SOLUTIONS

MAT 544 Fall 2011 Midterm 1

Name: ______________________ SB ID number: ______________________

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Instructions: *Please write your name at the top of every page of the exam.* This exam is closed book, closed notes, calculators are not allowed, and all cellphones and other electronic devices must be turned off for the duration of the exam. You will have approximately 80 minutes for this exam. The point value of each problem is written next to the problem – use your time wisely. **Please show all work, unless instructed otherwise.** Partial credit will be given only for work shown. For results quoted from the text, state clearly and correctly the hypotheses and conclusions, and give the "name" if the result has a famous name.

You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., raise your hand.

In each part denote $S_N := \sum_{n=1}^{N} a_n$ and $\|S\|_N := \sum_{n=1}^{N} \|a_n\|$. Thus the series converges to S_∞ if & only if $(S_N) \to S_\infty$. And the series converges absolutely if and only if $(\|S\|_N)$ converges, or, equivalently, is bounded above.

Name: ______________________________ **Problem 1:** ______________ /45

Problem 1(45 points) In each of the following cases, state whether or not the given series $\sum_{n=1}^{\infty} a_n$ in the given normed vector space $(V, \|\bullet\|)$ is convergent. If the series is convergent, give the limiting value, and then say whether or not the series is absolutely convergent. Show all computations, but you need not quote theorems to justify your answer.

(a)(15 points) $(V, \|\bullet\|)$ is $\mathbb{R}$ with the absolute value norm and

$$a_n = \begin{cases} -1/(n+1), & n \text{ odd} \\ 1/n, & n \text{ even} \end{cases}$$

(b)(15 points) $(V, \|\bullet\|)$ is $(\ell^1, \|\bullet\|_{\ell^1})$ and each $\mathbf{a}_n$ is the sequence $(a_{n,k})_{k=1,2,\ldots} = (1/2^{nk})_{k=1,2,\ldots}$.

(c)(15 points) $(V, \|\bullet\|)$ is $(BC([0,1), \mathbb{R}), \|\bullet\|_{\text{uniform}})$, the vector space of bounded, continuous functions from $[0,1)$ (with its usual metric) to $\mathbb{R}$ and each $a_n(x)$ equals x^n.

(a) $a_{2m+1} + a_{2m} = 0$, thus $S_N = \begin{cases} -1/(N+1), & N \text{ odd} \\ 0, & N \text{ even} \end{cases}$. So $\lim_{N\to\infty} S_N = 0$, $\boxed{\text{the series converges to } 0.}$ However $\|S\|_{2m} = 2\left(\frac{1}{2\cdot 1} + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{2\cdot m}\right) = \sum_{k=1}^{m} \frac{1}{k}$.
And $\sum_{k=2}^{m} \frac{1}{k} \geq \int_{x=1}^{m} \frac{1}{x}\,dx = \ln(m)$. So $\|S\|_{2m} \geq 1 + \ln(m)$, and thus $(\|S\|_{2m})$ is unbounded. Thus the series is $\boxed{\text{not absolutely convergent.}}$

(b) For $r \neq 1$, $\sum_{n=1}^{N} r^n = \frac{1-r^N}{1-r} \cdot r$. So $\sum_{n=1}^{N} a_{n,k} = \sum_{n=1}^{N} \frac{1}{2^{nk}} = \frac{1 - 1/2^{Nk}}{2^k - 1}$, $\underline{S}_N = \left(\frac{1 - 1/2^{Nk}}{2^k - 1}\right)_k$.
Define $\underline{S}_\infty := \left(\frac{1}{2^k - 1}\right)_{k=1,2,\ldots}$. Note $\sum_{k=1}^{\infty} \left|\frac{1}{2^k - 1}\right| \leq \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = \frac{1-0}{1-\frac{1}{2}} = 2$.
Thus $\underline{S}_\infty$ is in ℓ^1. Since $S_{\infty,k} - S_{N,k}$ equals $\frac{-1/2^{Nk}}{2^k - 1}$, we have
$$\|\underline{S}_\infty - \underline{S}_N\|_{\ell^1} = \sum_{k=1}^{\infty} \frac{1}{2^k - 1} \frac{1}{2^{Nk}} \leq \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \cdot \frac{1}{2^{Nk}} = \frac{1}{2^N} \frac{1-0}{1 - 1/2^{N+1}} = \frac{1}{2^N - 1/2} \leq \frac{1}{2^{N-1}}.$$
So $\lim_{N\to\infty} \|\underline{S}_\infty - \underline{S}_N\|_{\ell^1} \leq \lim_{N\to\infty} \frac{1}{2^{N-1}} = 0$. $\boxed{\text{Therefore the series converges to } \underline{S}_\infty = \left(\frac{1}{2^k - 1}\right)_{k=1,2,\ldots}}$
Since all terms are positive, $\|S\|_N$ equals $\|\underline{S}_N\|_{\ell^1} = \sum_{k=1}^{\infty} \frac{1 - 1/2^{Nk}}{2^k - 1} \leq \sum_{k=1}^{\infty} \frac{1}{2^k - 1} = \|\underline{S}_\infty\|_{\ell^1} \leq 2$.
Thus $(\|S\|_N)_{N=1,2,\ldots}$ is bounded by $\|\underline{S}_\infty\|_{\ell^1} \leq 2$. $\boxed{\text{So the series is absolutely convergent.}}$

