MAT 544 Problem Set 6

Problems.

Problem 1 Let $(W, \| \bullet \|_W)$ be a Banach space (the most important case is $W = \mathbb{R}^n$). Let I be a bounded, open interval in $\mathbb{R}$ with closure $\overline{I}$, and let

$$F : \overline{I} \to L(W,W), \quad t \mapsto (F_t : W \to W)$$

be a bounded, continuous function. For every $t_0 \in I$, consider the initial value problem

$$\frac{dA}{dt}(t) = F_t \circ A(t), \quad A(t_0) = \mathrm{Id}_W$$

where A is a continuously differentiable function from some open neighborhood of t_0 in I to $L(W,W)$. As proved in lecture, there is a unique solution $A_{t_0}(t)$. Denote this by $A(t,t_0) = A_{t_0}(t)$, called a *Green's function*.

(a) For fixed $t_0, t_1 \in I$, check that both of the following functions

$$\tilde{A}(t) = A(t,t_0), \quad \widehat{A}(t) = A(t,t_1) \circ A(t_1,t_0).$$

solve the initial value problem

$$\frac{dA}{dt}(t) = F_t \circ A(t), \quad A(t_1) = A(t_1,t_0),$$

and thus are equal by uniqueness. In particular, conclude that $A(t_1,t_0)$ and $A(t_0,t_1)$ are inverse (bounded) linear operators.

(b) Let U be an element in $L(W,W)$ which has an inverse U^{-1} in $L(W,W)$. Check that $B(t,t_0) = U \circ A(t,t_0) \circ U^{-1}$ is a solution of the initial value problem

$$\frac{dB}{dt}(t) = (U \circ F_t \circ U^{-1}) \circ B(t), \quad B(t_0) = \mathrm{Id}_W.$$

(c) Now let $\vec{g} : \overline{I} \to W$ be a continuous function and consider the initial value problem

$$\frac{d\vec{x}}{dt} = F_t(\vec{x}(t)) + \vec{g}(t), \quad \vec{x}(t_0) = 0.$$

where $\vec{x}(t)$ is a continuous map $\overline{I} \to W$ which is continuously differentiable on I. This equation is called an *inhomogeneous* linear ODE. Check that the following formula gives one solution (which is unique)

$$\vec{x}(t) = A(t, t_0) \circ \int_{t_0}^{t} A(s, t_0)^{-1} \circ \vec{g}(s)ds = A(t, t_0) \circ \int_{t_0}^{t} A(t_0, s) \circ \vec{g}(s)ds =$$

$$\int_{t_0}^{t} A(t, s) \circ \vec{g}(s)ds.$$

Problem 2 Let $(W, \| \bullet \|_W)$ be a Banach space, e.g., $W = \mathbb{R}^n$. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an absolutely convergent series with positive radius of convergence R, i.e.,

$$\sum_{n=0}^{\infty} |c_n| R^n < \infty.$$

(a) For every A in the closed ball $B_{\leq R}(0)$ in $L(W, W)$, prove that the sequence of partial sums

$$\sum_{n=0}^{N} c_n A^n$$

converges to a limit. Call this limit $f_{L(W,W)}(A)$.

(b) Prove further that the associated map

$$f_{L(W,W)} : B_{\leq R}(0) \to L(W, W), \quad A \mapsto f_{L(W,W)}(A)$$

is continuous. (Consider this as a uniform limit of polynomial functions.)

(c) For every U in $L(W, W)$ with inverse U^{-1} also in $L(W, W)$, prove that $f_{L(W,W)}(UAU^{-1})$ equals $Uf_{L(W,W)}(A)U^{-1}$.

Problem 3 Read about how to find the Jordan normal form, particularly for 2×2, 3×3 and 4×4 matrices. One short review is available from the lecture notes at the following URL: `http://ocw.mit.edu/courses/mathematics/18-034-honors-differential-equations-spring-2004/`

Problem 4 Consider the following second order differential equation with initial values.

$$\frac{d^2}{dt^2}x(t) - 6\frac{d}{dt}x(t) + 9x(t) = 0, \quad x(t_0) = b_0, \ x'(t_0) = b_1$$

(a) Find a 2×2 matrix A such that for every choice of b_0,b_1, the unique solution of the initial value problem

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t), \quad \vec{x}(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix}, \quad \vec{x}(t_0) = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

gives a solution of the second order differential equation by $x(t) = x_0(t)$.

(b) Find an invertible 2×2 matrix U such that $AU = U\tilde{A}$ where $\tilde{A}$ is a 2×2 matrix of the form

$$\tilde{A} = \tilde{S} + \tilde{N}$$

where $\tilde{S}$ is a diagonal matrix, $\tilde{N}$ is a strictly upper triangular matrix (with zeroes on the diagonal), and $\tilde{S}\tilde{N} = \tilde{N}\tilde{S}$.

(c) Compute $\exp(\tilde{S}(t-t_0))$, $\exp(\tilde{N}(t-t_0))$ and

$$\exp(\tilde{A}(t-t_0)) = \exp(\tilde{S}(t-t_0))\exp(\tilde{N}(t-t_0)).$$

The compute $\exp(A(t-t_0))U = U\exp(\tilde{A}(t-t_0))$.

(d) Let c_0, c_1 be real numbers such that

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = U \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}.$$

Find the solution of the initial value problem with respect to c_0 and c_1, and use this to find the solution of the original second order differential equation with respect to c_0 and c_1.

Problem 5 For the same differential equation as in **Problem 4**, use **Problem 1(c)** to solve the following inhomogeneous equation

$$\frac{d^2}{dt^2}x(t) - 6\frac{d}{dt}x(t) + 9x(t) = e^{3t}, \quad x(t_0) = x'(t_0) = 0.$$

There are other methods to solve this problem (such as the method of undetermined coefficients). You may use these methods to check your work, but please write up the solution using the Green's function as in **Problem 1(c)**.