Jason Starr

MAT 544 Problem Set 1

Problems.

Problem 1. Let (X, d_X) and (Y, d_Y) be metric spaces. Define a function

$$d_{X\times Y} : (X \times Y) \times (X \times Y) \to \mathbb{R}$$

by $d_{X\times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$.

(a) Prove that this is a metric space.

(b) Denote by $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ the two projections. Prove that these functions are continuous, in fact even Lipschitz (hence uniformly continuous).

(c) If X and Y are each complete metric spaces, prove that also $X \times Y$ (with the above metric) is a complete metric space.

(d) Let (Z, d_Z) be a metric space and let $(f_X : Z \to X, f_Y : Z \to Y)$ be a pair of continuous functions. Prove that there exists a unique continuous function $f : Z \to X \times Y$ such that f_X equals $f \circ \pi_X$ and f_Y equals $f \circ \pi_Y$.

(e) Give an example of metric spaces X and Y and a metric d' on $X \times Y$ which is different from $d_{X\times Y}$ and which still satisfies the property from part (d). Conclude that this property does not characterize $d_{X\times Y}$ (however, it does characterize the *topology* induced by this metric).

Problem 2. Let (X, d_X) be a metric space. Give $X \times X$ the metric from **Problem 1**. Prove that the function $d_X : X \times X \to \mathbb{R}$ is Lipschitz for this metric.

Problem 3. For a metric space (X, d_X), an element x of X, and a real number $r \geq 0$, the *closed unit ball* is sometimes defined to be

$$B_{\leq r}(x) := \{x' \in X | d_X(x, x') \leq r\},$$

i.e., one uses "less than or equal to" rather than "less than" as in the definition of the open unit ball.

(a) For $r > 0$, prove that the closure of the open unit ball $B_r(x)$ is contained in the closed unit ball $B_{\leq r}(x)$.

(b) Give an example of a subset S of $\mathbb{R}^2$ (with the usual Euclidean metric), an element x of S and a real number $r > 0$, such that for the subspace metric on S, the closure of $B_r(x)$ in S is strictly contained in $B_{\leq r}(x)$.

Problem 4 Define sequence of integers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ by the recursive relation $a_0 = 2$, $b_0 = 1$ and for every $n \geq 0$,

$$a_{n+1} = a_n^2 + 2b_n^2, \quad b_{n+1} = 2a_nb_n.$$

Prove that every $b_n \neq 0$ so that $(a_n/b_n)_{n\geq 0}$ is a well-defined sequence in $\mathbb{Q}$, prove that this sequence is Cauchy, and prove that this sequence does not have a limit. Thus the Archimedean ordered field $\mathbb{Q}$ is not complete.

Problem 5 This is Exercise 4.3.14 of Loomis-Sternberg. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. Define the metric $d_{X\times Y}$ on $X \times Y$ as in **Problem 1**. Let $g : X \times Y \to Z$ be a function such that for every $x \in X$ the function

$$g_x : Y \to Z, \quad y \mapsto g(x, y)$$

is continuous and for every $y \in Y$ the function

$$g_y : X \to Z, \quad x \mapsto g(x, y)$$

is continuous *uniformly over* y, i.e., for every x_0 in X and for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x_0, x) < \delta \Rightarrow d_Z(g(x_0, y), g(x, y)) < \epsilon$$

for all values $y \in Y$ simultaneously. Prove that g is continuous.