Name: ______________________________ **Problem 1:** ____________ /40

Problem 1(40 points) Let $W = \mathbb{R}^3$ with the standard Euclidean inner product. For the following vectors $\vec{v}_1$ and $\vec{v}_2$, denote $\mathrm{span}(\vec{v}_1)$ by U and $\mathrm{span}(\vec{v}_1, \vec{v}_2)$ by V.

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 9 \\ 0 \\ 9 \end{bmatrix},$$

(a)(25 points) Find orthonormal vectors $\vec{b}_1$ and $\vec{b}_2$ such that U equals $\mathrm{span}(\vec{b}_1)$ and V equals $\mathrm{span}(\vec{b}_1, \vec{b}_2)$.

(b)(15 points) Compute the matrix M_V of the orthogonal projection to V (with respect to the standard ordered basis of $\mathbb{R}^3$).

(a). $\vec{b}_1$. $\langle \vec{v}_1, \vec{v}_1 \rangle = 2^2 + 1^2 + 2^2 = 9$, $\|\vec{v}_1\| = \sqrt{\langle \vec{v}_1, \vec{v}_1 \rangle} = 3$, $\vec{b}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \boxed{\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}$

$\vec{b}_2$. $\langle \vec{v}_2, \vec{b}_1 \rangle = \frac{1}{3}(9 \cdot 2 + 0 \cdot 1 + 9 \cdot 2) = 12$, $\vec{w}_2 := \vec{v}_2 - \langle \vec{v}_2, \vec{b}_1 \rangle \vec{b}_1 = \begin{bmatrix} 9 \\ 0 \\ 9 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$.

$\langle \vec{w}_2, \vec{w}_2 \rangle = 1^2 + (-4)^2 + 1^2 = 18$, $\|\vec{w}_2\| = \sqrt{\langle \vec{w}_2, \vec{w}_2 \rangle} = 3\sqrt{2}$, $\vec{b}_2 = \frac{1}{\|\vec{w}_2\|}\vec{w}_2 = \boxed{\frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}}$

(b). Since $(\vec{b}_1, \vec{b}_2)$ is an ON basis for V, orthogonal projection π_V is

$\pi_V(\vec{w}) = \langle \vec{w}, \vec{b}_1 \rangle \vec{b}_1 + \langle \vec{w}, \vec{b}_2 \rangle \vec{b}_2 = \frac{1}{9}\langle \vec{w}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \rangle \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{18}\langle \vec{w}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \rangle \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$.

$\pi_V\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \frac{1}{9}\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \rangle \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{18}\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \rangle \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 8 \\ 4 \\ 8 \end{bmatrix} + \frac{1}{18}\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 9 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$.

$\pi_V\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \frac{1}{9}\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \rangle \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{18}\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \rangle \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} -2 \\ 8 \\ -2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$\pi_V\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \frac{1}{9}\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \rangle \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{18}\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \rangle \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 8 \\ 4 \\ 8 \end{bmatrix} + \frac{1}{18}\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 9 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$.

Therefore M_V equals $\left[\pi_V\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \middle| \pi_V\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \middle| \pi_V\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\right]$

$$= \boxed{\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}}$$

Name: ______________________________ **Problem 2:** ______________ /60

Problem 2(60 points) Consider the following initial value problem

$$\begin{cases} dx/dt &= x + y \\ d^2y/dt^2 &= \qquad + dy/dt \end{cases}$$

$$x(0) = 0,\ y(0) = 0,\ \frac{dy}{dt}(0) = 1.$$

(a)(5 points) Find a 3×3-matrix A with real entries such that for every $(b_0, c_0, c_1)^\dagger \in \mathbb{R}^3$, the solution of the 1st order IVP

$$\frac{d\vec{x}}{dt} = A\vec{x},\ \vec{x}(t_0) = \begin{bmatrix} b_0 \\ c_0 \\ c_1 \end{bmatrix},\ \vec{x}(t) = \begin{bmatrix} x_0(t) \\ y_0(t) \\ y_1(t) \end{bmatrix}$$

gives a solution of the IVP

$$\begin{cases} dx/dt &= x + y \\ d^2y/dt^2 &= \qquad + dy/dt \end{cases}$$

$$x(0) = b_0,\ y(0) = c_0,\ \frac{dy}{dt}(0) = c_1.$$

by $x(t) = x_0(t)$ and $y(t) = y_0(t)$.

