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MAT 544 Problem Set 2

Problems.

Problem 1 A metric space is *separable* if it contains a dense subset which is finite or countably infinite. Prove that every totally bounded metric space X is separable. Also give an example of a bounded, separable metric space which is not totally bounded.

Problem 2 Recall from lecture that for a sequence $(x_n)_{n\in\mathbb{Z}_{>0}}$ with no convergent subsequence, the infinite set $S = \{x_n | n \in \mathbb{Z}_{>0}\}$ has a covering by balls $(B_{r(x)}(x))_{x\in S}$ such that every $B_{r(x)} \cap S = \{x\}$.

(a) Prove that the covering $(B_{r(x)/2}(x))_{x\in S}$ has the same property as above, and further for every pair of distinct points x, y of S, the balls $B_{r(x)/x}(x)$ and $B_{r(y)/2}(y)$ are disjoint in X.

(b) Define a continuous function $f : X \to \mathbb{R}$ which equals 0 on the complement of all of the balls $B_{r(x)/2}(x)$ and which is unbounded.

(c) Conclude that a metric space X is sequentially compact if and only if every continuous function $f : X \to \mathbb{R}$ is bounded.

Problem 3 Theorem 4.5.1 on p. 211 of the textbook proves that a continuous function $f : X \to Y$ is uniformly continuous if X is compact. Give a second proof by applying Lemma 4.5.3 to the inverse images of ϵ-balls of Y.

Problem 4 For a metric space X, recall the definition of the associated metric space,

$$C^\infty(X, \mathbb{R}) := \{f : X \to \mathbb{R} | f \text{ continuous and bounded }\},$$

$$\|f\| := \text{ least upper bound } (|f(x)|)_{x\in X}, \quad d(f, g) := \|f - g\|.$$

Prove that for bounded, continuous functions $(f_n : X \to \mathbb{R})_{n\in\mathbb{Z}_{>0}}$ and $f : X \to \mathbb{R}$, the sequence (f_n) converges to f in this metric space if and only if the sequence of functions converges uniformly to f. This justifies the names "uniform norm" and "uniform metric".

Problem 5 Let X be a metric space and let $(f_n : X \to \mathbb{R})_{n\in\mathbb{Z}_{>0}}$ be an **equicontinuous** sequence of bounded, continuous functions which converge pointwise to the zero function. Recall this means that for every $\epsilon > 0$ and for every $x \in X$, there exists $\delta = \delta(x, \epsilon) > 0$ such that $d_X(y, x) < \delta$ implies for all $n \in \mathbb{Z}_{>0}$ that $|f_n(y) - f_n(x)| < \epsilon$.

(a) If X is compact, prove that the sequence converges uniformly.

Hint. Show that for every $\epsilon > 0$ and for every $x \in X$, there exists a $\delta = \delta(x, \epsilon) > 0$ and a positive integer $N = N(x, \epsilon)$ such that $d_X(y, x) < \delta$ implies for all $n \geq N$ that $|f_n(y)| < \epsilon$. Then apply Theorem 4.5.3 to this covering of X by δ-balls.

(b) Let $(x_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in X with no convergent subsequence. With notation as above, find an equicontinuous sequence of bounded, continuous functions $(f_x : X \to \mathbb{R})_{x \in S}$ such that f_x is zero on the complement of $B_{r(x)/2}(x)$, and thus the sequence $(f_x)_{x \in S}$ automatically converges pointwise to the zero function, and yet the functions do not converge uniformly to the zero function.