MAT 544 Problem Set 1 Solutions

Problems.

Problem 1. Let (X, d_X) and (Y, d_Y) be metric spaces. Define a function

$$d_{X\times Y} : (X \times Y) \times (X \times Y) \to \mathbb{R}$$

by $d_{X\times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$.

(a) Prove that this is a metric space.

(b) Denote by $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ the two projections. Prove that these functions are continuous, in fact even Lipschitz (hence uniformly continuous).

(c) If X and Y are each complete metric spaces, prove that also $X \times Y$ (with the above metric) is a complete metric space.

(d) Let (Z, d_Z) be a metric space and let $(f_X : Z \to X, f_Y : Z \to Y)$ be a pair of continuous functions. Prove that there exists a unique continuous function $f : Z \to X \times Y$ such that f_X equals $f \circ \pi_X$ and f_Y equals $f \circ \pi_Y$.

(e) Give an example of metric spaces X and Y and a metric d' on $X \times Y$ which is different from $d_{X\times Y}$ and which still satisfies the property from part (d). Conclude that this property does not characterize $d_{X\times Y}$ (however, it does characterize the *topology* induced by this metric).

Problem 2. Let (X, d_X) be a metric space. Give $X \times X$ the metric from **Problem 1**. Prove that the function $d_X : X \times X \to \mathbb{R}$ is Lipschitz for this metric.

Problem 3. For a metric space (X, d_X), an element x of X, and a real number $r \geq 0$, the *closed ball* is sometimes defined to be

$$B_{\leq r}(x) := \{x' \in X | d_X(x, x') \leq r\},$$

i.e., one uses "less than or equal to" rather than "less than" as in the definition of the open unit ball.

(a) For $r > 0$, prove that the closure of the open unit ball $B_r(x)$ is contained in the closed unit ball $B_{\leq r}(x)$.

(b) Give an example of a subset S of $\mathbb{R}^2$ (with the usual Euclidean metric), an element x of S and a real number $r > 0$, such that for the subspace metric on S, the closure of $B_r(x)$ in S is strictly contained in $B_{\leq r}(x)$.

Problem 4. Define sequence of integers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ by the recursive relation $a_0 = 2$, $b_0 = 1$ and for every $n \geq 0$,

$$a_{n+1} = a_n^2 + 2b_n^2, \quad b_{n+1} = 2a_nb_n.$$

Prove that every $b_n \neq 0$ so that $(a_n/b_n)_{n\geq 0}$ is a well-defined sequence in $\mathbb{Q}$, prove that this sequence is Cauchy, and prove that this sequence does not have a limit. Thus the Archimedean ordered field $\mathbb{Q}$ is not complete.

Problem 5. This is Exercise 4.3.14 of Loomis-Sternberg. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. Define the metric $d_{X\times Y}$ on $X \times Y$ as in **Problem 1**. Let $g : X \times Y \to Z$ be a function such that for every $x \in X$ the function

$$g_x : Y \to Z, \quad y \mapsto g(x, y)$$

is continuous and for every $y \in Y$ the function

$$g_y : X \to Z, \quad x \mapsto g(x, y)$$

is continuous *uniformly over* y, i.e., for every x_0 in X and for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x_0, x) < \delta \Rightarrow d_Z(g(x_0, y), g(x, y)) < \epsilon$$

for all values $y \in Y$ simultaneously. Prove that g is continuous.

Solutions to Problems.

Solution to (1)

Solution to (1a) We must verify the three axioms: positive definiteness, symmetry and the triangle inequality. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ be elements. Since X is positive, $d_X(x_1, x_2) \geq 0$. Since Y is positive, $d_Y(y_1, y_2) \geq 0$. Thus $d_{X\times Y}((x_1, y_1), (x_2, y_2))$ is the sum of two nonnegative real numbers, $d_X(x_1, x_2) + d_Y(y_1, y_2)$. Therefore the sum is nonnegative. Moreover, the sum of two nonnegative numbers equals 0 if and only if both summands equal 0. Therefore $d_{X\times Y}((x_1, y_1), (x_2, y_2))$ equals 0 if and only if both $d_X(x_1, x_2) = 0$ and $d_Y(y_1, y_2) = 0$. Since d_X and d_Y are positive definite, this holds if and only if $x_1 = x_2$ and $y_1 = y_2$, i.e., if and only if $(x_1, y_1) = (x_2, y_2)$. Therefore $d_{X\times Y}$ is positive definite.

Next, since d_X and d_Y are symmetric, $d_X(x_2, x_1) = d_X(x_1, x_2)$ and $d_Y(y_2, y_1) = d_Y(y_1, y_2)$. Therefore we also have for the sum,

$$d_X(x_2, x_1) + d_Y(y_2, y_1) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

In other words, $d_{X\times Y}((x_2, y_2), (x_1, y_1)) = d_{X\times Y}((x_1, y_1), (x_2, y_2))$. Therefore $d_{X\times Y}$ is symmetric.