(b)(10 points) Find the characteristic polynomial of A, find the factorization into a product of linear factors (each of which will be real), and find all eigenvalues of A.

(c)(25 points) Find an invertible 3×3 matrix U, a diagonal 3×3 matrix $\tilde{S}$, and a 3×3 matrix $\tilde{N}$ which is upper triangular (or lower triangular if you prefer) such that $\tilde{S}\tilde{N} = \tilde{N}\tilde{S}$ and such that $AU = U(\tilde{S} + \tilde{N})$.

(d)(10 points) Compute $\exp(\tilde{S}t)$, $\exp(\tilde{N}t)$ and $\exp(At)$. In your answer, write out each entry of the matrix; do not leave matrix multiplications unevaluated. All entries of your matrices should involve only polynomials in t and exponentials in t, no unevaluated power series.

(e)(10 points) Find the general solution $\vec{x}(t)$ of the 1st order IVP above. Write out each component of $\vec{x}(t)$; do not leave matrix multiplications unevaluated. Both components should involve only polynomials in t and exponentials in t.

Bonus problem(5 bonus points) Solve the following inhomogeneous IVP.

$$\begin{cases} dx/dt &= x + y & & + e^t \\ d^2y/dt^2 &= & + dy/dt & \end{cases} \quad x(0) = 0,\ y(0) = 0,\ \frac{dy}{dt}(0) = 0.$$

(a) $\vec{x}(t)=\begin{bmatrix}x_0(t)\\y_0(t)\\y_1(t)\end{bmatrix}=\begin{bmatrix}x(t)\\y(t)\\\frac{dy}{dt}(t)\end{bmatrix}$, $\frac{d\vec{x}}{dt}=\begin{bmatrix}\frac{dx}{dt}\\\frac{dy}{dt}\\\frac{d^2y}{dt^2}\end{bmatrix}=\begin{bmatrix}x+y\\dy/dt\\dy/dt\end{bmatrix}=\begin{bmatrix}1x_0(t)+1y_0(t)+0y_1(t)\\0x_0(t)+0y_0(t)+1y_1(t)\\0x_0(t)+0y_0(t)+1y_1(t)\end{bmatrix}=\begin{bmatrix}1&1&0\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_0(t)\\y_0(t)\\y_1(t)\end{bmatrix}$

$$\boxed{A=\begin{bmatrix}1&1&0\\0&0&1\\0&0&1\end{bmatrix}}$$

(b) $\operatorname{Det}(\lambda Id_{3\times3}-A)=\operatorname{Det}\begin{bmatrix}\lambda-1&-1&0\\0&\lambda&-1\\0&0&\lambda-1\end{bmatrix}=\boxed{\lambda(\lambda-1)^2}$, Eigenvalues,
$\lambda_0=0$ with (algebraic) multiplicity $e_0=1$
$\lambda_1=1$ with (algebraic) multiplicity $e_1=2$.

(c) $\lambda_0=0$, $A-\lambda_0 Id_{3\times3}=\begin{bmatrix}1&1&0\\0&0&1\\0&0&1\end{bmatrix}\overset{\text{Row Equiv}}{\sim}\begin{bmatrix}1&1&0\\0&0&1\\0&0&0\end{bmatrix}$ Reduced Row Echelon (leading, free, leading), $\operatorname{Ker}(A-\lambda_0 Id_{3\times3})=\operatorname{span}\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}\right)$, $\dim\operatorname{Ker}(A-\lambda_0 I)=1=e_0$
Stop: $\vec{v}_1=\begin{bmatrix}-1\\1\\0\end{bmatrix}$.

