{S}_g)$ where $\mathcal{S}_g \subset \mathbb{P}$ is the codimension 3 linear subspace parametrized by g. And there is a dense open subset of U over which $\pi_G|_U : \mathcal{Y} \cap \pi_G^{-1}(U) \to U$ is smooth with geometrically irreducible fibers.*

So we see that up to replacing U by a dense open subset, the family $\pi_G|_U : \mathcal{Y} \cap \pi_G^{-1}(U) \to U$ is a smooth family of geometrically irreducible, projective surfaces. The next step is to compute the relative dualizing sheaf of $\pi_G|_U$ and show that the square of the relative dualizing sheaf is trivial on fibers of $\pi_G|_U$. As mentioned above, this reduces to the fact that the pullbacks of $\mathcal{O}_Z(D_+)$ and $\mathcal{O}_Z(D_-)$ are trivial on $\mathcal{Y} \cap \pi_G^{-1}(U)$.

By standard computations of the relative dualizing sheaf of a projective bundle, we have isomorphisms of dualizing sheaves

$$\begin{array}{lcl} \omega_{P_+/R} & \cong & \bigwedge_R^3 (V_+^\vee) \otimes_R \mathcal{O}_{P_+}(-3), \\ \omega_{P_-/R} & \cong & \bigwedge_R^3 (V_-^\vee) \otimes_R \mathcal{O}_{P_-}(-3), \\ \omega_{Z/P_+ \times P_-} & \cong & \pi^* \bigwedge^2 (\mathcal{E}^\vee) \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(-2) \end{array} \tag{24}$$

By base change, the dualizing sheaves $\omega_{P_+ \times U/U}$, $\omega_{P_- \times U/U}$ and $\omega_{Z \times U/P_+ \times P_- \times U}$ are just the pullbacks of these sheaves. Therefore the relative dualizing sheaf ω_{π_U} of $\pi_U : Z \times U \to U$ is isomorphic to

$$\left[\bigwedge_R^3 (V_+^\vee) \otimes_R \bigwedge_R^3 (V_-^\vee) \right] \otimes_R \mathrm{pr}_Z^* \left[\pi^* \mathrm{pr}_+^* \mathcal{O}_{P_+}(-1) \otimes \pi^* \mathrm{pr}_+^* \mathcal{O}_{P_-}(-1) \otimes \mathcal{O}_Z(-2) \right]. \tag{25}$$

And on $Z \times U$ we have the Koszul resolution of the structure sheaf of $\mathcal{Y} \cap \pi_G^{-1}(U)$ of the form

$$\begin{array}{ccccc} 0 & \to & \mathrm{pr}_Z^* \mathcal{O}_Z(-3) \otimes \mathrm{pr}_G^* \bigwedge^3 S_3 & \to & \mathrm{pr}_Z^* \mathcal{O}_Z(-2) \otimes \mathrm{pr}_G^* \bigwedge^2 S_3 \\ & \to & \mathrm{pr}_Z^* \mathcal{O}_Z(-1) \otimes \mathrm{pr}_G^* S_3 & \to & \mathcal{O}_{Z \times U} \end{array} \tag{26}$$

Since $\pi_G|_U$ is flat and $\mathcal{Y} \cap \pi_G^{-1}(U) \hookrightarrow Z \times U$ is a regular embedding of codimension 3, we have that $\omega_{\pi_G|_U} \cong \mathcal{E}xt^3_{\mathcal{O}_{Z \times U}}(\mathcal{O}_{\mathcal{Y}}, \omega_{Z \times U/U})$. Using the Koszul resolution above, the relative dualizing sheaf $\omega_{\pi_G|_U}$ is isomorphic to

$$\left[\bigwedge_R^3 (V_+^\vee) \otimes_R \bigwedge_R^3 (V_-^\vee) \right] \otimes_R \mathrm{pr}_Z^* \left[\pi^* \mathrm{pr}_+^* \mathcal{O}_{P_+}(-1) \otimes \pi^* \mathrm{pr}_-^* \mathcal{O}_{P_-}(-1) \otimes \mathcal{O}_Z(1) \right] \otimes \mathrm{pr}_G^* \bigwedge^3 (S_3^\vee).$$

Since $\mathcal{O}_Z(D_+) \cong \pi^*\mathrm{pr}_-^*\mathcal{O}_{P_-}(-2)\otimes\mathcal{O}_Z(1)$ and $\mathcal{O}_Z(D_-) \cong \pi^*\mathrm{pr}_+^*\mathcal{O}_{P_+}(-2)\otimes\mathcal{O}_Z(1)$, we have

$$\left[\pi^*\mathrm{pr}_+^*\mathcal{O}_{P_+}(-1)\otimes\pi^*\mathrm{pr}_-^*\mathcal{O}_{P_-}(-1)\otimes\mathcal{O}_Z(1)\right]^{\otimes 2} \cong \mathcal{O}_Z(D_+)\otimes\mathcal{O}_Z(D_-). \tag{27}$$

Since $\mathcal{Y}\cap\pi_G^{-1}(U)$ intersects neither $D_+\times U$ nor $D_-\times U$, we have that the restrictions of $\pi_Z^*\mathcal{O}_Z(D_+)$ and $\pi_Z^*\mathcal{O}_Z(D_-)$ are trivial. So the restriction of the pullback of the sheaf above is the trivial invertible sheaf. Therefore we have that $\omega_{\pi_G|_U}^{\otimes 2}$ is isomorphic to

$$\left[\bigwedge_R^3(V_+^\vee)\otimes_R\bigwedge_R^3(V_-^\vee)\right]^{\otimes 2}\otimes_R \mathrm{pr}_G^*\bigwedge^3(S_3^\vee)^{\otimes 2}. \tag{28}$$

