A NOTE ON FANO MANIFOLDS WHOSE SECOND CHERN CHARACTER IS POSITIVE

A. J. DE JONG AND JASON MICHAEL STARR

1. INTRODUCTION

This note about Fano manifolds $X$ for which $(c_1^2 - c_2)(T_X)$ is positive, lists what few examples are known, as well as giving many non-examples. Presumably there are many more examples. They do not seem easy to find.

**Notation 1.1.** Let $X$ be a projective variety over an algebraically closed field. For every integer $k \geq 0$, denote by $N_k(X)$ the finitely-generated free Abelian group of $k$-cycles modulo numerical equivalence, and denote by $N_k^*(X)$ the $k$th graded piece of the quotient algebra $A^*(X)/\text{Num}^*(X)$, cf. [Ful98, Example 19.3.9]. For every $\mathbb{Z}$-module $B$, denote $N_k^*(X)\otimes B$.

**Definition 1.2.** A class in $N_k^*(X)$ is nef if it pairs nonnegatively with every element in $NE_k(X)$. The corresponding cone is denoted $\text{Nef}^k(X)$. A class is weakly positive if it pairs positively with every element in $NE_k(X)$. The corresponding cone is denoted $\text{WPos}^k(X)$. A class is positive if it is contained in the interior of $\text{Nef}^k(X)$; the interior of $\text{Nef}^k(X)$ is denoted $\text{Pos}^k(X)$. For $k > 1$, it can happen that $\text{Ample}^k(X) \neq \text{Pos}^k(X)$; for instance, because $(N^1(X)\otimes k \to N^k(X)$ is not surjective. There are also examples where $\text{Pos}^k(X) \neq \text{WPos}^k(X)$ and $\text{WPos}^k(X) \neq \text{Nef}^k(X)$.

**Problem 1.4.** Find smooth, connected, projective varieties $X$ such that $c_1(T_X) = 0$ and $c_2(T_X) = 1/2(c_1^2 - 2c_2)(T_X)$ is ample, resp. positive, weakly positive, nef. More generally, allow $X$ to be a smooth, connected, proper Deligne-Mumford stack whose coarse moduli space is projective.

2. POSITIVE EXAMPLES

Following are examples of Fano manifolds with $c_2(T_X)$ ample or positive.
1. The simplest example is $\mathbb{P}^n$ for $n \geq 2$. Denote by $h \in N^1(\mathbb{P}^n)$ the first Chern class of $O_{\mathbb{P}^n}(1)$. Using the Euler sequence,

$$0 \rightarrow O_{\mathbb{P}^n} \rightarrow O_{\mathbb{P}^n}(1)^{\oplus (n+1)} \rightarrow T_{\mathbb{P}^n} \rightarrow 0,$$

the Chern character of $T_{\mathbb{P}^n}$ is $(n+1)e^h - 1$. In particular, $\text{ch}_k(T_X) = (n+1)h^k/k!$ for every $k = 1, \ldots, n$. So $\text{ch}_k(T_X)$ is ample for $k = 1, \ldots, n$.

2. Weighted projective spaces are also examples. The weighted projective space $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_n)$ is the coarse moduli space of a smooth Deligne-Mumford stack $X$, and $\text{ch}_k(T_X) = (n+1)h^k/k!$ where $h$ is the first Chern class of the invertible sheaf $O_X(1)$ on the stack. Some positive multiple of $h$ is the pullback of an ample class from the coarse moduli space, thus $h$ is an ample class.

3. Let $Y$ be a smooth complete intersection of divisors $D_1, \ldots, D_r$ in $\mathbb{P}$ of respective degrees $d_1, \ldots, d_r$. Using the exact sequences,

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{\mathbb{P}}(N) \rightarrow \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}}(d_i) \rightarrow 0,$$

the Chern character of $\mathcal{T}_Y$ is $(n+1)e^h - 1 - \sum_{i=1}^r e^{d_i h}$. Thus $\text{ch}_k(\mathcal{T}_Y) = 1/k!(n+1 - (d_1^k + \cdots + d_r^k))h^k$ for $k = 1, \ldots, n-r$. In particular, if $d_1^2 + \cdots + d_r^2 < n+1$ then $\text{ch}_1(\mathcal{T}_Y)$ and $\text{ch}_2(\mathcal{T}_Y)$ are both ample.

4. For every integer $k \geq 1$, the Grassmannians $G = \text{Grass}(k, 2k)$ and $G = \text{Grass}(k, 2k+1)$ have $\text{ch}_1(T_G)$ is ample and $\text{ch}_2(T_G)$ is positive. If $k > 1$, then $\text{ch}_2(T_G)$ is not in the 1-dimensional subspace spanned by $\sigma_1^2$. Therefore $\text{ch}_2(T_G)$ is positive, but not ample.

3. Nef Examples

Given a Fano manifold, there are a few methods of constructing a new Fano manifold $Y$ with $\text{ch}_2(T_Y)$ nef. Typically even if $\text{ch}_2(T_X)$ is positive, $\text{ch}_2(T_Y)$ is not weakly positive.

1. Let $X$ be a Fano manifold with $\text{ch}_2(T_X)$ nef. Let $Y$ be a smooth divisor in $X$. If $c_1(T_X) - [Y]$ is ample, and $\text{ch}_2(T_X) - [Y]^2/2$ is nef, then $Y$ is a Fano manifold and $\text{ch}_2(T_Y)$ is nef. This is essentially the same as Example 3 in Section 2.