$\lambda_1=1$, $A-\lambda_1 Id_{3\times3}=\begin{bmatrix}0&1&0\\0&-1&1\\0&0&0\end{bmatrix}\overset{\text{Row Equiv}}{\sim}\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$ (free), $\operatorname{Ker}(A-\lambda_1 Id_{3\times3})=\operatorname{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$, $\dim\operatorname{Ker}(A-\lambda_1 I)=1<e_1$
Continue.

$(A-\lambda_1 Id_{3\times3})^2=\begin{bmatrix}0&-1&1\\0&1&-1\\0&0&0\end{bmatrix}\overset{\text{Row Equiv}}{\sim}\begin{bmatrix}0&1&-1\\0&0&0\\0&0&0\end{bmatrix}$ (free, free), $\operatorname{Ker}(A-\lambda_1 Id_{3\times3})^2=\operatorname{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right)$, $\dim\operatorname{Ker}(A-\lambda_1 I)^2=2=e_1$
Stop.

$\vec{v}_3=\begin{bmatrix}0\\1\\1\end{bmatrix}$, $\operatorname{Ker}(A-\lambda_1 I)^2=\operatorname{Ker}(A-\lambda_1 I)\oplus\operatorname{span}(\vec{v}_3)$, $\vec{v}_2=(A-\lambda_1 I)\vec{v}_3=\begin{bmatrix}1\\0\\0\end{bmatrix}$.

$\boxed{\mathcal{U}=[\vec{v}_1\,|\,\vec{v}_2\,|\,\vec{v}_3]=\begin{bmatrix}-1&1&0\\1&0&1\\0&0&1\end{bmatrix}}$, $\mathcal{U}^{-1}$: $\left[\begin{array}{ccc|ccc}-1&1&0&1&0&0\\1&0&1&0&1&0\\0&0&1&0&0&1\end{array}\right]$ $\begin{matrix}R_1'=R_2-R_3\\R_2'=R_1+R_2-R_3\\R_3'=R_3\end{matrix}$ $\rightarrow\left[\begin{array}{ccc|ccc}1&0&0&0&1&-1\\0&1&0&1&1&-1\\0&0&1&0&0&1\end{array}\right]$, $\mathcal{U}^{-1}=\begin{bmatrix}0&1&-1\\1&1&-1\\0&0&1\end{bmatrix}$.

$$\tilde{S}=\begin{array}{c}\begin{bmatrix}\lambda_0&0&0\\0&\lambda_1&0\\0&0&\lambda_1\end{bmatrix}\end{array}\begin{array}{l}e_0\\e_1\\ \end{array}=\begin{array}{c}\begin{array}{ccc}e_0&e_1&\end{array}\\\boxed{\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}}\end{array}$$

$\mathcal{U}\tilde{S}=[\vec{v}_1|\vec{v}_2|\vec{v}_3]\begin{bmatrix}\lambda_0&&\\&\lambda_1&\\&&\lambda_1\end{bmatrix}=[\lambda_0\vec{v}_1\,|\,\lambda_1\vec{v}_2\,|\,\lambda_1\vec{v}_3]$, $A\mathcal{U}=[A\vec{v}_1|A\vec{v}_2|A\vec{v}_3]=\cancel{[\lambda_0\vec{v}_1|\lambda_1\vec{v}_2+\vec{v}_3|\lambda_1\vec{v}_3]}$
$=[\lambda_0\vec{v}_1\,|\,\lambda_1\vec{v}_2\,|\,\vec{v}_2+\lambda_1\vec{v}_3]$