And this invertible sheaf is trivial on fibers of $\pi_G|_U$. We summarize as follows.

lem-E5

Lemma 4.4. *The morphism $\pi_G|_U : \mathcal{Y}\cap\pi_G^{-1}(U)\to U$ is a flat, projective, local complete intersection morphism so that the relative dualizing sheaf $\omega_{\pi_G|_U}$ is invertible. Moreover the square $\omega_{\pi_G|_U}^{\otimes 2}$ is trivial on fibers of $\pi_G|_U$.*

According to the classification of surfaces in all characteristics, a surface whose dualizing sheaf is numerically trivial is either a $K3$-surface, an Abelian surface or an Enriques surface (c.f. [1, Thm. 1.1.2]). For any point $g\in U$, we can compute the Euler characteristic of the structure sheaf of $\mathcal{Y}_g$ using the Koszul resolution above and the Hirzebruch-Riemann-Roch theorem on Z. If we denote by h_+ and h_- the first Chern classes of $\pi^*\mathrm{pr}_+^*\mathcal{O}_{P_+}(1)$ and $\pi^*\mathrm{pr}_+^*\mathcal{O}_{P_-}(1)$ respectively, and if we denote by ζ the first Chern class of $\mathcal{O}_Z(1)$, then the Chow ring of Z is simply $\mathbb{Z}[h_+,h_-,\zeta]/\langle h_+^3, h_-^3, \zeta^2-2(h_++h_-)\zeta+4h_+h_-\rangle$. In the Chow ring we have

$$\begin{array}{rcrr} \zeta^2 &=& 2(h_++h_-)\zeta & -4h_+h_-, \\ \zeta^3 &=& 4(h_+^2+h_+h_-+h_-^2)\zeta & -8(h_++h_-)h_+h_-, \\ \zeta^4 &=& 8(h_+^2h_-+h_+h_-^2)\zeta & -16h_+^2h_-^2, \\ \zeta^5 &=& 16h_+^2h_-^2\zeta & \end{array} \tag{29}$$

The class of a point in Z is just $h_+^2h_-^2\zeta$. So among the monomials of degree 5 in h_+, h_-, ζ, we have

class	deg
$h_+^2h_-^2\zeta$	1
$h_+^2h_-\zeta^2$	2
$h_+h_-^2\zeta^2$	2
$h_+^2\zeta^3$	4
$h_+h_-\zeta^3$	4
$h_-^2\zeta^3$	4
$h_+\zeta^4$	8
$h_-\zeta^4$	8
ζ^5	16

(30)

Using the Koszul resolution, the Chern character of the structure sheaf of $\mathcal{Y}_g$ is $\mathrm{ch}(\mathcal{Y}_g) = (1-e^{-\zeta})^3$. And the Todd class of Z is

$$\mathrm{Td}(Z) = \frac{-2h_+-2h_-+2\zeta}{1-e^{2h_++2h_--2\zeta}}\left(\frac{h_+}{1-e^{-h_+}}\right)^3\left(\frac{h_-}{1-e^{-h_-}}\right)^3. \tag{31}$$

Expanding out the power series, the terms of degree 5 in h_+, h_-, and ζ which occur in $\mathrm{ch}(\mathcal{Y}_g)\mathrm{Td}(Z)$ are

$$\frac{1}{12}\zeta^5 + \frac{1}{12}(h_+ + h_-)\zeta^4 - \frac{1}{12}(2h_+^2 + h_+h_- + 2h_-^2)\zeta^3. \tag{32}$$

Evaluating this using the table above, the degree is exactly 1. So by the Hirzebruch-Riemann-Roch theorem, the Euler characteristic of the structure sheaf of $\mathcal{Y}_g$ is 1. By [1, Prop. 1.1.1], we conclude that when $\mathcal{Y}_g$ is smooth, then $\mathcal{Y}_g$ is an Enriques surface. We summarize the results of this section in the following proposition.

prop-E

Proposition 4.5. *The morphism $\pi_G : \mathcal{Y} \to G$ is a surjective, projective morphism of smooth R-schemes. The open set $U \subset G$ is dense in the fiber of G over any point in R. The morphism $\pi_G : \mathcal{Y} \cap \pi_G^{-1}(U) \to U$ is flat of relative dimension 2. Over an open subset of U dense in every fiber over R, the morphism π_G is smooth and all geometric fibers are connected Enriques surfaces.*

subsec-K3

4.1. **The $K3$ covers.** In characteristic 2 there are three different classes of Enriques surfaces: classical, μ_2 and α_2 (c.f. [1, Section 1.2]). We will show that the Enriques surfaces $\mathcal{Y}_g$ are all classical Enriques surfaces and we will construct the $K3$-cover of $\mathcal{Y}_g$.