2. Let $X$ be a Fano manifold and let $L$ be a nef line bundle such that $c_1(T_X) - c_1(L)$ is ample and $\text{ch}_2(T_X) + c_1(L)^2/2$ is nef. Then the projective bundle $\mathbb{P}E = \mathbb{P}(L^\vee \oplus O_X)$ is a Fano manifold and $\text{ch}_2(T_{\mathbb{P}E})$ equals $\pi^*(\text{ch}_2(T_X) + c_1(L)^2/2)$. This is nef, but not weakly positive; its restriction to $\pi^{-1}(C)$ is zero for every curve $C \subset X$. Note $\text{ch}_2(T_X)$ need not be nef, e.g., for integers $(n, d, a)$ satisfying $1 \leq d \leq \lfloor (n^2 + n + 1)/2n \rfloor$ and $\lfloor \max(0, d^2 - n - 1) \rfloor \leq a \leq n - d$, for every smooth degree $d$ hypersurface $X \subset \mathbb{P}^n$, the projective bundle $\mathbb{P}(O_X(-a) \oplus O_X)$ is a Fano manifold with $\text{ch}_2(T_{\mathbb{P}E})$ nef.

3. Let $X$ and $Y$ be Fano manifolds such that $\text{ch}_2(T_X)$ and $\text{ch}_2(T_Y)$ are nef. The product $X \times Y$ is Fano and $\text{ch}_2(T_{X \times Y}) = \pi_X^* \text{ch}_2(T_X) + \pi_Y^* \text{ch}_2(T_Y)$, which is nef. For rational curves $C_X \subset X$ and $C_Y \subset Y$, the pairing of $\text{ch}_2(T_{X \times Y})$ with $C_X \times C_Y$ is zero, thus $\text{ch}_2(T_{X \times Y})$ is not weakly positive.
4. Projective bundles

One way to produce new examples of Fano manifolds is to form the projective bundle of a vector bundle of “low degree” over a given Fano manifold.

**Lemma 4.1.** Let $E$ be a vector bundle on $X$ of rank $r$. Denote by $π : \mathbb{P}E → X$ the associated projective bundle. The graded pieces of the Chern character of $T_{\mathbb{P}E}$ are, $c_1(T_{\mathbb{P}E}) = rζ + π^*(c_1(T_X) + c_1(E))$ and $\text{ch}_2(T_{\mathbb{P}E}) = rζ^2 / 2 + π^*(c_1(T_X) + ch_2(E))$, where $ζ$ equals $c_1(O_{\mathbb{P}E}(1))$.

**Proof.** There is an Euler sequence,

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}E} \rightarrow π^*E \otimes \mathcal{O}_{\mathbb{P}E}(1) \rightarrow T_{\mathbb{P}E/X} \rightarrow 0.
\]

Therefore $\text{ch}(T_{\mathbb{P}E/X}) = π^*\text{ch}(E)ζ - 1$, i.e.,

\[
(r + π^*c_1(E) + π^*ch_2(E) + \ldots)(1 + ζ + ζ^2 / 2 + \ldots) - 1 =
\]

\[
[r - 1] + [rζ + π^*c_1(E)] + [rζ^2 / 2 + π^*(c_1(T_X) + c_1(E))] + \ldots
\]

Using the exact sequence,

\[
0 \rightarrow T_{\mathbb{P}E/X} \rightarrow T_{\mathbb{P}E} \rightarrow \pi^*T_X \rightarrow 0,
\]

$\text{ch}(T_{\mathbb{P}E})$ equals $\text{ch}(T_{\mathbb{P}E/X}) + π^*\text{ch}(T_X)$. Thus $\text{ch}_1(T_{\mathbb{P}E/X}) = rζ + π^*(c_1(T_X) + c_1(E))$ and,

\[
\text{ch}_2(T_{\mathbb{P}E}) = rζ^2 / 2 + π^*(c_1(T_X) + ch_2(E)).
\]

\[\square\]

**Proposition 4.2.** Let $X$ be a smooth Fano manifold and let $E$ be a vector bundle on $X$ of rank $r$. The projective bundle $\mathbb{P}E$ is Fano iff there exists $ε > 0$ such that for every irreducible curve $B \subset X$,

\[
\mu^1_B(E|_B) - \mu_B(E|_B) ≤ (1 - ε)\text{deg}_B(-K_X)/r,
\]

where $\mu_B$ and $\mu^1_B$ are the slopes from Definition 6.2, resp. Definition 6.3.

**Proof.** The invertible sheaf $ω^\vee_{\mathbb{P}E}$ is $π$-relatively ample. By hypothesis, $ω^\vee_X$ is ample. By Lemma 6.4, $ω^\vee_{\mathbb{P}E}$ is ample iff there exists a real number $ε > 0$ such that

\[
\text{deg}_B(g^*ω^\vee_{\mathbb{P}E}) ≥ r\text{deg}_B(g^*π^*ω^\vee_X),
\]

for every finite morphism $g : B \rightarrow \mathbb{P}E$ of a smooth, connected curve to $X$ for which $π ◦ g$ is also finite. Using the universal property of $\mathbb{P}E$, this holds iff for every finite morphism $f : B \rightarrow X$ and every invertible quotient $f^*E^\vee → L^\vee$,

\[
\text{deg}_B(g^*ω^\vee_{\mathbb{P}E}) ≥ r\text{deg}_B(g^*π^*ω^\vee_X),
\]

where $g : B \rightarrow \mathbb{P}E$ is the associated morphism. By Lemma 4.1, $\text{deg}_B(ω^\vee_{\mathbb{P}E})$ equals $r[c_1(L^\vee) + c_1(f^*E) + c_1(f^*T_X)]$, i.e.,

\[
r[c_1(f^*T_X)/r - (\mu_B(L) - \mu_B(f^*E))].
\]