$A\mathcal{U}-\mathcal{U}\tilde{S}=\cancel{[\lambda_0\vec{v}_1|\lambda_1\vec{v}_2+\vec{v}_3|\lambda_1\vec{v}_3]-[\lambda_0\vec{v}_1|\lambda_1\vec{v}_2|\lambda_1\vec{v}_3]=[\vec{0}|\vec{v}_3|\vec{0}]=[\vec{v}_1|\vec{v}_2|\vec{v}_3]\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}}$
$=[\lambda_0\vec{v}_1\,|\,\lambda_1\vec{v}_2\,|\,\vec{v}_2+\lambda_1\vec{v}_3]-[\lambda_0\vec{v}_1\,|\,\lambda_1\vec{v}_2\,|\,\lambda_1\vec{v}_3]=[\vec{0}\,|\,\vec{0}\,|\,\vec{v}_2]$
$=[\vec{v}_1\,|\,\vec{v}_2\,|\,\vec{v}_3]\boxed{\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}}=\mathcal{U}\tilde{N}$

(d) $\tilde{S}t=\begin{bmatrix}0&0&0\\0&t&0\\0&0&t\end{bmatrix}$, $\exp(\tilde{S}t)=\begin{bmatrix}e^0&0&0\\0&e^t&0\\0&0&e^t\end{bmatrix}=\boxed{\begin{bmatrix}1&0&0\\0&e^t&0\\0&0&e^t\end{bmatrix}}$

$\tilde{N}t=\begin{bmatrix}0&0&0\\0&0&t\\0&0&0\end{bmatrix}$, $(\tilde{N}t)^2=0_{3\times3}$, $\exp(\tilde{N}t)=Id_{3\times3}+\frac{1}{1!}\tilde{N}t=\boxed{\begin{bmatrix}1&0&0\\0&1&t\\0&0&1\end{bmatrix}}$. Since $\tilde{S}\tilde{N}=\tilde{N}\tilde{S}$,

$\exp(\tilde{S}t+\tilde{N}t)=\exp(\tilde{S}t)\exp(\tilde{N}t)=\begin{bmatrix}1&0&0\\0&e^t&0\\0&0&e^t\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&t\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&e^t&te^t\\0&0&e^t\end{bmatrix}$. $At=\mathcal{U}(\tilde{S}t+\tilde{N}t)\mathcal{U}^{-1}$, thus

$\exp(At)=\mathcal{U}\exp(\tilde{S}t+\tilde{N}t)\mathcal{U}^{-1}=\begin{bmatrix}-1&1&0\\1&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&e^t&te^t\\0&0&e^t\end{bmatrix}\begin{bmatrix}0&1&-1\\1&1&-1\\0&0&1\end{bmatrix}=\begin{bmatrix}-1&e^t&te^t\\1&0&e^t\\0&0&e^t\end{bmatrix}\begin{bmatrix}0&1&-1\\1&1&-1\\0&0&1\end{bmatrix}=\boxed{\begin{bmatrix}e^t&-1+e^t&1-e^t+te^t\\0&1&-1+e^t\\0&0&e^t\end{bmatrix}}$

(e) $\vec{x}(t)=\exp(At)\cdot\vec{x}(0)=\begin{bmatrix}e^t&-1+e^t&+1-e^t+te^t\\0&1&-1+e^t\\0&0&e^t\end{bmatrix}\begin{bmatrix}b_0\\c_0\\c_1\end{bmatrix}=\begin{bmatrix}(-c_0+c_1)+(b_0+c_0-c_1)e^t+c_1te^t\\(c_0-c_1)+c_1e^t\\c_1e^t\end{bmatrix}$

