MAT 544 Real Analysis I — Jason Starr
Stony Brook University — Fall 2011
Midterm 2, Thurs. 11/03/2011

MAT 544 Midterm 2 Review

The policies regarding exams are posted on the exams part of the course webpage. The exam is closed book, closed notes, no electronic devices are allowed, and you need only bring a writing implement.

Review Topics. Midterm 2 may involve any topics which appeared through the lecture of October 25th. But the emphasis will be on topics which were not already tested on Midterm 1. Please be familiar with all of the following concepts.

1. Differentiability and derivatives for maps between domains in normed vector spaces. Criterion for continuous differentiability in terms of continuous differentiability of partial derivatives. The Chain Rule. The Mean Value Theorem.

2. Banach's Contraction Mapping Fixed Point Theorem. Continuous variation of the fixed point with respect to parameters. Computation of several iterates of the approximating sequence to a fixed point.

3. The Implicit Function Theorem and its special case, the Inverse Function Theorem. Relation to the Contraction Mapping Fixed Point Theorem. Computation of several iterates of the approximating sequence to the implicit function. Continuity and differentiability of the implicit function. Application to differentiable variation of the fixed point in the Contraction Mapping Fixed Point Theorem.

4. The Picard-Lindelöf Theorem on existence and uniqueness of solutions of a first-order ODE whose defining equation is locally Lipschitz. Relation to the Contraction Mapping Fixed Point Theorem. Maximal extensions.

5. Diagonalization of diagonalizable matrices; Jordan form for matrices of small size. The characteristic polynomial. Eigenvalues. Eigenvectors, eigenspaces and generalized eigenspaces. The semisimple-nilpotent decomposition of a matrix. Algorithm for 2×2, 3×3 and 4×4 matrices.

6. Matrix exponentials. Definition via convergent power series. Compatibility with $M \mapsto UMU^{-1}$; compatibility for commuting matrices. Computation given the Jordan form.

7. Solutions of constant coefficient linear systems of ODEs, both homogeneous and inhomogeneous. Reduction of order. Solution of first-order systems by matrix exponentials. "Green's functions" / "fundamental solutions"; solution of the inhomogeneous equation.

8. Algebras and σ-algebras in the power set of a set. The Borel σ-algebra. Product σ-algebras.

9. Measures, outer measures and premeasures. Finite and countable additivity. Subadditivity. Monotonicity. Continuity from above and below. Caratheordory's Theorem constructing a measure from an outer measure. Compatibility with the premeasure used to construct an outer measure.

10. Borel measures on $\mathbb{R}$ which are finite on bounded sets. Distribution functions. The Lebesgue-Stieltjes measure. Characterization of Borel sets in terms of G_δ-sets and null sets.

Some Practice Problems.

Problem 1 For normed vector space $(V, \|\bullet\|_V)$ and $(W, \|\bullet\|_W)$, for an open subset $\tilde{V} \subset V$, and for a function $F : \tilde{V} \to W$, define "differentiability" and the "derivative" of F at a point $\vec{v}_0 \in \tilde{V}$. For a direct sum decomposition $V = V_1 \times \cdots \times V_n$ into closed subspaces, define the "partial derivatives" of F in the directions $V_1, \dots, V_n$. State a criterion for the continuous differentiability of F in terms of existence and continuity of the partial derivatives.

Problem 2 Consider the function $F : \mathbb{R}^3 \to \mathbb{R}^3$ by $F(x, y, z) = (yz, xz, xy)$. Compute the partial derivatives with respect to x, y and z. Compute the total derivative. Determine all points at which the total derivative is invertible. At each such point, find the inverse. Find a radius $r > 0$ such that the restriction of F to the ball $B_r(\vec{0})$ (with respect to the product norm) is a contraction (use the Mean Value Theorem).