So, finally, $ω^\vee_{\mathbb{P}E}$ is ample iff there exists $ε > 0$ such that for every finite morphism $f : B \rightarrow X$ and every invertible quotient $f^*E^\vee → L^\vee$,

\[
\mu_B(L) - \mu_B(f^*E) ≤ (1 - ε)\text{deg}_B(f^*c_1(T_X))/r.
\]

Taking the supremum over covers of $B$ and invertible quotients of the pullback of $E$, this is,

\[
\mu^1_B(f^*E) - \mu_B(f^*E) ≤ (1 - ε)\text{deg}_B(-f^*K_X)/r.
\]
Since every finite morphism \( f : B \to X \) factors through its image, it suffices to consider only irreducible curves \( B \) in \( X \).

For \( r = 2 \), there is a necessary and sufficient condition for \( \text{ch}_2(T_{\mathbb{P}E}) \) to be nef.

**Proposition 4.3.** Let \( E \) be a vector bundle on \( X \) of rank 2. Denoting by \( \pi : \mathbb{P}E \to X \) the projection, \( \text{ch}_2(T_{\mathbb{P}E}) = \pi^*(\text{ch}_2(T_X) + 1/2(c_1^2 - 4c_2)(E)) \). Therefore \( \text{ch}_2(T_{\mathbb{P}E}) \) is nef iff \( \text{ch}_2(T_X) + 1/2(c_1^2 - 4c_2)(E) \) is nef. If \( \dim(X) > 0 \), \( \text{ch}_2(T_{\mathbb{P}E}) \) is not weakly positive.

**Proof.** By Lemma 4.1, \( \text{ch}_2(T_{\mathbb{P}E}) = \zeta^2 + \pi^*c_1(E)\zeta + \pi^*(\text{ch}_2(T_X) + \text{ch}_2(E)) \). By definition of the Chern classes of \( E \), \( \zeta^2 + \pi^*c_1(E)\zeta + \pi^*c_2(E) \) equals 0. So the class above is \( -\pi^*c_2(E) + \pi^*(\text{ch}_2(T_X) + \text{ch}_2(E)) \). Finally, \( \text{ch}_2(E) - c_2(E) \) equals \( 1/2(c_1^2 - 2c_2)(E) - c_2(E) = 1/2(c_1^2 - 4c_2)(E) \).

Applying Proposition 4.2 and Proposition 4.3 to the vector bundle \( E = L^r \oplus \mathcal{O}_X \) gives Example 2 in Section 3.

Finally, for \( r > 2 \), there is a necessary condition for \( \text{ch}_2(T_{\mathbb{P}E}) \) to be nef.

**Proposition 4.4.** Let \( E \) be a vector bundle of rank \( r > 2 \) on \( X \). If \( \text{ch}_2(T_{\mathbb{P}E}) \) is nef, then the pullback of \( E \) to every smooth, projective, connected curve is semistable. Also, \( \text{ch}_2(T_{\mathbb{P}E}) \) is not weakly positive if \( \dim(X) > 0 \) and if the pullback of \( E \) to some curve is strictly semistable, e.g., if \( X \) contains a rational curve.

**Proof.** If the pullback of \( E \) to some smooth, projective, connected curve is not semistable, then by Corollary 6.11 there exists a smooth, projective, connected curve \( B \), a morphism \( f : B \to X \), and a rank 2 locally free subsheaf \( F \) of \( f^*E \) such that \( f^*E/F \) is locally free and \( \mu_B(F) > \mu_B(E) \). There is an induced morphism \( g : \mathbb{P}F \to \mathbb{P}E \) such that \( \pi \circ g = f \circ \pi \). By Lemma 4.1, \( g^*\text{ch}_2(T_{\mathbb{P}E}) = r\xi^2/2 + \pi^*f^*c_1(E)\xi + \pi^*f^*(\text{ch}_2(T_X) + \text{ch}_2(E)) \), where \( \xi \) equals \( c_1(\mathcal{O}_{\mathbb{P}F}(1)) \). Since \( B \) is a curve, \( f^*(\text{ch}_2(T_X) + \text{ch}_2(E)) \) equals 0. Also, by definition of the Chern classes of \( F \), \( \xi^2 + \pi^*c_1(F)\xi = 0 \). Substituting in,

\[
g^*\text{ch}_2(T_{\mathbb{P}E}) = 1/2\pi^*(2c_1(f^*E) - r\xi^2)\xi.
\]

In particular, \( \deg_{\mathbb{P}F}(g^*\text{ch}_2(T_{\mathbb{P}E})) = 1/2(2\deg_B(c_1(f^*E)) - r\deg_B(F)) \). This equals \( r(\mu_B(f^*E) - \mu_B(F)) \), which is negative by construction. Therefore \( \text{ch}_2(T_{\mathbb{P}E}) \) is not nef.

**Remark 4.5.** A vector bundle on a product of projective spaces whose restriction to every curve is semistable is of the form \( L^{2r} \), where \( L \) is an invertible sheaf, [OSS80 Thm. 3.2.1]. In this case, \( \mathbb{P}E \) is also a product of projective spaces.

**Corollary 4.6.** Let \( X \) be a Fano manifold. For every vector bundle \( E \) on \( X \) of rank \( r > 1 \), \( \text{ch}_2(T_{\mathbb{P}E}) \) is not weakly positive.