Bonus $\frac{d\vec{x}}{dt}=A\vec{x}+\vec{g}$, $\vec{x}(0)=\vec{0}$, $\vec{g}(t)=\begin{bmatrix}e^t\\0\\0\end{bmatrix}$, $\exp(A(-s))\vec{g}(s)=\begin{bmatrix}e^{-s}&-1+e^{-s}&1-e^{-s}-se^{-s}\\0&1&-1+e^{-s}\\0&0&e^{-s}\end{bmatrix}\begin{bmatrix}e^s\\0\\0\end{bmatrix}=\boxed{\begin{bmatrix}1-e^t+te^t\\-1+e^t\\e^t\end{bmatrix}}$ $\boxed{\begin{matrix}x(t)=1-e^t+te^t\\y(t)=-1+e^t\end{matrix}}$ $=\begin{bmatrix}1\\0\\0\end{bmatrix}$

$\vec{x}(t)=\exp(At)\cdot\int_0^t\exp(A(-s))\vec{g}(s)\,ds$
$=\exp(At)\cdot\int_0^t\begin{bmatrix}1\\0\\0\end{bmatrix}ds=\exp(At)\cdot\begin{bmatrix}t\\0\\0\end{bmatrix}=\begin{bmatrix}e^t&-1+e^t&1-e^t+te^t\\0&1&-1+e^t\\0&0&e^t\end{bmatrix}\begin{bmatrix}t\\0\\0\end{bmatrix}=\boxed{\begin{bmatrix}te^t\\0\\0\end{bmatrix}}$ $\boxed{\begin{matrix}x(t)=te^t\\y(t)=0\end{matrix}}$

Name: ______________________________ **Problem 3:** ____________ /35

Problem 3(35 points) Let $(X, \mathcal{M}, \mu)$ be a complete measure space.

(a)(10 points) For every measurable $g : (X, \mathcal{M}) \to [0, \infty)$ with $\int_X g d\mu$ finite, prove that the subset

$$\mathrm{Supp}(g) := \{x \in X | g(x) \neq 0\}$$

is σ-finite, i.e., there exists a sequence of measurable subsets $S_1 \subseteq S_2 \subseteq \dots$ such that $\mu(S_n) < \infty$ for every n and $\cup_n S_n$ equals $\mathrm{Supp}(g)$.

(b)(10 points) For g as above, prove that there exists a sequence $(g_n)_{n=1}^{\infty}$ of measurable functions $g_n : (X, \mathcal{M}) \to [0, \infty)$ with $g_n \leq g_{n+1}$ for every n, with $(g_n(x))_{n=1}^{\infty}$ converging to $g(x)$ for every $x \in X$, and with $\mathrm{Supp}(g_n)$ a set of finite measure for every n.

(c)(15 points) For f in $L^1_{\mathbb{C}}(X, \mathcal{M}, \mu)$, prove that there exists a sequence $(f_n)_{n=1}^{\infty}$ of measurable functions $f_n : (X, \mathcal{M}) \to \mathbb{C}$ with $\mathrm{Supp}(f_n)$ a set of finite measure for every n and with $(f_n)_{n=1}^{\infty}$ convergent to f in L^1. Therefore the set of functions with finite measure support are dense in L^1.

(a) For integers $n > 0$, define $S_n := \{x \in X \mid g(x) \geq \frac{1}{n}\}$. Since g is measurable, $S_n = g^{-1}([\frac{1}{n}, \infty))$ is in $\mathcal{M}$. Moreover, $\int_X g\, d\mu \underset{\text{monotonicity in } X \,\&\, S_n}{\geq} \int_{S_n} g\, d\mu \underset{\text{monotonicity in } g \,\&\, \frac{1}{n}\chi_{S_n}}{\geq} \int_{S_n} \frac{1}{n}\chi_{S_n}\, d\mu = \frac{1}{n}\mu(S_n)$.
Therefore $\mu(S_n) \leq n \int_X g\, d\mu$ is finite.

Clearly $g^{-1}([\frac{1}{n}, \infty)) \subseteq g^{-1}([\frac{1}{n+1}, \infty))$ since $[\frac{1}{n}, \infty) \subseteq [\frac{1}{n+1}, \infty)$. Finally $(0, \infty) = \bigcup_{n=1}^{\infty} [\frac{1}{n}, \infty)$ so $\mathrm{Supp}(g) = \bigcup_{n=1}^{\infty} S_n$.

