MAT 544 Problem Set 1

Problems.

Problem 1. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. Define a function

\[
d_{X \times Y} : (X \times Y) \times (X \times Y) \to \mathbb{R}
\]

by

\[
d_{X \times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).
\]

(a) Prove that this is a metric space.

(b) Denote by \(\pi_X : X \times Y \to X\) and \(\pi_Y : X \times Y \to Y\) the two projections. Prove that these functions are continuous, in fact even Lipschitz (hence uniformly continuous).

(c) If \(X\) and \(Y\) are each complete metric spaces, prove that also \(X \times Y\) (with the above metric) is a complete metric space.

(d) Let \((Z, d_Z)\) be a metric space and let \((f_X : Z \to X, f_Y : Z \to Y)\) be a pair of continuous functions. Prove that there exists a unique continuous function \(f : Z \to X \times Y\) such that \(f_X = f \circ \pi_X\) and \(f_Y = f \circ \pi_Y\).

(e) Give an example of metric spaces \(X\) and \(Y\) and a metric \(d'_X\) on \(X \times Y\) which is different from \(d_{X \times Y}\) and which still satisfies the property from part (d). Conclude that this property does not characterize \(d_{X \times Y}\) (however, it does characterize the topology induced by this metric).

Problem 2. Let \((X, d_X)\) be a metric space. Give \(X \times X\) the metric from Problem 1. Prove that the function \(d_X : X \times X \to \mathbb{R}\) is Lipschitz for this metric.

Problem 3. For a metric space \((X, d_X)\), an element \(x\) of \(X\), and a real number \(r \geq 0\), the closed unit ball is sometimes defined to be

\[
B_{\leq r}(x) := \{ x' \in X | d_X(x, x') \leq r \},
\]

i.e., one uses “less than or equal to” rather than “less than” as in the definition of the open unit ball.

(a) For \(r > 0\), prove that the closure of the open unit ball \(B_r(x)\) is contained in the closed unit ball \(B_{\leq r}(x)\).
(b) Give an example of a subset $S$ of $\mathbb{R}^2$ (with the usual Euclidean metric), an element $x$ of $S$ and a real number $r > 0$, such that for the subspace metric on $S$, the closure of $B_r(x)$ in $S$ is strictly contained in $B_{\leq r}(x)$.

**Problem 4** Define sequence of integers $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ by the recursive relation $a_0 = 2$, $b_0 = 1$ and for every $n \geq 0$,

$$a_{n+1} = a_n^2 + 2b_n^2, \quad b_{n+1} = 2a_nb_n.$$

Prove that every $b_n \neq 0$ so that $(a_n/b_n)_{n \geq 0}$ is a well-defined sequence in $\mathbb{Q}$, prove that this sequence is Cauchy, and prove that this sequence does not have a limit. Thus the Archimedean ordered field $\mathbb{Q}$ is not complete.

**Problem 5** This is Exercise 4.3.14 of Loomis-Sternberg. Let $(X,d_X)$, $(Y,d_Y)$ and $(Z,d_Z)$ be metric spaces. Define the metric $d_{X \times Y}$ on $X \times Y$ as in Problem 1. Let $g : X \times Y \to Z$ be a function such that for every $x \in X$ the function

$$g_x : Y \to Z, \quad y \mapsto g(x,y)$$

is continuous and for every $y \in Y$ the function

$$g_y : X \to Z, \quad x \mapsto g(x,y)$$

is continuous *uniformly over $y$*, i.e., for every $x_0$ in $X$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x_0,x) < \delta \Rightarrow d_Z(g(x_0,y),g(x,y)) < \epsilon$$

for all values $y \in Y$ simultaneously. Prove that $g$ is continuous.