5. Blowings up

Let \( X \) be a smooth, connected, projective variety, let \( i : Y \hookrightarrow X \) be the closed immersion of a smooth, connected subvariety of \( X \) of codimension \( c \). Denote by \( \nu : \tilde{X} \to X \) the blowing up of \( X \) along \( Y \). Denote by \( \pi : E \to Y \) the exceptional divisor. Denote by \( j : E \to \tilde{X} \) the obvious inclusion. Then \( E = \mathbb{P}N_{Y/X} \) and \( i^*\mathcal{O}_{\tilde{X}}(E) \) is canonically isomorphic to \( \mathcal{O}_{\mathbb{P}N}(-1) \).
Lemma 5.1. The graded pieces of the Chern character of $\tilde{X}$ are, $c_1(T_{\tilde{X}}) = \nu^*c_1(T_X) - (c - 1)|E|$ and $\text{ch}_2(T_{\tilde{X}}) = \nu^*\text{ch}_2(T_X) + (c + 1)|E|^2/2 - i_*\pi^*c_1(N_{Y/X})$

Proof. Using the short exact sequence,

\[ 0 \rightarrow \nu^*\Omega_X \rightarrow \Omega_{\tilde{X}} \rightarrow j_*\Omega_\pi \rightarrow 0, \]

$\text{ch}(\Omega_{\tilde{X}})$ equals $\nu^*\text{ch}(\Omega_X) + \text{ch}(j_*\Omega_\pi)$. Grothendieck-Riemann-Roch for the morphism $j$ gives,

\[ \text{ch}(R_j a) = j_*(\text{ch}(a))(1 - e^{-|E|})/|E|. \]

Using the Euler sequence for $\Omega_\pi$,

\[ 0 \rightarrow \Omega_\pi \rightarrow \pi^*N_{Y/X}^\vee \otimes \mathcal{O}_{\mathbb{P}N}(-1) \rightarrow \mathcal{O}_E \rightarrow 0, \]

$\text{ch}(\Omega_\pi)$ equals $\pi^*\text{ch}(N_{Y/X}^\vee)i^*(1 + e^{|E|}) - 1$. Putting the pieces together gives the lemma. \hfill \Box

When is $\tilde{X}$ Fano? Denote by $C_1$ the collection of finite morphisms $g : B \rightarrow X$ from a smooth, connected curve to $X$ whose image is not contained in $Y$. Denote by $C_2$ the collection of finite morphisms $g : B \rightarrow Y$ from a smooth, connected curve to $Y$.

Proposition 5.2. Let $h$ be the first Chern class of an ample invertible sheaf on $X$, e.g., $h = c_1(T_X)$ if $X$ is Fano. The blowing up $\tilde{X}$ is Fano iff there exists $\epsilon > 0$ such that,

(i) for every $g : B \rightarrow X$ in $C_1$,

\[ \deg_B(g^{-1}Y) \leq \frac{1}{c - 1}(\deg_B(g^*c_1(T_X)) - \epsilon\deg_B(g^*h)), \]

and

(ii) for every $g : B \rightarrow Y$ in $C_2$,

\[ \mu_B^1(g^*N_{Y/X}) \leq \frac{1}{c - 1}(\deg_B(g^*c_1(T_X)) - \epsilon\deg_B(g^*h)). \]

6. THEOREMS ABOUT VECTOR BUNDLES ON CURVES

There are two theorems in this section. The first goes back to Shou-Wu Zhang, though possibly it is older. The second is a variation of the first.

Definition 6.1. Let $B$ be a smooth, projective curve. A cover of $B$ is a finite, flat morphism $f : C \rightarrow B$ of constant, positive degree. A vector bundle on $B$ is a locally free $\mathcal{O}_B$-module of constant rank.

Definition 6.2. Let $B$ be a smooth, projective curve. For every non-zero vector bundle $E$ on $B$, the slope is,

\[ \mu_B(E) = \deg(E)/\text{rank}(E) = \chi(B, E)/\text{rank}(E) - \chi(B, \mathcal{O}_B). \]

For every cover $f : C \rightarrow B$ and every non-zero vector bundle $E$ on $C$, the $B$-slope is,

\[ \mu_B(f, E) := \deg(E)/(\deg(f)\text{rank}(E)) = \mu_B(f_*E) - \mu_B(f_*\mathcal{O}_C). \]

When there is no chance of confusion, this is denoted simply $\mu_B(E)$.

For every cover $g : C' \rightarrow C$, $f \circ g : C' \rightarrow B$ is a cover and $\mu_B(f \circ g, g^*E)$ equals $\mu_B(f, E)$. \hfill \Box
**Definition 6.3.** Let $B$ be a smooth, projective curve and let $E$ be a vector bundle on $B$ of rank $r > 0$. For every integer $1 \leq k \leq r$, define $\mu_B^k(E)$ to be,

$$
\sup \{ -\mu_B(f, F^\vee) | f : C \to B \text{ a cover, } f^*E^\vee \to F^\vee \text{ a rank } k \text{ quotient} \}
$$

$$
= \sup \{ \mu_B(f, F) | f : C \to B \text{ a cover, } F \subset f^*E \text{ a rank } k \text{ subbundle whose cokernel is locally free} \}.
$$

Let $f : X \to Y$ be a morphism of projective varieties. Denote by $C$ the collection of all irreducible curves in $X$ not contained in a fiber of $f$. Denote by $C_2$ the collection of finite morphisms $g : C \to X$ occurring as the normalization of an irreducible curve in $X$ not contained in a fiber of $f$. Finally, denote by $C_3$ the collection of all finite morphisms from smooth, connected curves to $X$ whose image is not contained in a fiber of $f$.