(c) If $(S_N(x))_{N=1,2,\ldots}$ converges, then the sequence $(a_N(x))$ must converge to 0.
But $\|a_n(x)\|_{un} = \operatorname{lub}_{0 \leq x < 1} x^n = \lim_{x \to 1} x^n = 1$ for all n. So $\lim_{N\to\infty} \|a_n(x)\|_{un} = 1 \neq 0$.
Thus the series $\boxed{\text{does not converge.}}$ Moreover $\|S\|_N = \|a_1\| + \cdots + \|a_N\| = 1 + \cdots + 1 = N$, which is unbounded. Thus $\boxed{\text{not abs. conv.}}$

Name: ______________________ Problem 2: ____________ /30

Problem 2(30 points) In this exercise, an *exhaustion* of a metric space (X, d_X) is a sequence of open subsets $(U_n)_{n=1,2,\ldots}$ such that $U_n \subset U_{n+1}$ for every $n = 1, 2, \ldots$, and such that $X = \cup_{n=1}^{\infty} U_n$. The exhaustion *stabilizes* if there exists n such that $U_n = X$. Without quoting theorems from the text, prove that for every sequentially compact metric space, every exhaustion stabilizes.

By way of contradiction assume there exists an exhaustion which does not stabilize, i.e., $X \setminus \mathcal{U}_n$ is nonempty for every $n=1,2,\ldots$. Using the countable axiom of choice, there exists a sequence $(x_n)_{n=1,2,\ldots}$ such that $x_n \in X \setminus \mathcal{U}_n$ for every $n=1,2,\ldots$ Since X is sequentially compact there exists a subsequence $(x_{n_k})_{k=1,2,\ldots}$ which converges to some $x_\infty \in X$. Since $X = \bigcup_{n=1}^{\infty} \mathcal{U}_n$, there exists an integer n such that $x_\infty \in \mathcal{U}_n$. Since $(x_{n_k}) \to x_\infty$, there exists an integer K such that for all $k \geq K$, x_{n_k} is in $\mathcal{U}_n$. For $k \geq \max(K, n)$, x_{n_k} is in $\mathcal{U}_n$ and $n_k \geq n$. Thus $\mathcal{U}_{n_k}$ contains $\mathcal{U}_n$. So x_{n_k} is in $\mathcal{U}_n \subseteq \mathcal{U}_{n_k}$ contradicting the hypothesis that $x_{n_k} \in X \setminus \mathcal{U}_{n_k}$. Therefore, by contradiction, every exhaustion stabilizes.

Name: ______________________________ Problem 3: ____________ /50

Problem 3(50 points) Let (X, d_X) be a metric space. Let $BC(X, \mathbb{R})$ be the vector space of bounded, continuous functions with the uniform norm $\| \bullet \|_{\text{uniform}}$.

(a)(10 points) State carefully what it means for a subset $\mathcal{F} \subset BC(X, \mathbb{R})$ to be *equicontinuous*.

(b)(20 points) State carefully the Arzela-Ascoli theorem.

(c)(20 points) Give a sequence $(f_n(x))_{n=1,2,\ldots}$ of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ (for the standard metric) such that $\|f_n\|_{\text{un}} \leq 1$ for every n, and such that $\{f_n | n = 1, 2, \ldots\}$ is equicontinuous (or even all 1-Lipschitz), yet no subsequence converges uniformly. Explain why your example does not contradict the Arzela-Ascoli theorem.

(a) A subset $\mathcal{F} \subset BC(X, \mathbb{R})$ is equicontinuous if for every real $\varepsilon > 0$ and for every $x \in X$ there exists a real number $\delta > 0$ (dependent on ε and x) such that for every $f \in \mathcal{F}$ and for every $x' \in B_\delta(x)$, we have $|f(x') - f(x)| < \varepsilon$.