(b). Define $g_n := g \cdot \chi_{S_n}$. Since $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq S_{n+1} \subseteq \cdots$, also $g_1 \leq g_2 \leq \cdots \leq g_n \leq g_{n+1}$ ...
For every $x \in X \setminus \mathrm{Supp}(g)$, both $g(x)$ and every $g_n(x)$ equals 0, so $\lim_{n\to\infty} g_n(x) = 0 = g(x)$.
For every $x \in \mathrm{Supp}(g)$, since $\mathrm{Supp}(g) = \bigcup_{m=1}^{\infty} S_m$, $\exists m$ s.t. $x \in S_m$. So $\forall n \geq m$, $x \in S_n$ and thus $g_n(x) = g(x)$. Thus $\lim_{n\to\infty} g_n(x) = \lim_{\substack{n\to\infty \\ n\geq m}} g_n(x) = \lim_{\substack{n\to\infty \\ n\geq m}} g(x) = g(x)$.
Finally, $\mathrm{Supp}(g_n) = \mathrm{Supp}(g) \cap \mathrm{Supp}(\chi_{S_n})$ equals S_n, which has finite measure.

(c). Define $g(x) = |f(x)|$. Then g is as in (a). Define $f_n(x) = f(x) \cdot \chi_{S_n}(x)$.
Each of these is measurable & $|f_n(x)| \leq g(x)$. So $|f(x) - f_n(x)| \leq g(x) + g(x) = 2g(x)$ by the triangle inequality.
By the same argument as in (b), $f_n \to f$ pointwise,
i.e. $|f - f_n| \to 0$ pointwise.
Since all $|f - f_n|$ are dominated by $2g$, which is integrable, by the Dominated Convergence Theorem, $\lim_{n\to\infty} \int_X |f - f_n|\, d\mu \to \int_X 0\, d\mu = 0$, i.e. $(f_n)_{n=1}^{\infty}$ converges to f in L^1.

Name: ______________________________ **Problem 4:** ____________ /25

Problem 4(25 points) Let (X, d_X) and (Y, d_Y) be separable metric spaces. Denote by $\mathcal{B}_X$ and $\mathcal{B}_Y$ the corresponding Borel algebras. You may use without proof that the Borel algebra of $(X \times Y, d_{X\times Y})$ is simply $\mathcal{B}_X \otimes \mathcal{B}_Y$ (this uses the hypothesis that X and Y are separable). Let $f : X \to Y$ be a continous function.

(a)(5 points) Prove that the set $\Gamma_f := \{(x, y) \in X \times Y | y = f(x)\}$ is in $\mathcal{B}_X \otimes \mathcal{B}_Y$. (**Hint.** What kind of set is Γ_f? What kind of set is its complement?)

(b)(10 points) Let $\mu_X : \mathcal{B}_X \to [0, \infty]$ and $\mu_Y : \mathcal{B}_Y \to [0, \infty]$ be σ-finite measure functions. Assume that $\mu_Y(\{y\})$ equals 0 for every $y \in Y$. Compute the measure of Γ_f with respect to the product measure $\mu_{X\times Y}$.

(c)(10 points) Prove that the set $\{y \in Y |\, \mu_X(f^{-1}(\{y\})) > 0\}$ in Y has measure 0. (**Hint**. Compute the measure from (b) in a second way.)

(a) The map $(f, Id_Y) : X\times Y \to Y \times Y$ by $(x,y) \mapsto (f(x), y)$ is continuous since f is continuous. Also the map $d_Y : Y \times Y \to \mathbb{R}$ is continuous. So the composition $d_Y \circ (f, Id_Y) : X\times Y \to \mathbb{R}$ is continuous. The set Γ_f is the inverse image under this continuous map of the closed subset $\{0\} \subseteq \mathbb{R}$. Hence also Γ_f is closed in $X\times Y$. As the complement of an open set, Γ_f is Borel.