**Lemma 6.4.** Let $f : X \to Y$ be a morphism of projective varieties and let $L$ be an ample invertible $\mathcal{O}_Y$-module. An $f$-ample invertible $\mathcal{O}_X$-module $M$ is ample iff there exists a real number $\epsilon > 0$ such that for every morphism $g : C \to X$ in $C_1$, resp. $C_2, C_3$, $\deg_C(g^*M) \geq \epsilon \deg_C(g^*f^*L)$.

**Proof.** Because $M$ is $f$-ample and $L$ is ample, there exists an integer $n > 0$ such that $M \otimes f^*L^\otimes n$ is ample. By Kleiman’s criterion, $M$ is ample iff there exists a real number $0 < \delta < 1$ such that for every irreducible curve $C$ in $X$,

$$
\deg_C(M) \geq \delta \deg_C(M \otimes f^*L^\otimes n).
$$

Simplifying, this is equivalent to,

$$
\deg_C(M) \geq \frac{n\delta}{1-\delta} \deg_C(f^*L).
$$

As $M$ is $f$-ample, this holds if $C$ is contained in a fiber of $f$. So $M$ is ample iff the inequality holds for every curve in $C_1$. Setting $\epsilon = n\delta/(1-\delta)$, $\delta = \epsilon/(n+\epsilon)$, gives the lemma.

Since $C_2 \subset C_3$, the condition for $C_3$ implies the condition for $C_2$. Since degrees on a curve can be computed after pulling back to the normalization, the condition for $C_2$ implies the condition for $C_1$. Finally, for every morphism $g : C \to X$ in $C_3$, $g(C)$ is in $C_1$. The inequality for $g(C)$ implies the inequality for $C$. Thus the condition for $C_1$ implies the condition for $C_3$. \qed

**Lemma 6.5.** Let $B$ be a smooth, connected, projective curve. A nonzero vector bundle $E$ on $B$ is ample iff there exists a positive real number $\delta$ such that for every cover $f : C \to B$ and every invertible quotient $f^*E \to L$, $\mu_B(L) \geq \delta$. In other words, $E$ is ample iff $\mu_B^1(L^\vee) < 0$.

**Proof.** Denote by $\pi : \mathbb{P}E^\vee \to B$ the projective bundle associated to $E^\vee$, and denote by $\pi^*E \to \mathcal{O}_{\mathbb{P}E^\vee}(1)$ the tautological invertible quotient. By definition, $E$ is ample iff $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is an ample invertible sheaf. Of course $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is $\pi$-relatively ample. Let $M$ be an invertible $\mathcal{O}_B$-module of degree 1. Then $M$ is ample. By Lemma 6.4, $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is ample iff there exists $\epsilon > 0$ such that for every smooth, connected curve $C$ and every finite morphism $g : C \to \mathbb{P}E^\vee$ such that $\pi \circ g$ is finite, $\deg_C(g^*\mathcal{O}_{\mathbb{P}E^\vee}(1)) \geq \epsilon \deg_C(g^*\pi^*M)$. Of course $\deg_C(g^*\pi^*M) = \deg(\pi \circ g)$. Using
the universal property of $\mathbb{P}E^\vee$, this holds iff for every cover $f : C \to B$ and every invertible quotient $f^*E \to L$,
\[
\deg_C(L) \geq c \deg(f) \iff \mu_B(L) \geq \epsilon.
\]

□

Lemma 6.6. For every ample vector bundle $E$ on $B$, there exists a cover $f : C \to B$, invertible $\mathcal{O}_C$-modules $L_1, \ldots, L_r$, and a morphism of $\mathcal{O}_C$-modules, $\phi : f^*E \to (L_1 \oplus \cdots \oplus L_r)$ such that,

(i) the support of $\text{coker}(\phi)$ is a finite set,
(ii) for every $i = 1, \ldots, r$, the projection $f^*E \to \oplus_{j \neq i} L_j$ is surjective, and
(iii) for every $i = 1, \ldots, r$, $\mu_B(L_i) = \deg_B(E)$.

Proof. Denote $r = \text{rank}(E)$. The claim is that for every $k = 1, \ldots, r$, there exists a cover $f_k : C_k \to B$, invertible $\mathcal{O}_{C_k}$-modules $L_{k,1}, \ldots, L_{k,k}$, and a morphism of $\mathcal{O}_{C_k}$-modules, $\phi_k : f_k^*E \to (L_{k,1} \oplus \cdots \oplus L_{k,k})$ satisfying (ii) and (iii) above and the following variant of (i): for $k < r$, $\phi_k$ is surjective and for $k = r$, the support of $\text{coker}(\phi_k)$ is a finite set. The lemma is the case $k = r$. The claim is proved by induction on $k$.

The base case is $k = 1$. Denote by $\pi : \mathbb{P}E^\vee \to B$ the projective bundle associated to $E^\vee$, and denote by $\pi^*E \to \mathcal{O}_{\mathbb{P}E^\vee}(1)$ the tautological invertible quotient. By hypothesis, $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is ample. By Bertini’s theorem, for $d_1, \ldots, d_{r-1} \gg 0$, there exist effective Cartier divisors $D_1, \ldots, D_{r-1}$ with $D_i \in |\mathcal{O}_{\mathbb{P}E^\vee}(d_i)|$ such that the intersection $C_1 = D_1 \cap \cdots \cap D_{r-1}$ is a smooth, connected curve, cf. [Jou83]. Denote by $f_1 : C_1 \to B$ the restriction of $\pi$. Denote by $\phi_1 : f_1^*E \to L_{1,1}$ the restriction of $\pi^*E \to \mathcal{O}_{\mathbb{P}E^\vee}(1)$. This satisfies (i) because $\pi^*E \to \mathcal{O}_{\mathbb{P}E^\vee}(1)$ is surjective. It satisfies (ii) trivially. Finally, $\deg(f)$ equals $d_1 \times \cdots \times d_{r-1}$, and $\deg_C(L_{1,1})$ equals $d_1 \times \cdots \times d_{r-1} \times |C_1(\mathcal{O}_{\mathbb{P}E^\vee}(1))|^r$, i.e., $d_1 \times \cdots \times d_{r-1} \times \deg_B(E)$. Therefore $\mu_B(L_{1,1}) = \deg_B(E)$, i.e., this satisfies (iii).