Consider the locally free sheaf on $P_+ \times P_-$ which is $\mathcal{F} = \mathrm{pr}_+^*\mathcal{O}_{P_+}(-1) \oplus \mathrm{pr}_-^*\mathcal{O}_{P_-}(-1)$. Form the relative projective bundle $\nu : P \to P_+ \times P_-$ where $P = \mathbb{P}(\mathcal{F})$ is $\underline{\mathrm{Proj}}_{P_+ \times P_-}\mathrm{Sym}^*(\mathcal{F}^\vee)$. Let $\delta : \nu^*\mathcal{F}^\vee \to \mathcal{O}_P(1)$ denote the tautological surjection on P. Let $\upsilon_+ : P_+ \times P_- \to P$ and $\upsilon_- : P_+ \times P_- \to P$ be the two sections of ν such that the pullback of δ are respectively the two projection morphisms

$$\mathcal{F}^\vee = \mathrm{pr}_+^*\mathcal{O}_{P_+}(1) \oplus \mathrm{pr}_-^*\mathcal{O}_{P_-}(1) \to \mathrm{pr}_+^*\mathcal{O}_{P_+}(1)$$
$$\mathcal{F}^\vee = \mathrm{pr}_+^*\mathcal{O}_{P_+}(1) \oplus \mathrm{pr}_-^*\mathcal{O}_{P_-}(1) \to \mathrm{pr}_-^*\mathcal{O}_{P_-}(1)$$

Define F_+ and F_- to be the Cartier divisors in P which are the images of υ_+ and υ_- respectively.

Define $V = V_+ \oplus V_-$ and consider $\mathbb{P}(V)$, i.e. $\mathrm{Proj}\mathrm{Sym}_R^*(V^\vee)$. There are canonical inclusions $P_+ \to \mathbb{P}(V)$ and $P_- \to \mathbb{P}(V)$. There is a unique morphism $u : P \to \mathbb{P}(V)$ such that the pullback of the tautological surjection on $\mathbb{P}(V)$ is

$$V^\vee \otimes_R \mathcal{O}_P = H^0(P, \mathcal{O}_P(1)) \otimes_R \mathcal{O}_P \to \mathcal{O}_P(1). \tag{33}$$

Clearly $u^{-1}(P_+) = F_+$ and $u^{-1}(P_-) = F_-$. By comparing universal properties it follows that the induced morphism $u : P - (F_+ \cup F_-) \to \mathbb{P}(V) - (P_+ \cup P_-)$ is an isomorphism of R-schemes.

There is a unique morphism $t : P \to Z$ such that $\pi \circ t = \nu$ and such that the pullback $t^*\alpha$ of the tautological surjection is the composition of the pullback by ν of the natural map

$$\mathcal{E}^\vee = \mathrm{pr}_+^*\mathcal{O}_{P_+}(2) \oplus \mathrm{pr}_-^*\mathcal{O}_{P_-}(2) \hookrightarrow \mathrm{pr}_+^*\mathcal{O}_{P_+}(2) \oplus \mathrm{pr}_+^*\mathcal{O}_{P_+}(1) \otimes \mathrm{pr}_-^*\mathcal{O}_{P_-}(1) \oplus \mathrm{pr}_-^*\mathcal{O}_{P_-}(2) = \mathrm{Sym}^2(\mathcal{F}^\vee) \tag{34}$$

with the map

$$\mathrm{Sym}^2(\delta) : \nu^*\mathrm{Sym}^2(\mathcal{F}^\vee) \to \mathrm{Sym}^2(\mathcal{O}_P(1)) = \mathcal{O}_P(2). \tag{35}$$

Clearly $t^{-1}(D_+) = F_+$ and $t^{-1}(D_-) = F_-$.

Recall that the group scheme $\mu_{R,2}$ is defined to be Spec $R[\zeta]/\langle\zeta^2-1\rangle$ where the multiplication morphism is given by

$$R[\zeta]/\langle\zeta^2-1\rangle \to R[\zeta]/\langle\zeta^2-1\rangle \otimes_R R[\zeta]/\langle\zeta^2-1\rangle, \quad \zeta \mapsto \zeta\otimes\zeta. \tag{36}$$

To give an action $s:\mu_{R,2}\times P\to P$ commuting with the map ν, it suffices to give a map of $\mathcal{O}_{P_+\times P_-}$-algebras

$$\mathrm{Sym}^*(\mathcal{F}^\vee) \to R[\zeta]/\langle\zeta^2-2\rangle \otimes_R \mathrm{Sym}^*(\mathcal{F}^\vee) \tag{37}$$

which preserves the grading and satisfies the obvious axioms. There is a unique map of $\mathcal{O}_{P_+\times P_-}$-algebras as above such that for $a_+\in \mathrm{pr}_+^*\mathcal{O}_{P_+}(1)$ we have $a_+\mapsto a_+\otimes 1$ and for $a_-\in\mathrm{pr}_-^*\mathcal{O}_{P_-}(1)$ we have $a_-\mapsto a_-\otimes\zeta$. Let $s:\mu_{R,2}\times P\to P$ be the associated action of $\mu_{R,2}$. Denoting by $\mathrm{pr}_P\to\mu_{R,2}\times P\to P$ the obvious projection, there is a unique $\mu_{R,2}$-linearization $s^*\mathcal{O}_P(1)\to\mathrm{pr}_P^*\mathcal{O}_P(1)$ such that the induced action of $\mu_{R,2}$ on $\oplus_d\nu_*\mathcal{O}_P(d)$ is just the sheaf map defined above.