Problem 3 Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^4$ by $F(s, t) = (s^3, s^2t, st^2, t^3)$. Compute the partial derivatives with respect to s and t. Compute the total derivative. Determine all points at which the total derivative is injective, hence has a left inverse. At each such point, find a left inverse.

Problem 4 Consider the function $F : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ by $F(A) = A \cdot A$. Compute the total derivative. Prove that the total derivative at A is invertible if and only if $\mathrm{Det}(A) \neq 0$ and $\mathrm{Tr}(A) \neq 0$. Find a radius $r > 0$ such that the restriction of F to the ball $B_r(0_{2\times 2})$ (with respect to the operator norm) is a contraction.

Problem 5 Consider the function $G : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ by $G(B) = \mathrm{Det}(B)$. Compute the total derivative. Prove that the total derivative at B is surjective if and only if $\mathrm{Det}(B)$ is nonzero. Then use the Chain Rule to compute the total derivative of $G \circ F$, where F is as in **Problem 4**. Next, use a property of the determinant to find a function $H : \mathbb{R} \to \mathbb{R}$ such that $G \circ F$ equals $H \circ G$. Compute the total derivative of $H \circ G$ using the Chain Rule. Compare your answers for the total derivative of $G \circ F$ and $H \circ G$.

Problem 6 Consider the function $C : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathrm{Poly}_2(\mathbb{R})$ by $C(A) = c_A(t) = \mathrm{Det}(tI_{2\times 2} - A)$. Compute the total derivative of C. Is there any point at which the derivative is injective, resp. surjective? Compute the derivative of $C \circ F$ using the Chain Rule, where F is as in **Problem 4**.

Problem 7 In **Problem 2**, **Problem 4** and **Problem 5**, at each point where the derivative of the function is surjective, write down the contraction mapping whose fixed point gives an implicit function for the fiber of the function. Then compute the zeroth, first and second terms in the approximating sequence to the implicit function. In **Problem 2** and **Problem 4**, at each point where the derivative of the function is invertible, also write down the contraction mapping whose fixed point gives a local inverse function. The compute the zeroth, first and second terms in the approximating sequence to the inverse function.

Problem 8 Let (X, d_X) be a complete metric space. Let $F : X \to X$ be a continuous function such that some power F^n is a contraction. Prove that F has a unique fixed point. (Hint. If x_0 is a fixed point of F^n, what about $x_1 = F(x_0)$?)

Problem 9 Let (X, d_X) be a complete metric space. Let $F : X \to X$ and $G : X \to X$ be continuous functions such that $G \circ F$ is a contraction, which thus has a unique fixed point x_0. Prove that also $F \circ G$ has a fixed point, and every fixed point y_0 satisfies $G(y_0) = x_0$.

Problem 10 Let $(V, \| \bullet \|_V)$ be a Banach space and let $F : V \to V$ be a contraction. For every $\vec{v}_0 \in V$, denote by $T_{\vec{v}_0} : V \to V$ the translation $\vec{v} \mapsto \vec{v} + \vec{v}_0$. Prove that both $T_{\vec{v}_0} \circ F$ and $F \circ T_{\vec{v}_0}$ are contractions. Prove that there exist continuous functions $\lambda : V \to V$ and $\rho : V \to V$ such that for every $\vec{v}_0 \in V$, $\lambda(\vec{v}_0)$ is a fixed point of $T_{\vec{v}_0} \circ F$ and $\rho(\vec{v}_0)$ is a fixed point of $F \circ T_{\vec{v}_0}$, i.e., $F(\lambda(\vec{v}_0)) + \vec{v}_0 = \lambda(\vec{v}_0)$ and $F(\rho(\vec{v}_0) + \vec{v}_0) = \rho(\vec{v}_0)$.