Finally, let $(x_3, y_3) \in X \times Y$ be a third element. Since d_X and d_Y satisfy the triangle inequality, we have both

$$d_X(x_1, x_3) \leq d_X(x_1, x_2) + d_X(x_2, x_3), \quad d_Y(y_1, y_3) \leq d_Y(y_1, y_2) + d_Y(y_2, y_3).$$

Therefore, taking the sum gives

$$d_X(x_1, x_3) + d_Y(y_1, y_3) \leq (d_X(x_1, x_2) + d_X(x_2, x_3)) + (d_Y(y_1, y_3) \leq d_Y(y_1, y_2) + d_Y(y_2, y_3)) = (d_X(x_1, x_2) + d_Y(y_1$$

where we have used associativity of addition. In other words,

$$d_{X\times Y}((x_1, y_1), (x_3, y_3)) \leq d_{X\times Y}((x_1, y_1), (x_2, y_2)) + d_{X\times Y}((x_2, y_2), (x_3, y_3)).$$

Therefore $d_{X\times Y}$ also satisfies the triangle inequality. Since it is positive definite, symmetric and satisfies the triangle inequality, $d_{X\times Y}$ is a metric function.

Solution to (1b) As above, let (x_1, y_1) and (x_2, y_2) be elements in $X \times Y$. Then $\pi_X(x_i, y_i)$ equals x_i for $i = 1, 2$. Therefore, $d_X(\pi_X(x_1, y_1), \pi_X(x_2, y_2))$ equals $d_X(x_1, x_2)$. Since d_Y is positive definite, $0 \leq d_Y(y_1, y_2)$ so that

$$d_X(x_1, x_2) \leq d_X(x_1, x_2) + d_Y(y_1, y_2) = d_{X\times Y}((x_1, y_1), (x_2, y_2)).$$

Putting the pieces together,

$$d_X(\pi_X(x_1, y_1), \pi_X(x_2, y_2)) \leq d_{X\times Y}((x_1, y_1), (x_2, y_2)).$$

Therefore π_X is Lipschitz with Lipschitz constant 1. By an exactly similar argument, also π_Y is Lipschitz with Lipschitz constant 1.

Solution to (1c) Let $((x_n, y_n))_{n\in\mathbb{Z}_{>0}}$ be a Cauchy sequence in $X\times Y$. Since π_X and π_Y are uniformly continuous by **(b)**, both $(x_n)_{n\in\mathbb{Z}_{>0}}$ is Cauchy in X and $(y_n)_{n\in\mathbb{Z}_{>0}}$ is Cauchy in Y. Since X and Y are each complete, both sequences converge in their respective metric spaces. Call the limits x_∞, resp. y_∞. Then for every real $\epsilon > 0$, there exist integers $N_X > 0$, respectively $N_Y > 0$, such that for every integer $n \geq N_X$, resp. $n \geq N_Y$, we have $d_X(x_n, x_\infty) < \epsilon/2$, resp. $d_Y(y_n, y_\infty) < \epsilon/2$. Setting $N = \max(N_X, N_Y)$, then for every integer $n \geq N$, we have

$$d_{X\times Y}((x_n, y_n), (x_\infty, y_\infty)) = d_X(x_n, x_\infty) + d_Y(y_n, y_\infty) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore the sequence $((x_n, y_n))_{n\in\mathbb{Z}_{>0}}$ converges to (x_∞, y_∞) in $X\times Y$. Since every Cauchy sequence converges, $X \times Y$ is complete.

Solution to (1d) First of all, there exists a unique function $f : Z \to X \times Y$ such that both $\pi_X \circ f = f_X$ and $\pi_Y \circ f = f_Y$, namely $f(z) = (f_X(z), f_Y(z))$. So the problem is to prove that f is continuous. Let $\epsilon > 0$ be a real number, and let z be an element in Z. Since f_X is continuous there exists a real number $\delta_X > 0$ such that $d_Z(z, z') < \delta_X$ implies $d_X(f_X(z), f_X(z')) < \epsilon/2$. Similarly, since f_Y is continuous there exists a real number $\delta_Y > 0$ such that $d_Z(z, z') < \delta_Y$ implies

$d_Y(f_Y(z), f_Y(z')) < \epsilon/2$. Denote $\delta = \min(\delta_X, \delta_Y)$, which is still a positive real number. Then $d_Z(z, z') < \delta$ implies that

$$d_{X\times Y}(f(z), f(z')) = d_X(f_X(z), f_X(z')) + d_Y(f_Y(z), f_Y(z')) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus also f is continuous. In fact this argument also clearly implies that if f_X and f_Y are both uniformly continuous (so that δ_X and δ_Y depend only on ϵ, not on z), then so is f. And if f_X, resp. f_Y, is K_X-Lipschitz, resp. K_Y-Lipschitz, then also f is K-Lipschitz for $K = K_X + K_Y$.

Solution to (1e) For any positive real number c, the function $(cd_{X\times Y})((x_1, y_1), (x_2, y_2)) = c \cdot d_{X\times Y}((x_1, y_1), (x_2, y_2))$ is another metric which satisfies **(d)**. For a less trivial example, for $X = Y = \mathbb{R}$ with the usual metric, the Euclidean metric on $X \times Y = \mathbb{R}^2$ satisfies **(d)**.