(b) Theorem [Arzela-Ascoli] Let (X, d_X) be a compact metric space. Let $\mathcal{F}$ be a subset of $BC(X, \mathbb{R})$. Assume that $\mathcal{F}$ is equicontinuous and pointwise bounded, i.e., for every $x \in X$ the subset $\{f(x) \mid f \in \mathcal{F}\}$ of $\mathbb{R}$ is bounded. Then $\mathcal{F}$ is totally bounded, or, equivalently, the closure of $\mathcal{F}$ in $BC(X, \mathbb{R})$ is compact.

(c) There are many examples. For instance, define $f_n(x) = \begin{cases} |n-x| - 1, & |n-x| \leq 1 \\ 0, & |n-x| > 1 \end{cases}$.

This is a uniformly bounded sequence of 1-Lipschitz functions. And $(f_n(x))_{n=1,2,\ldots}$ does converge pointwise to the zero function. Yet $\|f_n\|_{\text{uniform}} = \|f_n - 0\|_{\text{uniform}} = 1$ for all n, so (f_n) does not converge to 0 uniformly.

This does not contradict Arzela-Ascoli since $\mathbb{R}$ is not compact (in fact $\mathbb{R}$ is not even totally bounded).

Name: ______________________________ **Problem 4:** ____________ /75

Problem 4(75 points) Let $(V, \langle \bullet, \bullet \rangle)$ be a real inner product space. Let $U \subset V$ and $W \subset V$ be linear subspaces which are closed. Let $\pi_U : V \to U$ and $\pi_W : V \to W$ denote the corresponding orthogonal projections.

(a)(15 points) Let $V = \mathbb{R}^2$ with the standard Euclidean inner product. Give examples of proper, nontrivial subspaces U and W and a vector $\vec{v} \in \mathbb{R}^2$ such that $(\pi_U \circ \pi_W)(\vec{v})$ does not equal $(\pi_W \circ \pi_U)(\vec{v})$.

(b)(20 points) Again in the general case, assume that $U \subset W$. Prove that $\pi_U \circ \pi_W$ equals $\pi_W \circ \pi_U$.

(c)(40 points) Let $V = \mathbb{R}^3$ with the standard Euclidean inner product. For the following vectors $\vec{v}_1$ and $\vec{v}_2$, let $U = \text{span}(\vec{v}_1)$ and let $W = \text{span}(\vec{v}_1, \vec{v}_2)$.

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

Find orthonormal vectors $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$ such that $U = \text{span}(\vec{b}_1)$ and $W = \text{span}(\vec{b}_1, \vec{b}_2)$. Then give the matrices M_U and M_W for the linear operators $\pi_U : V \to U \subset V$ and $\pi_W : V \to W \subset V$ (with respect to the standard basis of $\mathbb{R}^3$). For fun, contemplate computing $M_W M_U - M_U M_W$ directly (without a calculator).

Nota bene. Every coordinate or matrix entry in (c) will be a rational number or a rational number times $1/\sqrt{5}$.

(a) For nonzero vectors $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} c \\ d \end{bmatrix}$, $\pi_U\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{a^2+b^2}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $\pi_W\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{c^2+d^2}\begin{bmatrix} c^2 & cd \\ cd & d^2 \end{bmatrix}$

and $(\pi_U \pi_W - \pi_W \pi_U)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{(ad-bc)(ac+bd)}{(a^2+b^2)(c^2+d^2)}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\det(\vec{u},\vec{w})\,(\vec{u}\cdot\vec{w})}{(\vec{u}\cdot\vec{u})(\vec{w}\cdot\vec{w})}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

So if $\det(\vec{u},\vec{w}) \neq 0$, i.e., $(\vec{u},\vec{w})$ are linearly independent, and if $\vec{u}\cdot\vec{w} \neq 0$, i.e., $(\vec{u},\vec{w})$ are not orthogonal, then $\pi_U\pi_W - \pi_W\pi_U$ is nonzero on every $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Example. $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{w} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $\pi_U(\vec{v}) = \vec{0}$, so $\pi_W\pi_U(\vec{v}) = \vec{0} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ yet $\pi_W(\vec{v}) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ and $\pi_U\pi_W(\vec{v}) = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$.

(b) Note that $\pi_U(\vec{v})$ is in U, which is in W by hypothesis. Since $\pi_U(\vec{v})$ is already in W, $\pi_W(\pi_U(\vec{v}))$ equals $\pi_U(\vec{v})$. Thus $\boxed{\pi_W \circ \pi_U \text{ equals } \pi_U.}$