Problem 11 Let $(V, \| \bullet \|_V)$ be a Banach space and let $F : V \to V$ be a contraction. For every $S \in L(V, V)$ with operator norm ≤ 1, i.e., for every S in the ball $B_{\leq 1}(0)$ in $L(V, V)$, prove that both $S \circ F$ and $F \circ S$ are contractions. Prove that there exists a continuous function $g : B_{\leq 1}(0) \times V \to V$ such that for every $(S, \vec{v}_0) \in B_{\leq 1}(0) \times V$, $g(S, \vec{v}_0)$ is a fixed point of $T_{\vec{v}_0} \circ S \circ F$.

Problem 12 In **Problem 11** when F is an linear transformation with operator norm < 1, explicitly compute the function g as a convergent power series in the operator $S \circ F$.

Problem 13 Let $n \geq 1$ be an integer. Starting with the function $f_0(t) = x_0$, compute the first three Picard iterates f_0, f_1 and f_2 approximating the solution of the initial value problem

$$\frac{dx}{dt} = x^n, \quad x(t_0) = x_0.$$

Problem 14 In **Problem 2** and **Problem 4**, starting with the constant function $f_0 = \vec{v}_0$, compute the first three Picard iterates f_0, f_1 and f_2 approximating the solution of the initial value problem

$$\frac{d\vec{x}}{dt} = F(\vec{x}), \quad \vec{x}(t_0) = \vec{x}_0.$$

Problem 15 Let A be a real $n \times n$ matrix. If λ is a complex eigenvalue for A, prove that the complex conjugate $\overline{\lambda}$ is also a complex eigenvalue for A. Moreover, if the real and imaginary parts are given by $\lambda = \lambda_R + i\lambda_I$, and if $\vec{v}$ is a complex λ-eigenvector with real and imaginary parts $\vec{v} = \vec{v}_R + i\vec{v}_I$, prove that

$$\left\{ \begin{array}{rcccc} A\vec{v}_R & = & \lambda_R\vec{v}_R & - & \lambda_I\vec{v}_I \\ A\vec{v}_I & = & \lambda_I\vec{v}_R & + & \lambda_R\vec{v}_I \end{array} \right.$$

Problem 16 Consider the following matrix,

$$M = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}.$$

Compute the determinant, the trace, the characteristic polynomial and all eigenvalues. For each eigenvalue, find a basis for the eigenspace. If the matrix is diagonaliable, find an invertible matrix U and a diagonal matrix $\tilde{S}$ such that $MU = U\tilde{S}$.

 Jason Starr

Problem 17 Consider the following matrix,

$$M = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 5 & -3 \\ 6 & -6 & 2 \end{bmatrix}.$$

Compute the determinant, the trace, the characteristic polynomial and all eigenvalues. For each eigenvalue, find a basis for the eigenspace. If the matrix is diagonaliable, find an invertible matrix U and a diagonal matrix $\tilde{S}$ such that $MU = U\tilde{S}$.

Problem 18 Consider the following matrix,

$$M = \begin{bmatrix} 1 & 3 \\ -3 & -5 \end{bmatrix}.$$

Compute the determinant, the trace, the characteristic polynomial and all eigenvalues. For each eigenvalue, find a basis for the eigenspace. If the dimension of an eigenspace is less than the multiplicity of the corresponding eigenvalue, compute a vector in the generalized eigenspace which is not in the eigenspace. Find an invertible matrix U, a diagonal matrix $\tilde{S}$ and a strictly upper triangular matrix $\tilde{N}$ such that $\tilde{S}\tilde{N} = \tilde{N}\tilde{S}$ and such that $MU = U(\tilde{S} + \tilde{N})$.

Problem 19 Consider the following matrix,

$$M = \begin{bmatrix} 7 & 1 & 5 \\ 5 & 3 & -1 \\ 6 & -6 & 2 \end{bmatrix}.$$

Compute the determinant, the trace, the characteristic polynomial and all eigenvalues. For each eigenvalue, find a basis for the eigenspace. If the dimension of an eigenspace is less than the multiplicity of the corresponding eigenvalue, compute a vector in the generalized eigenspace which is not in the eigenspace. Find an invertible matrix U, a diagonal matrix $\tilde{S}$ and a strictly upper triangular matrix $\tilde{N}$ such that $\tilde{S}\tilde{N} = \tilde{N}\tilde{S}$ and such that $MU = U(\tilde{S} + \tilde{N})$.