By way of induction, assume the result is known for $k < r$, and consider the case $k + 1$. Since $\phi_k$ is surjective, there is an induced closed immersion $\mathbb{P}(L_{k,1} \oplus \cdots \oplus L_{k,k})^\vee \hookrightarrow \mathbb{P}(f_k^*E)^\vee$. The image is irreducible and has dimension $r - k \geq 1$. For every $i = 1, \ldots, k$, the image of $\mathbb{P}(\oplus_{j \neq i} L_{k,j})^\vee$ is irreducible and has dimension $r - k + 1 \geq 2$. Associated to the finite morphism $f_k$, there is a finite morphism $\mathbb{P}(f_k^*E)^\vee \to \mathbb{P}E^\vee$. The pullback of an ample invertible sheaf by a finite morphism is ample; hence $\mathcal{O}_{\mathbb{P}(f_k^*E)^\vee}(1)$ is ample. By Bertini’s theorem, for $d_1, \ldots, d_{r-1} \gg 0$, there exist effective Cartier divisors $D_1, \ldots, D_{r-1}$ with $D_i \in |\mathcal{O}_{\mathbb{P}(f_k^*E)^\vee}(d_i)|$ such that the intersection $C_{k+1} = D_1 \cap \cdots \cap D_{r-1}$ is a smooth, connected curve, disjoint from $\mathbb{P}(\oplus_{j \neq i} L_{k,j})^\vee$ for every $i = 1, \ldots, k$, and either disjoint from $\mathbb{P}(\oplus_{i} L_i)^\vee$ if $k < r - 1$, or else intersecting $\mathbb{P}(\oplus_{i} L_i)^\vee$ in finitely many points if $k = r - 1$. Define $g_{k+1} : C_{k+1} \to C_k$ to be the restriction of the projection. Define $f_{k+1} = f_k \circ g_{k+1}$, define $L_{k+1,i} = g_{k+1}^*L_{k,i}$ for $i = 1, \ldots, k$, and define $L_{k+1,k+1}$ to be the restriction of $\mathcal{O}_{\mathbb{P}(f_k^*E)^\vee}(1)$. Define $\phi_{k+1}$ to be the obvious morphism.

The cokernel of $\phi_{k+1}$ is supported on the intersection of $C_{k+1}$ with $\mathbb{P}(L_{k,1} \oplus \cdots \oplus L_{k,k})^\vee$. By construction, this is empty if $k < r - 1$, and is a finite set if $k = r - 1$. Thus $\phi_{k+1}$ satisfies (i). By the induction hypothesis, $f_{k+1}^*E \to (L_{k+1,1} \oplus \cdots \oplus L_{k+1,k})$, which is the pullback under $g_{k+1}$ of $\phi_k$, is surjective. For $i = 1, \ldots, k$, the cokernel of $f_{k+1}^*E \to \oplus_{j \neq i} L_{k+1,j}$ is supported on the intersection of $C_{k+1}$ with the
image of $P(\oplus_{j \neq i} L_{k,j})$. By construction, this is empty, i.e., $f^*_{k+1} E \rightarrow \oplus_{j \neq i} L_{k+1,j}$ is surjective. Thus $\phi_{k+1}$ satisfies (ii). Finally, $\phi_{k+1}$ satisfies (iii) by the same argument as in the base case. The claim is proved by induction on $k$. □

**Theorem 6.7.** For every non-zero vector bundle $E$ on $B$, for every $\epsilon > 0$, there exists a cover $f : C \rightarrow B$ and a invertible quotient $f^* E \rightarrow L$ such that $\mu_B(L) < \mu_B(E) + \epsilon$. In other words, $\mu_B(E^\vee) \geq \mu_B(E^\vee) - \epsilon$.

**Proof.** Denote $r = \text{rank}(E)$. If $r = 1$, set $f = \text{Id}_B$ and $L = E$. Then $L$ is an invertible quotient of $f^* E$, and $\mu_B(L)$ equals $\mu_B(E)$ which is less than $\mu_B(E) + \epsilon$. Therefore assume $r > 1$.

Certainly an effective version of the following argument can be given, but a simpler argument is by contradiction.

**Hypothesis 6.8.** For every cover $f : C \rightarrow B$ and every invertible quotient $f^* E \rightarrow L$, $\mu_B(L)$ is $\geq \mu_B(E) + \epsilon$, i.e., $\mu^1_B(E^\vee) < \mu_B(E^\vee) - \epsilon$.

By way of contradiction, assume Hypothesis 6.8. Let $f : C \rightarrow B$ be a connected, smooth cover of degree $d$. For every $a/d \in \frac{1}{\mathbb{Z}}$, there exists an invertible sheaf $M$ on $C$ of degree $a$, and thus $\mu_B(M) = a/d$. In particular, for $d$ sufficiently large, there exists an invertible quotient $M$ such that $0 < \mu_B(E) - \mu_B(M) < \epsilon/(r-1)$. Denote $\delta = \mu_B(E) - \mu_B(M)$. Denote $F = f^* E \otimes M'$. Then $\mu_B(F)$ equals $\delta$, and $0 < \delta < \epsilon/(r-1)$.