On the other hand, $\vec{v} = \pi_W(\vec{v}) + \pi_{W^\perp}(\vec{v})$ where $\pi_W(\vec{v}) \in W$ and $\pi_{W^\perp}(\vec{v}) \in W^\perp$. Since U is in W, also $W^\perp$ is in $U^\perp$. Since $\pi_{W^\perp}(\vec{v})$ is in $U^\perp$, $\pi_U(\pi_{W^\perp}(\vec{v}))$ equals 0, i.e. $\pi_U \circ \pi_{W^\perp}$ equals 0. So $\pi_U(\vec{v}) = \pi_U(\pi_W(\vec{v}) + \pi_{W^\perp}(\vec{v}))$ $= \pi_U(\pi_W(\vec{v})) + \pi_U(\pi_{W^\perp}(\vec{v})) = \pi_U(\pi_W(\vec{v})) + \vec{0}$. Thus $\boxed{\pi_U \text{ equals } \pi_U \circ \pi_W.}$ Therefore, $\boxed{\pi_W\pi_U = \pi_U = \pi_U\pi_W.}$

(c) To apply Gram-Schmidt we need a third vector $\vec{v_3} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. It will be simplest if we choose $\vec{v_3}$ orthogonal to $\vec{v_1}$ and $\vec{v_2}$, i.e., $\begin{cases} 2x+3y+6z=0 \\ 2y-z=0 \end{cases}$. A basis of the solution space is $\vec{v_3} = \begin{bmatrix} -15 \\ 2 \\ 4 \end{bmatrix}$. So apply Gram-Schmidt to $\vec{v_1}, \vec{v_2}, \vec{v_3} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -15 \\ 2 \\ 4 \end{bmatrix}$.

$\boxed{\vec{b_1}}$ $\vec{v_1}\cdot\vec{v_1} = 2^2+3^2+6^2 = 4+9+36 = 49$, $\|\vec{v_1}\| = \sqrt{49} = 7$. So $\vec{b_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \boxed{\frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}} = \vec{b_1}$

$\boxed{\vec{b_2}}$ $\vec{v_2}\cdot\vec{b_1} = 0\cdot 2 + 2\cdot 3 + (-1)6 = 0$. So $\vec{v_2}^{\perp} := \vec{v_2} - (\vec{v_2}\cdot\vec{b_1})\vec{b_1}$ equals $\vec{v_2}$.

And $\vec{v_2}\cdot\vec{v_2} = 0^2+2^2+(-1)^2 = 5$. So $\|\vec{v_2}^{\perp}\| = \sqrt{5}$. So $\vec{b_2} = \frac{\vec{v_2}^{\perp}}{\|\vec{v_2}^{\perp}\|} = \boxed{\frac{1}{\sqrt{5}}\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}} = \vec{b_2}$

$\boxed{\vec{b_3}}$ By construction $\vec{v_3}$ is orthogonal to $\text{span}(\vec{v_1}, \vec{v_2}) = \text{span}(\vec{b_1}, \vec{b_2})$. Thus $\vec{v_3}^{\perp} = \vec{v_3}$. And $\vec{v_3}\cdot\vec{v_3} = (15)^2 + 2^2 + 4^2 = 245 = 5\cdot 49$

So $\|\vec{v_3}^{\perp}\| = 7\sqrt{5}$. Thus $\vec{b_3} = \frac{\vec{v_3}^{\perp}}{\|\vec{v_3}^{\perp}\|} = \boxed{\frac{1}{7\sqrt{5}}\begin{bmatrix} -15 \\ 2 \\ 4 \end{bmatrix}} = \vec{b_3}$

$\pi_V(\vec{v}) = (\vec{b_1}\cdot\vec{v})\vec{b_1} = \frac{2x+3y+6z}{49}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 4 & 6 & 12 \\ 6 & 9 & 18 \\ 12 & 18 & 36 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $\boxed{M_V = \frac{1}{49}\begin{bmatrix} 4 & 6 & 12 \\ 6 & 9 & 18 \\ 12 & 18 & 36 \end{bmatrix}}$.

Define $\tilde{\pi}(\vec{v}) = (\vec{b_2}\cdot\vec{v})\vec{b_2} = \frac{2y-z}{5}\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $\tilde{M} = \frac{1}{5}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 1 \end{bmatrix}$.

Then $\pi_W(\vec{v}) = (\vec{b_1}\cdot\vec{v})\vec{b_1} + (\vec{b_2}\cdot\vec{v})\vec{b_2} = \pi_V(\vec{v}) + \tilde{\pi}(\vec{v}) = (M_V + \tilde{M})\vec{v}$.

Thus $M_W = M_V + \tilde{M} = \frac{1}{5\cdot 49}\begin{bmatrix} 20 & 30 & 60 \\ 30 & 45 & 90 \\ 60 & 90 & 180 \end{bmatrix} + \frac{1}{5\cdot 49}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 196 & -98 \\ 0 & -98 & 49 \end{bmatrix}$;

$$\boxed{M_W = \frac{1}{245}\begin{bmatrix} 20 & 30 & 60 \\ 30 & 241 & -8 \\ 60 & -8 & 229 \end{bmatrix}}.$$