Solution to (2) Let (x_1, y_1) and (x_2, y_2) be elements in $X \times X$. By the triangle inequality, we have

$$d_X(x_2, y_2) \le d_X(x_2, x_1) + d_X(x_1, y_1) + d_X(y_1, y_2).$$

Using symmetry of d_X, this says that

$$d_X(x_2, y_2) - d_X(x_1, y_1) \le d_X(x_1, x_2) + d_X(y_1, y_2).$$

By permuting x_1 with x_2 and y_1 with y_2, the same argument also implies that

$$d_X(x_1, y_1) - d_X(x_2, y_2) \le d_X(x_2, x_1) + d_X(y_2, y_1) = d_X(x_1, x_2) + d_X(y_1, y_2).$$

Therefore we have,

$$|d_X(x_2, y_2) - d_X(x_1, y_1)| \le d_X(x_1, x_2) + d_X(y_1, y_2) = d_{X\times X}((x_1, y_1), (x_2, y_2)).$$

Therefore d_X is Lipschitz with Lipschitz constant 1.

Solution to (3) Define the function $d_{X,x} : X \to \mathbb{R}$ by $d_{X,x}(x') := d_X(x, x')$. Observe that the open unit ball $B_{<r}(x)$ is simply the inverse image of the open interval $(-r, r)$ under the continuous function $d_{X,x}$. Similarly, the closed unit ball is the inverse image of the closed unit interval $[-r, r]$. Since d_X is continuous by **Problem 2**, also $d_{X,x}$ is continuous. Therefore the inverse image of the closed subset $[-r, r]$ of $\mathbb{R}$ is a closed subset of X, i.e., $B_{\le r}(x)$ is a closed subset of X. And it contains the open unit ball $B_{<r}(x)$. Since $B_{\le r}(x)$ is one closed set containing $B_{<r}(x)$, it contains the smallest closed set containing $B_{<r}(x)$, i.e., it contains the closure of $B_{<r}(x)$.

On the other hand, the closed unit ball need not equal the closure of the open unit ball. For instance, for any metric space X and any positive real number ϵ, define a new metric function on X by

$$d'_X(x, y) = \min(d_X(x, y), \epsilon).$$

It is straightforward to verify that d' is still a metric function. For every real number $r \le \epsilon$, the open ball $B'_{<r}(x)$ for this new metric equals the open ball $B_{<r}(x)$ for the original metric. In particular, both metrics give the same convergent sequences, and thus the same topology. So the closure of

$B'_{<\epsilon}(x)$ equals the closure of $B_{<\epsilon}(x)$. And by part (a), this is contained in $B_{\leq\epsilon}(x)$. On the other hand, the closed ball $B'_{\leq\epsilon}(x)$ is the entire metric space X. So if $B_{\leq\epsilon}(x)$ is properly contained in X, e.g., if X is unbounded as is $X = \mathbb{R}$, then $B'_{\leq\epsilon}(x)$ strictly contains the closure of $B'_{<\epsilon}(x)$.

For a similar example with S a subset of $\mathbb{R}^2$, consider the complement S of the open annulus $A := \{(x,y) \in \mathbb{R}^2 | 1 < x^2 + y^2 < 4\}$. Then $B_{<2}(0,0) \cap S$ equals $B_{\leq 1}(0,0)$, which is already closed. Hence the closure equals $B_{\leq 1}(0,0)$. On the other hand, $B_{\leq 2}(0,0) \cap S$ contains the circle of radius 2 centered at $(0,0)$. So it strictly contains the closure of $B_{<2}(0,0) \cap S$.

Solution to (4) The solution to this exercise was discussed in lecture.

Solution to (5) Let ϵ be a positive real number. Let (x_0, y_0) be an element of $X \times Y$. Since $g_{x_0} : Y \to Z$ is continuous, there exists a positive real δ_Y such that $d_Y(y_0, y_1) < \delta_Y$ implies that $d_Z(g(x_0, y_0), g(x_0, y_1)) < \epsilon/2$. And since $g_{y_1} : X \to Z$ is equicontinuous at x_0, there exists a positive real δ_X such that for all $y_1 \in Y$ simultaneously, $d_X(x_0, x_1) < \delta_X$ implies that $d_Z(g(x_0, y_1), g(x_1, y_1)) < \epsilon/2$. Therefore, by the triangle inequality, for all (x_1, y_1) with $d_X(x_0, x_1) < \delta_X$ and with $d_Y(y_0, y_1) < \delta_Y$, we have

$$d_Z(g(x_0, y_0), g(x_1, y_1)) \leq d_Z(g(x_0, y_0), g(x_0, y_1)) + d_Z(g(x_0, y_1), g(x_1, y_1)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Finally, choosing $\delta = \min(\delta_X, \delta_Y)$, then $d_{X\times Y}((x_0, y_0), (x_1, y_1)) < \delta$ implies that both $d_X(x_0, x_1) < \delta_X$ and $d_Y(y_0, y_1) < \delta_Y$, and thus $d_Z(g(x_0, y_0), g(x_1, y_1)) < \epsilon$. So g is continuous at (x_0, y_0).