Problem 20 In each of **Problem 16**, **Problem 17**, **Problem 18**, and **Problem 19**, compute the matrix exponentials $\exp(\tilde{S}(t - t_0))$, $\exp(\tilde{N}(t - t_0))$, and $\exp(M(t - t_0))$.

Problem 21 In each of **Problem 16**, **Problem 17**, **Problem 18**, and **Problem 19**, find the solution of the initial value problem

$$\frac{dA}{dt} = MA, \quad A(t_0) = \mathrm{Id}.$$

Then find the solution of the initial value problem

$$\frac{d\vec{x}}{dt} = M\vec{x}, \quad \vec{x}(t_0) = \vec{x}_0.$$

Finally, for a constant vector $\vec{g}$, find the solution of the initial value problem

$$\frac{d\vec{x}}{dt} = M\vec{x} + \vec{g}, \quad \vec{x}(t_0) = \vec{0}.$$

Problem 22 Let X and Y be disjoint sets. Let $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$ be algebras, resp. σ-algebras. Prove that the smallest algebra, resp. σ-algebra, in $\mathcal{P}(X \sqcup Y)$ containing $\mathcal{A} \cup \mathcal{B}$ is $\{E \cup F | E \in \mathcal{A},\ F \in \mathcal{B}\}$.

Problem 23 Let X be a set and let $(\mathcal{A}_i)_{i=1}^{\infty}$ be a sequence of algebras on X with $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ for every i. Prove that the union $\mathcal{A} = \cup_{i=1}^{\infty}\mathcal{A}$ is again an algebra on X.

Problem 24 With notation as in the previous exercise, find a nested sequence of algebras $(\mathcal{A}_i)_{i=1}^{\infty}$ where every $\mathcal{A}_i$ is a σ-algebra and yet $\mathcal{A}$ is not a σ-algebra. (Hint. For every finite subset $S \subset X$, the collection of subsets of X which are contained in S or whose complement is contained in S is a σ-algebra.)

Problem 25 Let $(X, \mathcal{M}, \mu)$ be a measure space which is σ-finite. Prove that it is semifinite in the following strong sense: for every $E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists a sequence $(E_i)_{i=1}^{\infty}$ of sets $E_i \in \mathcal{M}$ with $E_1 \subset E_2 \subset \cdots \subset E$, with $E = \cup_i E_i$, with $\mu(E_i) < \infty$ and with $\lim_{i\to\infty} \mu(E_i) = \infty$.

Problem 26 Let $(X, \mathcal{M}, \mu)$ be a semifinite measure space. Prove that either there exists a subset $E \in \mathcal{M}$ with $\mu(E) = \infty$ which is σ-finite or else there exists $E \in \mathcal{M}$ with $\mu(E) < \infty$ and such that for every $F \in \mathcal{M}$ with $\mu(F) < \infty$, the set $F \setminus E$ is a null set (so E is the "finite" part of X).

Problem 27 Let X be a set, let $\mathcal{M}$ be a σ-algebra on X, and let $(\mu_i)_{i=1}^{\infty}$ be a sequence of measure functions $\mu_i : \mathcal{M} \to [0, +\infty]$ such that for every $E \in \mathcal{M}$, $\mu_1(E) \leq \mu_2(E) \leq \dots$. Prove that $\mu : \mathcal{M} \to [0, \infty]$ by $\mu(E) := \sup\{\mu_i(E)\}_{i=1}^{\infty}$ gives a measure function.

Problem 28 Exercises 1.18-1.24 on pp. 32-33 of the textbook.