(b) Since Γ_f is a Borel set, the characteristic function $\chi_{\Gamma_f} : (X\times Y, \mathcal{B}_{X\times Y}) \to [0,\infty)$ is measurable. Thus, by Tonelli's theorem,

$$\mu_{X\times Y}(\Gamma_f) \overset{\text{Defn.}}{=} \int_{X\times Y} \chi_{\Gamma_f}\, d\mu_{X\times Y} \overset{\text{Tonelli}}{=} \int_X \left(\int_Y \chi_{\Gamma_f}(x,y)\, d\mu_Y(y) \right) d\mu_X(x).$$

For every $x \in X$, $\chi_{\Gamma_f}(x, \cdot) : Y \to [0,\infty)$ is the characteristic function of the singleton set $\{f(x)\}$. So $\int_Y \chi_{\Gamma_f}(x,y)\,d\mu_Y(y)$ equals $\mu_Y(\{f(x)\})$, which is 0 by hypothesis. Thus $\int_X \left(\int_Y \chi_{\Gamma_f}(x,y)\,d\mu_Y(y)\right) d\mu_X(x)$ equals $\int_X 0\, d\mu_X(x) = 0$. Therefore $\mu_{X\times Y}(\Gamma_f)$ equals 0.

(c) Again by Tonelli's theorem, $\int_{X\times Y} \chi_{\Gamma_f}\, d\mu_{X\times Y}$ equals $\int_Y \left(\int_X \chi_{\Gamma_f}(x,y)\,d\mu_X(x)\right) d\mu_Y(y)$. For fixed y, $\chi_{\Gamma_f}(\cdot, y) : X \to [0,\infty)$ is $\chi_{f^{-1}(\{y\})}$. Thus $\int_X \chi_{\Gamma_f}(x,y)\,d\mu_X(x)$ equals $\mu_X(f^{-1}(\{y\}))$.
For every integer $n > 0$, setting $S_n = \{y \in Y |\, \mu_X(f^{-1}(\{y\})) \geq \frac{1}{n}\}$, we have S_n is measurable and

$$0 \leq \frac{1}{n}\mu_Y(S_n) \overset{\text{Def}}{=} \int_{S_n} \frac{1}{n} \chi_{S_n}\, d\mu_Y \underset{\text{Monotonicity}}{\leq} \int_{S_n} \mu_X(f^{-1}(\{y\}))\,d\mu_Y(y) \underset{\text{Monot.}}{\leq} \int_Y \mu_X(f^{-1}(\{y\}))\,d\mu_Y(y) = \mu_{X\times Y}(\Gamma_f) = 0.$$

So every $\mu_Y(S_n)$ equals 0. Since μ_Y is continuous from below, $\mu_Y(\{y \in Y | \mu_X(f^{-1}(\{y\})) > 0\}) = \lim_{n\to\infty} \mu_Y(S_n) = 0$.

<u>Bonus</u>. $g_n(x) = \sum_{\substack{0\le k<2^n \\ k \text{ even}}} \chi_{[\frac{k}{2^n}, \frac{k+1}{2^n}]}$. Then $(f_n(x))_{n=1}^\infty$ converges uniformly to $f(x)=\frac{1}{2}x$.

For $m>n$, $g_n(x)-g_m(x) = \sum_{\substack{\frac14 \text{ of all} \\ k=0,\dots,2^m-1}} \pm \chi_{[\frac{k}{2^m}, \frac{k+1}{2^m}]}$, so $|g_n(x)-g_m(x)|$ equals 1 on a set of measure $\frac14$.