Using the $\mu_{R,2}$-linearization $s^*\mathcal{O}_P(2)\to\mathrm{pr}_P^*\mathcal{O}_P(2)$, we can compare the two surjections $s^*t^*\alpha$ and $\mathrm{pr}_P^*t^*\alpha$. Since $\zeta^2=1$, the map of $\mathcal{O}_{P_+\times P_-}$-algebras from above maps any element a in $\mathcal{F}^\vee\subset\mathrm{Sym}^2(\mathcal{E}^\vee)$ to the element $a\otimes 1$. Therefore we have that $s^*t^*\alpha=\mathrm{pr}_P^*t^*\alpha$. By the universal property of projective bundles, we conclude that $t\circ s=t\circ\mathrm{pr}_P$. In particular, F_+, F_- and $P-(F_+\cup F_-)$ are invariant schemes for the action of $\mu_{R,2}$, i.e. there is an induced action of $\mu_{R,2}$ on each of these schemes such that the locally closed immersion into P is $\mu_{R,2}$-equivariant. Moreover, since $t\circ s=\mathrm{pr}_P\circ s$, we have an induced morphism of schemes

$$(s,\mathrm{pr}_P):\mu_{R,2}\times P\to P\times_Z P \tag{38}$$

Using local coordinates, it is not hard to show that $t:P\to Z$ is a universal categorical quotient and a uniform geometric quotient. More important for us, the restriction

$$(s,\mathrm{pr}_P):\mu_{R,2}\times(P-(F_+\cup F_-))\to(P-(F_+\cup F_-))\times_Z(P-(F_+\cup F_-)) \tag{39}$$

is an isomorphism, so that $t:P-(F_+\cup F_-)\to Z-(D_+\cup D_-)$ together with the action s make $P-(F_+\cup F_-)$ a $\mu_{R,2}$-torsor over $Z-(D_+\cup D_-)$ (with respect to the flat topology). The invertible sheaf on $Z-(D_+\cup D_-)$ associated to the standard representation $\mu_{R,2}$ is just the kernel of the trace map $t_*\mathcal{O}_P\to\mathcal{O}_Z$ which turns out to be the restriction to $Z-(D_+\cup D_-)$ of the invertible sheaf:

$$\pi^*\mathrm{pr}_+^*\mathcal{O}_{P_+}(1)\otimes\pi^*\mathrm{pr}_-^*\mathcal{O}_{P_-}(1)\otimes\mathcal{O}_Z(-1). \tag{40}$$

We define $\mathcal{X}\subset P\times G$ to be the closed subscheme which is the fiber product of $\mathcal{Y}$ and $P\times G$ over $Z\times G$. Since $\mathcal{Y}\cap\pi_G^{-1}(U)$ is contained in $(Z-(D_+\cup D_-))\times U$, it follows that $\mathcal{X}\cap\pi_G^{-1}(U)$ is contained in $(P-(F_+\cup F_-))\times U$ and that the induced morphism $q:\mathcal{X}\cap\pi_G^{-1}(U)\to\mathcal{Y}\cap\pi_G^{-1}(U)$ is a $\mu_{R,2}$-torsor. We could equivalently have defined $\mathcal{X}$ as the fiber product of $\mathcal{Y}$ with P over Z. Since $\mathrm{pr}_Z:\mathcal{Y}\to Z$ is a Grassmannian bundle, also $\mathrm{pr}_P:\mathcal{X}\to X$ is a Grassmannian bundle. In particular, $\mathcal{X}$ is smooth and projective over R and every fiber over R is geometrically irreducible. It follows from Bertini's theorem [5, Thm. 6.3] that the generic fiber of $\mathrm{pr}_G:\mathcal{X}\to G$ is geometrically irreducible. Away from characteristic 2, we have that $\mathcal{X}\cap\pi_G^{-1}(U)\to\mathcal{Y}\cap\pi_G^{-1}(U)$ is étale so that also the generic fiber of $\mathrm{pr}_G:\mathcal{X}\to G$ is smooth. So the generic fiber is smooth and geometrically irreducible.

Incidentally, away from characteristic 2 this gives a second proof that the generic fiber of $\mathrm{pr}_G|_U : \mathcal{Y} \cap \mathrm{pr}_G^{-1}(U) \to U$ is an Enriques surface. By the computation in the last paragraph, the $\mu_{R,2}$-torsor $q : \mathcal{X} \cap \pi_G^{-1}(U) \to \mathcal{Y} \cap \pi_G^{-1}(U)$ is associated to the invertible sheaf which is just $\omega^\vee_{\pi_G|_U}$ twisted by an invertible sheaf which is trivial on fibers of $\pi_G|_U$. Since the generic fiber of $\mathcal{X} \to G$ is smooth and irreducible, the torsor is non-trivial which implies that $\omega_{\pi_G|_U}$ is a 2-torsion invertible sheaf which is non-trivial. This property characterizes Enriques surfaces (away from characteristic 2).

Now suppose that $p \in R$ is a point with $\mathrm{char}(\kappa(p)) = 2$. Let $\overline{\eta} \in G_p$ denote the geometric generic point. Then in fact $\mathcal{X}_{\overline{\eta}}$ is singular, but it is geometrically reduced. Moreover the morphism $u : \mathcal{X}_{\overline{\eta}} \to \mathbb{P}(V) \otimes_R \kappa(\overline{\eta})$ is a closed immersion whose image is a complete intersection of three quadrics. So in fact $\mathcal{X}_{\overline{\eta}}$ is a complete intersection which is generically reduced, so it is reduced. At any rate, the induced μ_2-torsor $q : \mathcal{X}_{\overline{\eta}} \to \mathcal{Y}_{\overline{\eta}}$ is nontrivial. The Cartier dual group to μ_2 is the separable group scheme $\mathbb{Z}/2\mathbb{Z}$. Since there is a nontrivial μ_2-torsor, it follows that there is a nontrivial group homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Pic}(\mathcal{Y}_{\overline{\eta}})$ which shows that $\mathcal{Y}_{\overline{\eta}}$ is a *classical* Enriques surface.