Let $g : C' \rightarrow C$ be any cover and let $g^* F \rightarrow N$ be any invertible quotient. Then $f \circ g : C' \rightarrow B$ is a cover and $(f \circ g)^* E = g^* F \otimes g^* M \rightarrow N \otimes g^* M$ is an invertible quotient. By Hypothesis 6.8

$$\mu_C(N) = \deg(f) \mu_B(N) = \deg(f)(\mu_B(N \otimes g^* M) - \mu_B(M))$$

$$\geq \deg(f)((\mu_B(E) + \epsilon) - \mu_B(M)) > \deg(f)\epsilon.$$ 

By Lemma 6.5 $F$ is an ample vector bundle on $C$. By Lemma 6.6 there exists a cover $g : C' \rightarrow C$ and an invertible quotient $g^* F \rightarrow P$ such that $\mu_B(P) = r \mu_B(F) = r\delta$. Therefore $L := g^* M \otimes P$ is an invertible quotient of $g^* f^* E$ and,

$$\mu_B(L) = \mu_B(g^* M \otimes P) = \mu_B(M) + r\delta = \mu_B(E) + (r-1)\delta.$$ 

By hypothesis, $(r-1)\delta < \epsilon$. So $\mu_B(L) < \mu_B(E) + \epsilon$, contradicting Hypothesis 6.8. The proposition is proved by contradiction. □

**Corollary 6.9.** For every non-zero vector bundle $E$ on $B$, for every $\epsilon > 0$, there exists a cover $f : C \rightarrow B$ and a sequence of vector bundle quotients,

$$f^* E = E^r \rightarrow E^{r-1} \rightarrow \cdots \rightarrow E^1,$$

such that each $E^k$ is a vector bundle of rank $k$ and $\mu_B(E^k) < \mu_B(E) + \epsilon$.

**Proof.** The proof is by induction on the rank $r$ of $E$. If $\text{rank}(E) = 1$, defining $f = \text{Id}_B$ and $E^1 = E$, the result follows. Thus, assume $r > 1$ and the result is known for smaller values of $r$. By Theorem 6.7 there exists a cover $g : B' \rightarrow B$ and a rank 1 quotient $g^* E \rightarrow L$ such that $\mu_B(L) < \mu_B(E) + \epsilon$. Denote by $K$ the kernel of $g^* E \rightarrow L$. Then $\text{rank}(K) = r - 1$ and $\mu_B(K) = (r \mu_B(E) - \mu_B(L))/(r-1)$. By the induction hypothesis, there exists a cover $h : C \rightarrow B'$ and a sequence of vector bundle quotients,

$$h^* K = K^{r-1} \rightarrow \cdots \rightarrow K^1,$$
such that each $K^k$ is a vector bundle of rank $k$, and $\mu_B(K^k) \leq \mu_B(K) + \deg(g) \epsilon$.
Of course $\mu_B(F) = \mu_B(F)/\deg(g)$ for every $F$. Thus $\mu_B(K^k) \leq \mu_B(K) + \epsilon$.

Define $f = h \circ g$, define $E^1 = h^*L$, and for every $k = 2, \ldots, r$, define $f^*E \to E^k$ to
be the unique quotient whose kernel is contained in $h^*K$ and such that $h^*K \to E^k$
has image $K^{k-1}$. Then $\mu_B(E^1) = \mu_B(L) \leq \mu_B(E) + \epsilon$, and for $k = 2, \ldots, r$,
\[
\mu_B(E^k) = 1/k(\mu_B(L) + (k-1)\mu_B(K^{k-1})) < 1/k(\mu_B(L) + (k-1)\mu_B(K) + (k-1)\epsilon) = 
\]
\[
\frac{r(k-1)}{(r-1)k} \mu_B(E) + \frac{r-k}{(r-1)k} \mu_B(L) + \frac{(r-1)(k-1)}{(r-1)k} \epsilon < \mu_B(E) + \frac{r-k}{(r-1)k} \epsilon < \mu_B(E) + \epsilon.
\]

For semistable bundles in characteristic zero, there is a more precise result.

**Theorem 6.10** (Zhang). Let $B$ be a smooth, projective curve over an algebraically
closed field of characteristic 0. Let $E$ be a semistable vector bundle on $B$. Let $\epsilon$ be a
positive real number. There exists a cover $f : C \to B$, invertible sheaves $L_1, \ldots, L_r$
on $C$, and a morphism of $\mathcal{O}_C$-modules, $\phi : f^*E \to (L_1 \oplus \cdots \oplus L_r)$ such that,
\begin{itemize}
  \item[(i)] the support of $\text{coker}(\phi)$ is a finite set,
  \item[(ii)] for every $i = 1, \ldots, r$, the projection $f^*E \to \oplus_{j \neq i} L_j$ is surjective,
  \item[(iii)] for every $i = 1, \ldots, r$, $\mu_B(L_i) \leq \mu_B(E) + \epsilon$.
\end{itemize}

**Proof.** Denote $r = \text{rank}(E)$. If $r$ equals 1, the theorem is trivial. Thus assume
$r > 1$. As in the proof of Theorem 6.7, there exists a cover $g : C' \to B$ and an
invertible sheaf $M$ on $C'$ such that $0 < \mu_B(E) - \mu_B(M) < \epsilon/(r-1)$. Denote
$\delta = \mu_B(E) - \mu_B(M)$ and denote $F = g^*E \otimes M^\vee$. Then $\mu_B(F)$ equals $\delta$, and
$0 < \delta < \epsilon/(r-1)$.