Thus $(g_{n_k}(x))_{k=1}^\infty$ is not Cauchy in measure for every subsequence. Note, however, that for every cts. $u:[0,1]\to[-R,R]$, $\int_{[0,1]} u g_n\, dm \to \int_{[0,1]} ug\, dm$, $g(x)=\frac12$. So g_n converges to g "weakly"!

Name: ______________________________ Problem 5: ____________ /40

Problem 5(40 points) Let $(g_n)_{n=1}^\infty$ be a sequence of Riemann integrable functions $g_n : [0,1] \to [-c,c]$. Define the sequence $(f_n)_{n=1}^\infty$ of functions on $[0,1]$ by

$$f_n(x) = \int_{[0,x]} g_n dm,$$

where m is Lebesgue measure.

(a)(10 points) Prove that f_n is c-Lipschitz. More generally, if $L-\epsilon < g_n(y) < L+\epsilon$ for every y in $(x-\delta, x+\delta)$, prove that

$$|f_n(y) - f_n(x) - L(y-x)| \le \epsilon|y-x|$$

for every y in $(x-\delta, x+\delta)$.

(b)(10 points) Prove that f_n is differentiable on the complement of a set of Lebesgue measure 0.

(c)(20 points) Prove that some subsequence $(f_{n_k})_{k=1}^\infty$ converges uniformly on $[0,1]$ to a c-Lipschitz function f.

Bonus problem(5 bonus points) Find an example as above where no subsequence of $(g_n)_{n=1}^\infty$ converges in measure. **Not to be written up.** For your example, what is the limit f? On the set where f is differentiable, what is the derivative g?

(a) Since $g_n - L\chi_{(x-\delta,x+\delta)}$ is in $(-\varepsilon,\varepsilon)$, we have for $x\le y<x+\delta$, by monotonicity of $\int dm$,

$$\int_{[x,y]} -\varepsilon\chi_{(x-\delta,x+\delta)}\, dm \le \int_{[x,y]} g_n\, dm - L\int_{[x,y]}\chi_{(x-\delta,x+\delta)}\,dm \le \int_{[x,y]}\varepsilon\chi_{(x-\delta,x+\delta)}\,dm, \text{ i.e.,}$$

$$-\varepsilon(y-x)\le f_n(y)-f_n(x)-L(y-x)\le +\varepsilon(y-x) \iff |f_n(y)-f_n(x)-L(y-x)|\le\varepsilon|y-x|.$$

A similar argument applies for $x-\delta<y\le x$. In particular, taking $L=0$ and $\varepsilon=c$, $|f_n(y)-f_n(x)|\le c\cdot|y-x|$, i.e., f_n is c-Lipschitz.

(b) Since g_n is <u>Riemann integrable</u>, $\mathrm{Disc}(g_n)$ has Lebesgue measure zero. For every $x\in[0,1]\setminus \mathrm{Disc}(g_n)$, since g_n is continuous at x, $\forall$ real $\varepsilon>0$ there exists real $\delta>0$ s.t. $\forall y\in(x-\delta,x+\delta)$, we have $g_n(x)-\varepsilon<g_n(y)<g_n(x)+\varepsilon$. So by (a), $|f_n(y)-f_n(x)-g_n(x)(y-x)|\le\varepsilon|y-x|$. Therefore f_n is differentiable at x with derivative $g_n(x)$.

(c) Since every f_n is c-Lipschitz, $\{f_n\}_{n=1}^\infty$ is <u>equicontinuous</u>. And since $f_n(0)=0$, also $|f_n(x)|\le c|x|$, so also $\{f_n\}_{n=1}^\infty$ is <u>pointwise bounded</u>. Thus, by <u>Arzela-Ascoli</u>, some subsequence of $(f_n)_{n=1}^\infty$ converges uniformly to continuous f. Also $|f(y)-f(x)| = \lim_{k\to\infty}|f_{n_k}(y)-f_{n_k}(x)|\le\lim_{k\to\infty} c|y-x| = c|y-x|$. So f is c-Lipschitz.