We just give a hint as to how to prove that $\mathcal{X}_{\overline{\eta}}$ is generically reduced. The image $u(\mathcal{X}_{\overline{\eta}}) \subset \mathbb{P}(V)$ is a complete intersection of three quadrics which are invariant under the action of μ_2. We can write these three quadrics as:

$$\begin{array}{lcllll} F_{110} & = & G_{110}(X_i^2, Y_j^2)+ & X_0X_1+ & C^{110}_{110}Y_0Y_1 + C^{101}_{110}Y_0Y_2 + C^{011}_{110}Y_1Y_2 \\ F_{101} & = & G_{101}(X_i^2, Y_j^2)+ & X_0X_2+ & C^{110}_{101}Y_0Y_1 + C^{101}_{101}Y_0Y_2 + C^{011}_{101}Y_1Y_2 \\ F_{011} & = & G_{011}(X_i^2, Y_j^2)+ & X_1X_2+ & C^{110}_{011}Y_0Y_1 + C^{101}_{011}Y_0Y_2 + C^{011}_{011}Y_1Y_1 \end{array} \tag{41}$$

The scheme in $\mathbb{P}(V)$ where the Jacobian matrix of this set does not have rank 3 is the complete intersection Δ of the three bihomogeneous polynomials:

$$\begin{array}{ll} (X_0C^{110}_{011} + X_1C^{110}_{101} + X_2C^{110}_{110})Y_1+ & (X_0C^{101}_{011} + X_1C^{101}_{101} + X_2C^{101}_{110})Y_2, \\ (X_0C^{110}_{011} + X_1C^{110}_{101} + X_2C^{110}_{110})Y_0+ & (X_0C^{011}_{011} + X_1C^{011}_{101} + X_2C^{011}_{110})Y_2, \\ (X_0C^{101}_{011} + X_1C^{101}_{101} + X_2C^{101}_{110})Y_0+ & (X_0C^{011}_{011} + X_1C^{011}_{101} + X_2C^{011}_{110})Y_1 \end{array} \tag{42}$$

This scheme Δ is, in fact, a threefold. But over a field of characteristic 2, Euler's identity does not hold! The singular scheme of $\mathcal{X}_{\overline{\eta}}$ is the intersection of Δ with the zero locus of F_{110}, F_{101} and F_{011}. Computing one special case (particularly simple is when $C^{ijk}_{ijk} = 1$ and otherwise $C^{lmn}_{ijk} = 0$), one sees that this intersection is a zero-dimensional scheme. Therefore $\mathcal{X}_{\overline{\eta}}$ is smooth away from codimension 2, so it is generically reduced (and even normal). Let us also remark that since the singular locus of $\mathcal{X}_{\overline{\eta}}$ is zero-dimensional, it follows from the proof of [1, Cor. 1.3.1] that $\mathcal{X}_{\overline{\eta}}$ is a rational surface (probably this can be verified directly), and thus $\mathcal{Y}_{\overline{\eta}}$ is a *Zariski surface*.

5. The Enriques surface construction *sec-Enriques*

This section is mostly independent of the last section. At the beginning of the section we quickly review how the Grassmannian G is a parameter space for octic $K3$-surfaces invariant under the action of μ_2. Throughout this section we assume that $\mathrm{char}(k) \neq 2$.

Let B be a curve isomorphic to $\mathbb{P}^1$ and let $(g : D \to C, f : C \to B)$ be a pair of maps $\mathbb{P}^1 \to \mathbb{P}^1$ as in Proposition 2.6 such that $\deg(f) = 3$, $\deg(g) = 2$ and the

Galois group $\Gamma_{f\circ g}$ of $f \circ g$ acts as the full group $W_{3,2}$ on the geometric generic fiber of $f \circ g$. In particular, since g has order 2, $g : D \to C$ is already Galois. Let $\gamma : D \to D$ be a generator for the Galois group Γ_g of g.

We can choose homogeneous coordinates $[X_0 : X_1]$ on D such that the action of γ on D is just $\gamma \cdot [X_0 : X_1] = [X_0 : -X_1]$. We define $V = V_+ \oplus V_-$ to be a direct sum of two 3-dimensional vector spaces with ordered bases $(\mathbf{e}_{+,1}, \mathbf{e}_{+,2}, \mathbf{e}_{+,3})$ and $(\mathbf{e}_{-,1}, \mathbf{e}_{-,2}, \mathbf{e}_{-,3})$ respectively. We define an action of Γ_g on V which acts diagonally as the identity on V_+ and as -1 on V_-. Let $(Y_{+,1}, Y_{+,2}, Y_{+,3}), (Y_{-,1}, Y_{-,2}, Y_{-,3})$ be ordered bases on $V_+^\vee, V_-^\vee$ respectively which are dual to the ordered bases above. Observe that $\mathbf{e}_{+,i}, \mathbf{e}_{-,i}$ give homogeneous coordinates on the projective space $\mathbb{P}(V^\vee) = \mathrm{Proj}\mathrm{Sym}^*(V)$. The action of Γ_g on V induces an action on $\mathbb{P}(V^\vee)$. We define a closed immersion $u : D \to \mathbb{P}(V^\vee)$ with $u^*\mathcal{O}_{\mathbb{P}(V^\vee)}(1) = \mathcal{O}_D(5)$ by

$$u^*\mathbf{e}_{+,1} = X_0^5, u^*\mathbf{e}_{+,2} = X_0^3X_1^2, u^*\mathbf{e}_{+,3} = X_0X_1^4,$$
$$u^*\mathbf{e}_{-,1} = X_1^5, u^*\mathbf{e}_{-,2} = X_0^2X_1^3, u^*\mathbf{e}_{-,3} = X_0^4X_1$$

Observe that this morphism is equivariant for the action of $\Gamma(L/L')$.