Let $h : C \to C'$ be any cover and let $h^*F \to N$ be an invertible quotient.
The composition $g \circ h : C \to B$ is a cover. By Kempf’s theorem, [Kem92], which
ultimately relies on the theorem that every stable vector bundle admits a Hermite-
Einstein metric, $(g \circ h)^*E$ is semistable. (Note, there are counterexamples in positive
characteristic.) Therefore $h^*F$ is semistable. So $\mu_C(L) \geq \mu_C(h^*F)$, i.e., $\mu_C(L) \geq \mu_{C'}(F) = \delta$.
Thus by Lemma 6.6, $F$ is an ample vector bundle on $C'$. Thus by
Lemma 6.6, there exists a cover $h : C \to C'$, invertible $\mathcal{O}_C$-modules $N_1, \ldots, N_r$, and
a morphism of $\mathcal{O}_C$-modules $\psi : h^*F \to (N_1 \oplus \cdots \oplus N_r)$ satisfying (i), (ii) and
(iii) of Lemma 6.6. Define $f = g \circ h$, $L_i = N_i \otimes h^*M$ and $\phi$ is the twist of $\psi$ by
$\text{Id}_{h^*M}$. Then $\phi$ satisfies (i) and (ii). And for every $i = 1, \ldots, r$,
\[
\mu_B(L_i) = \mu_B(N_i) + \mu_B(M) = \mu_{C'}(N_i)/\deg(g) + \mu_B(E) - \delta = 
\mu_B(E) + r\delta/\deg(g) - \delta \leq \mu_B(E) + (r - 1)\delta/\deg(g) < \mu_B(E) + \epsilon.
\]

Of course, $\mu_B^r(E)$ equals $\mu_B(E)$. The other values are more interesting.

**Corollary 6.11.** The slopes $\mu_B^k(E)$ satisfy $\mu_B^1(E) \leq \mu_B^2(E) \leq \cdots \leq \mu_B^r(E) =
\mu_B(E)$. For each $1 \leq k < r$, $\mu_B^k(E) = \mu_B(E)$ iff $f^*E$ is semistable for every cover
$f : C \to B$. 

9
Proof. By Corollary [6.9] for every $\epsilon > 0$, there exists a cover $f : C \to B$ and a rank $k$ quotient $f^*E \to E^k$ such that $\mu_B(E^k) < \mu_B(E) + \epsilon$. Thus $\mu^k_B(E) \geq \mu_B(E)$. Applying the same reasoning to rank $k - 1$ quotients of rank $k$ quotients of $f^*E$, $\mu^{k-1}_B(E) \geq \mu^k_B(E)$.

If $f^*E$ is semistable for every cover $f : C \to B$, then every vector bundle quotient of $f^*E$ has slope $\geq \mu_C(f^*E)$, and thus has $B$-slope $\geq \mu_B(f^*E)$. Therefore $\mu^k_B(E) \leq \mu_B(E)$, i.e., $\mu^k_B(E) = \mu_B(E)$.

Conversely, suppose there is a cover $f : C \to B$ such that $f^*E$ is not semistable. Then there exists a vector bundle quotient $f^*E \to F$ such that $\mu_B(F) < \mu_B(E)$. Denote the rank by $l$. Suppose first that $l \geq k$, and define $\epsilon = \deg(f)(\mu_B(E) - \mu_B(F))$. Then by Corollary [6.9] there exists a cover $g : C' \to C$ and a rank $k$ quotient $g^*F \to G$ such that $\mu_C(G) < \mu_C(F) + \epsilon$. Therefore $g^*f^*E \to g^*F \to G$ is a rank $k$ quotient of $g^*f^*E$ and $\mu_B(G) < \mu_B(F) + (\mu_B(E) - \mu_B(F)) = \mu_B(E)$. Therefore $\mu^k_B(E) > \mu_B(E)$.

Next suppose that $l < k$. Denote by $K$ the kernel of $f^*E \to F$. Then $r\mu_B(E) = l\mu_B(F) + (r-l)\mu_B(K)$. Define,

$$\epsilon = \frac{(r-k)\deg(f)(\mu_B(E) - \mu_B(F))}{(r-l)(k-l)}.$$

By Corollary [6.9] there exists a cover $g : C' \to C$ and a rank $k-l$ quotient $g^*K \to G'$ such that $\mu_C(G') < \mu_C(K) + \epsilon$. Therefore $\mu_B(G') < \mu_B(K) + \epsilon/\deg(f)$. Define $g^*f^*E \to G$ to be the unique vector bundle whose kernel is contained in $g^*K$ and such that the image of $g^*K \to G$ equals $G'$. Then,

$$k\mu_B(G) = l\mu_B(F) + (k-l)\mu_B(G') < l\mu_B(F) + (k-l)\mu_B(K) + (k-l)\epsilon/\deg(f) = l\mu_B(F) + \frac{k-l}{\deg(f)}(r\mu_B(E) - l\mu_B(F)) + \frac{k-l}{\deg(f)}\epsilon = k\mu_B(E) - \frac{(r-k)l}{r-l}(\mu_B(E) - \mu_B(F)) + \frac{(r-k)l}{r-l}(\mu_B(E) - \mu_B(F)) = k\mu_B(E).$$

Thus $\mu_B(G) < \mu_B(E)$, and therefore $\mu^k_B(E) > \mu_B(E)$.

References