As in Section 3, there is an induced map $v : C \to \mathbb{P}(\mathrm{Sym}^2(V^\vee))$. Because $\mathrm{char}(k) \neq 2$, we can identify $\mathrm{Sym}^2(V)$ with the dual vector space to $\mathrm{Sym}^2(V^\vee)$. Also we can choose homogeneous coordinates $[T_0 : T_1]$ on C such that $D \to C$ pulls back T_0 to X_0^2 and pulls back T_1 to X_1^2. Then the map v satisfies $v^*\mathcal{O}(1) = \mathcal{O}_C(5)$ and is given explicitly by:

$$v^*\mathbf{e}_{+,1}^2 = T_0^5, v^*\mathbf{e}_{+,1}\mathbf{e}_{+,2} = 2T_0^4T_1, v^*\mathbf{e}_{+,1}\mathbf{e}_{+,3} = 2T_0^3T_1^2$$
$$v^*\mathbf{e}_{+,2}^2 = T_0^3T_1^2, v^*\mathbf{e}_{+,2}\mathbf{e}_{+,3} = 2T_0^2T_1^3, v^*\mathbf{e}_{+,3}^2 = T_0T_1^4$$
$$v^*\mathbf{e}_{-,1}^2 = T_1^5, v^*\mathbf{e}_{-,1}\mathbf{e}_{-,2} = 2T_0T_1^4, v^*\mathbf{e}_{-,1}\mathbf{e}_{-,3} = 2T_0^2T_1^3$$
$$v^*\mathbf{e}_{-,2}^2 = T_0^2T_1^3, v^*\mathbf{e}_{-,2}\mathbf{e}_{-,3} = 2T_0^3T_1^2, v^*\mathbf{e}_{-,3}^2 = T_0^4T_1.$$

And $v^*\mathbf{e}_{+,i}\mathbf{e}_{-,j} = 0$ for all i, j. So in fact the image of v is contained in the sub-projective space $\mathbb{P}(W)$ where $W \subset \mathrm{Sym}^2(V^\vee)$ is the subspace $\mathrm{Sym}^2(V_+^\vee) \oplus \mathrm{Sym}^2(V_-^vee)$.

Associated to v is the pullback of the canonical surjection on $\mathbb{P}(W)$, namely $W^\vee \otimes_k \mathcal{O}_C \to v^*\mathcal{O}_{\mathbb{P}(W)}(1)$. Since $f : C \to B$ is finite and flat, the pushforward gives a surjection $W^\vee \otimes_k f_*\mathcal{O}_C \to f_*(v^*\mathcal{O}_{\mathbb{P}(W)}(1))$. Composing with the map $\mathrm{Id} \otimes f^\sharp : W^\vee \otimes_k \mathcal{O}_B \to W^\vee \otimes_k f_*\mathcal{O}_C$ we get a map $W^\vee \otimes_k \mathcal{O}_B \to f_*(v^*\mathcal{O}_{\mathbb{P}(W)}(1))$. Similar to the case in Section 3, one checks that this map is surjective. Therefore there is an induced map $w : B \to G$ where $G = \mathrm{Grass}(3, W)$ is the Grassmannian variety parametrizing 3-dimensional subspaces of W. Speaking loosely, if $b \in B$ is a geometric point and c_1, c_2, c_3 are distinct geometric points in $f^{-1}(b)$, then $w(b) = \mathrm{Span}(v(c_1), v(c_2), v(c_3))$. Let $S \subset W \otimes_k \mathcal{O}_G$ denote the tautological locally free subsheaf of rank 2 and let $\mathcal{O}_G(1)$ denote $\bigwedge^3(S^\vee)$, i.e. the pullback under the Plücker embedding of $\mathcal{O}(1)$.

It is straightforward to compute that $w^*\mathcal{O}_G(1) \cong \bigwedge^3 f_*(v^*\mathcal{O}(1)) \cong \mathcal{O}_B(3)$. In fact, we can compute that $h^0(B, f_*(v^*\mathcal{O}_{\mathbb{P}(W)}(1))) = h^0(C, v^*\mathcal{O}_{\mathbb{P}(W)}(1)) = 6$ and $h^1(B, f_*(v^*\mathcal{O}_{\mathbb{P}(W)}(1)) \otimes \mathcal{O}_B(-2))$ equals $h^1(B, v^*\mathcal{O}_{\mathbb{P}(W)}(1) \otimes \mathcal{O}_C(-6))$ which is zero. This follows because $v^*\mathcal{O}_{\mathbb{P}(W)}(1) \cong \mathcal{O}_C(5)$ and $f^*\mathcal{O}_B(1) \cong \mathcal{O}_C(3)$. Combined with Grothendieck's lemma [REFERENCE], we conclude that $f_*(v^*\mathcal{O}_{\mathbb{P}(W)}(1)) \cong \mathcal{O}_B(1)^{\oplus 3}$.

Now there is a closed subscheme $\mathcal{X} \subset \mathbb{P}(V) \times G$ whose ideal sheaf is the image of the map $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}(V)}(-2) \otimes \mathrm{pr}_2^* S \to \mathcal{O}_{\mathbb{P}(V)\times G}$ which is the adjoint of the canonical map $\mathrm{pr}_2^* S \to W \otimes_k \mathcal{O}_{\mathbb{P}(V)\times G} \to \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}(V)}(2)$. Let $\mathcal{K}$ denote the locally free sheaf of rank 11 on $\mathbb{P}(V)$ which is the kernel of the surjection $W \otimes_k \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(2)$. Denote by $\mathcal{X}_w \subset \mathbb{P}(V) \times B$ the preimage of $\mathcal{X}$ under $\mathrm{Id} \times w$. $\mathrm{Grass}_{\mathbb{P}(V)}(3, \mathcal{K})$.

Now consider some closed point $b \in B$ and let $\Delta = [c_1] + [c_2] + [c_3]$ be a divisor of degree 3 on C. Up to pre-composing f with an automorphism of C, we may suppose that $f^{-1}([b]) = \Delta$. And we may choose $c_1, c_2, c_3 \in C$ so that if $\{d_{+,1}, d_{-,1}\}, \{d_{+,2}, d_{-,2}\}, \{d_{+,3}, d_{-,3}\}$ are the three preimages of c_1, c_2, c_3 in B, then the six points $u(d_{+,i}), u(d_{-,i})$ in $\mathbb{P}(V^\vee)$ parametrize hyperplanes $H_{+,i}, H_{-,i} \subset \mathbb{P}(V)$ such that $H_{+,1} \cap H_{-,1} \cap H_{+,2} \cap H_{-,2} \cap H_{+,3} \cap H_{-,3}$ is empty. But now the set of points $b \in B$ where this holds is clearly an open set $B^o \subset B$ which is dense since it is non-empty. So B^o contains the generic point of B.

Let $\zeta \in B$ denote the generic point, and let $\overline{\zeta}$ denote the geometric generic point of B. Let $\mathcal{X}_{\overline{\zeta}} \subset \mathbb{P}(V) \times \mathrm{Spec}\ \kappa(\overline{\zeta})$ denote the pullback of the universal family $\mathcal{X} \subset \mathbb{P}(V) \times G$. Let $H_{+,i}, H_{-,i}$ be as in the last paragraph. Then we have

$$\mathcal{X}_{\overline{\zeta}} = (H_{+,1} \cup H_{-,1}) \cap (H_{+,2} \cup H_{-,2}) \cap (H_{+,3} \cup H_{-,3})\,. \tag{43}$$

The dual graph of $\mathcal{X}_{\overline{\zeta}}$ is a cube Π where the 8 vertices of the cube correspond to the 8 triples $(d_{+/-,1}, d_{+/-,2}, d_{+/-,3})$ and the incidence relation between two such triples is defined by having two of the three elements of the triple in common. The irreducible component of $\mathcal{X}_{\overline{\zeta}}$ corresponding to a triple, say $(d_{+,1}, d_{-,2}, d_{+,3})$ is simply the 2-plane $H_{+,1} \cap H_{-,2} \cap H_{-,3}$. The condition that the 6 hyperplanes $H_{+/-,i}$ have no common intersection guarantees that the dual graph of $\mathcal{X}_{\overline{\zeta}}$ is indeed the cube Π. In particular, the irreducible components corresponding to two opposite vertices of the cubic have no common intersection.

It is straightforward to work out the action of $\Gamma_{g\circ f} = W_{n,m}$ on the cube Π. It acts as the full group of rigid motions of Π (not necessarily orientation-preserving). So $W_{n,m}$ acts transitively on the set of vertices, edges and faces of Π, which have cardinalities 8, 12 and 6 respectively. Let $M \subset \mathbb{Z}_{>0}$ denote the additive sub-semigroup generated by 6, 8, and 12, i.e. $M = \{n \in \mathbb{Z}_{>0} | n \text{even}, n \geq 6\}$. By the same argument as in Section 3, we deduce the following.

prop-Emulti

Proposition 5.1. *Let $Y \subset \mathcal{X}_w$ be an irreducible multisection of degree e. Then e is divisible by 6 or 8. Let $Y \subset \mathcal{X}_w$ be a (possibly reducible) multisection of degree e. Then $e \in M$.*

Consider the affine space associated to the vector space $\mathrm{Hom}_{\mathcal{O}_B}(W^\vee \otimes_k \mathcal{O}_B, \mathcal{O}_B(1)^{\oplus 3})$, and let H denote the open subscheme of this affine space which parametrizes surjective maps $\phi : W^\vee \otimes_k \mathcal{O}_B \to \mathcal{O}_B(1)^{\oplus 3}$. There is a universal morphism $\Omega : B \times H \to \mathrm{Grass}(3, W)$ induced by the surjection $W^\vee \otimes_k \mathcal{O}_{B\times H} \to \mathrm{pr}_1^*\mathcal{O}_B(1)^{\oplus 3}$. Let $\mathcal{X}_\Omega \subset \mathbb{P}(V) \times B \times H$ denote the pullback by Ω of the universal subscheme $\mathcal{X} \subset \mathbb{P}(V) \times G$. The projection morphism $\mathrm{pr}_3 : \mathcal{X}_\Omega \to H$ is projective, and the invertible sheaf $\mathcal{L} = \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}(V)}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_B(1)$ is pr_3-relatively ample. For each integer p and each pair of positive integers e, e' denote by $\mathcal{H}_{e,e',p} \to H$ the relative Hilbert scheme $Hilb(\mathcal{X}_\Omega/H)$ parametrizing closed subschemes $Y \subset (\mathcal{X}_\Omega)_{w'}$ whose Hilbert polynomial with respect to $\mathcal{L}$ is $e't + 1 - p$ and whose Hilbert polynomial with respect to $\mathrm{pr}_2^*\mathcal{O}_B(1)$ is $e+1-p$. Observe that $\mathcal{H}_{e,e',p} \to H$ is a projective morphism. The proof of the following theorem is identical to the proof of Theorem 3.4.

thm-Emulti

Theorem 5.2. *If e is not in the sub-semigroup $M = \{6, 8, 10, 12, \ldots\}$, then the morphism $\mathcal{H}_{e,e',p} \to H$ is not surjective, and therefore it is not dominant.*

cor-Emulti

Theorem 5.3. *If k is an uncountable, algebraically closed field with $char(k) \neq 2$, then there exists a map $h : \mathbb{P}^1 \to G$ with $h^*\mathcal{O}_G(1) \cong \mathcal{O}_{\mathbb{P}^1}(3)$ such that for the pullback $\mathcal{Y}_h \subset Z \times \mathbb{P}^1$ of the universal family of Enriques surfaces $\mathcal{Y} \subset Z \times G$, every multisection of $\mathcal{Y}_h \to \mathbb{P}^1$ has degree at least 3. Moreover for any three very general Enriques surface Y_0, Y_1 and Y_∞, we can find h as above so that Y_0, Y_1 and Y_∞ are isomorphic to the fibers of $\mathcal{Y}_h \to \mathbb{P}^1$ over $0, 1, \infty \in \mathbb{P}^1$. For the relative dualizing sheaf ω of $pr_{\mathbb{P}^1} : \mathcal{Y}_h \to \mathbb{P}^1$, we have that the degree of $(pr_{\mathbb{P}^1})_*(\omega^{\otimes 2})$ is 6.*

Proof. By Theorem 5.2, for a very general $h : \mathbb{P}^1 \to G$ with $h^*\mathcal{O}_G(1) \cong \mathcal{O}_{\mathbb{P}^1}(3)$, we have that every multisection of the corresponding family of $K3$-covers $\mathcal{X}_h \to \mathbb{P}^1$ has degree at least 6. Since a multisection of $\mathcal{Y}_h \to \mathbb{P}^1$ of degree e pulls back to a multisection of $\mathcal{X}_h \to \mathbb{P}^1$ of degree $2e$, we conclude that every multisection of $\mathcal{Y}_h \to \mathbb{P}^1$ has degree at least 3.

Let $\Lambda_1, \Lambda_2, \Lambda_3 \subset V$ be three subspaces of dimension 3. If they are general such subspaces, we can choose a basis for V with respect to with $\Lambda_1 = \mathrm{span}(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$, $\Lambda_2 = \mathrm{span}(\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5)$ and $\Lambda_3 = \mathrm{span}(\mathbf{e}_0 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_5)$. Now consider the map of $\mathbb{P}^1$ to G:

$$[T_0 : T_1] \mapsto \mathrm{span}(T_0\mathbf{e}_0 + T_1\mathbf{e}_3, T_0\mathbf{e}_1 + T_1\mathbf{e}_4, T_0\mathbf{e}_2 + T_1\mathbf{e}_5). \tag{44}$$

From this it is clear that any general triple of points $(\Lambda_1, \Lambda_2, \Lambda_3) \in G \times G \times G$ lies on the image of a map $h : \mathbb{P}^1 \to G$ with $h^*\mathcal{O}_G(1) \cong \mathcal{O}_{\mathbb{P}^1}$. So a very general triple in $G \times G \times G$ lies on the image of a map $h : \mathbb{P}^1 \to G$ such that $\mathcal{Y}_h \to \mathbb{P}^1$ has no multisection of degree 1 or 2. Since every Enriques surface Y is isomorphic to an Enriques surface parametrized by a point of G, we conclude that any very general triple of Enriques surfaces Y_0, Y_1, Y_∞ is isomorphic to the triple coming from a very general point of $G \times G \times G$.

Finally the computation of $(\mathrm{pr}_{\mathbb{P}^1})_*(\omega^{\otimes 2})$ follows from the computations in Section 4. □

References

[1] F. Cossec and I. Dolgachev. *Enriques Surfaces I*, volume 76 of *Progress in Mathematics*. Birkhäuser, 1989.

[2] T. Graber, J. Harris, B. Mazur, and J. Starr. Rational connectivity and sections of families over curves. submitted Ann. Sci. Ecole Norm. Sup., preprint arXiv: math.AG/0210225.

[3] A. Grothendieck. *Revêtements etales et groupe fondamental (SGA 1)*, volume 224 of *Lecture notes in mathematics*. Springer-Verlag, 1971.

[4] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, 1977.

[5] J.-P. Jouanolou. *Théorèmes de Bertini et Applications*, volume 42 of *Progess in Mathematics*. Birkhäuser, 1983.

[6] J. Kollár. *Rational Curves on Algebraic Varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge*. Springer-Verlag, 1996.

[7] S. Lang. On quasi-algebraic closure. *Ann. Math.*, 55:373–390, 1952.