MAT 544 Final Exam Review

The policies regarding exams are posted on the exams part of the course webpage. The exam is closed book, closed notes, no electronic devices are allowed, and you need only bring a writing implement.

Review Topics. The final exam will be *cumulative*. Please look at the review sheets for Midterms 1 and 2, the problem sets and the solutions to the problem sets. There will be emphasis on topics covered since in Midterm 2. Please be familiar with all of the following concepts.

1. Measurable functions: Borel vs. Lebesgue, compatibility with composition, compatibility with product measure spaces, stable under sup, inf and limits, stable for positive and negative parts. Characteristic functions of measurable sets are measurable. Simple functions are measurable. Every nonnegative, measurable function is a pointwise limit of a monotone nondecreasing sequence of simple functions (and vice versa).

2. Integration of nonnegative, measurable functions: Integration of simple functions. Good properties include linearity (with nonnegative coefficients), monotonicity and countable additivity in the set E over which the simple function is integrated. Definition of the integral for general measurable functions. Monotonicity of the integral.

3. The Monotone Convergence Theorem. Countable additivity of the integral in the nonnegative, measurable function f. Fatou's Lemma.

4. Integration of real-valued, measurable functions. The set of integrable, real-valued functions is a real vector space, and integration is a linear functional. Integration of complex-value, measurable functions. The set of integrable, complex-valued functions is a complex vector space, and integration is a linear functional. The norm of the integral is less than or equal to the integral of the norm. The "complex measure" associated to an integrable, complex-valued function f is zero if and only if the L^1-norm equals 0 if and only if f equals 0 almost everywhere.

5. The Dominated Convergence Theorem. Completeness of $L^1_{\mathbb{C}}(X, \mu)$. Density of the subspace of simple functions, resp. span of characteristic functions of open sets in the Borel case, resp. continuous functions in the case of $(\mathbb{R}, m)$.

6. Comparison of Lebesgue measurable and Riemann integrable for bounded functions on a bounded closed interval. Riemann integrable implies Lebesgue measurable and both integrals agree. Riemann integrable if and only if the discontinuity set is a Lebesgue null set.

7. Modes of convergence: Cauchy in measure and convergence in measure. Convergence in L^1 implies convergence in measure. Cauchy in measure implies a subsequence converge to a measurable function both in measure and pointwise almost everywhere, moreover the entire sequence converges in measure (and the limit is unique off a null set). Egoroff's Theorem. Lusin's Theorem.

8. Product measures: Definition on rectangle sets. Well-definedness and countable additivity on the algebra of finite disjoint unions of rectangle sets. The Monotone Class Lemma. Tonelli's Theorem. Fubini's Theorem. The version of the Fubini-Tonelli Theorem for the associated complete measure space.

Some Practice Problems.

Problem 1 Problem 1.5, p. 24: For a subset $\mathcal{E}$ of $\mathcal{P}(X)$, the σ-algebra $\mathcal{M}(\mathcal{E})$ equals the union over all countable subset $\mathcal{F} \subset \mathcal{E}$ of $\mathcal{M}(\mathcal{F})$.

Problem 2 Problem 1.3, p. 24: For an algebra $\mathcal{A} \subset \mathcal{P}(X)$, even if $\mathcal{A}$ is countably infinite, the associated σ-algebra $\mathcal{M}(\mathcal{A})$ will be uncountable.

Problem 3 The intersection of any collections of σ-algebras is a σ-algebra.

Problem 4 Even for a linearly ordered collection of σ-algebras, the union may fail to be a σ-algebra. Describe the smallest σ-algebra containing the union.

Problem 5 For a finite, resp. σ-finite premeasure space, the associated measure space is also finite, resp. σ-finite.

Problem 6 The supremum of a linearly ordered collection of measure functions (on a fixed σ-algebra) is a measure function. If one of the constituent measures is complete, so is the supremum. If the supremum is finite, resp. σ-finite, then so is every constituent measure.

Problem 7 There exist non-linearly ordered collections of measure functions whose supremum is not a measure function.

Problem 8 For a collection Lebesgue-Stieltjes measures, what condition on the distribution functions corresponds to linear orderedness of the collection. Assuming the supremum measure is regular (i.e., finite on bounded sets), what is the distribution function?

Problem 9 If two premeasures $(X, \mathcal{A}, \mu_0)$ and $(X, \mathcal{B}, \nu_0)$ give the same outer measure, then for the algebra $\mathcal{C}$ generated by $\mathcal{A}$ and $\mathcal{B}$, there exists a premeasure function λ_0 on $\mathcal{C}$ which restricts to μ_0 on $\mathcal{A}$ and to ν_0 on $\mathcal{B}$.

Problem 10 Does the converse of the previous problem hold?

Problem 11 There exists a Borel measurable function $f : [0, 1] \to [0, 1]$ which is nowhere differentiable.

Problem 12 For every measurable function $f : (X, \mathcal{M}) \to (\mathbb{R} \setminus \{0\}, \mathcal{B})$, the function $1/f$ is also measurable.

Problem 13 For a metric space (Y, d_Y) with corresponding Borel algebra $\mathcal{B}_Y$, for every measurable function $f : (X, \mathcal{M}_X) \to (Y, \mathcal{B}_Y)$, for every point y of Y, the associated function

$$d_y(f) : (X, \mathcal{M}_X) \to (\mathbb{R}_{\geq 0}, \mathcal{B}_{\mathbb{R}}), \quad x \mapsto d_Y(y, f(x))$$

is measurable.

Problem 14 In the previous problem, assume that Y is separable. Then the converse holds: if $d_y(f)$ is measurable for every y in Y, then also f is measurable. Is there an example of a non-separable metric space where this fails?

Problem 15 For an integrable simple function ϕ, also ϕ^n is integrable for every integer $n \geq 1$.

Problem 16 For an integrable simple function ϕ, there exist nonnegative real numbers $m_1, \dots, m_r$ and complex numbers $z_1, \dots, z_r$ such that for every integer $n \geq 1$,

$$\int_X \phi^n d\mu = m_1 \cdot z_1^n + \cdots + m_r \cdot z_r^n.$$

Also show that this holds for an integrable function which equals a simple function almost everywhere.

Problem 17 Find an example of an integrable function f on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$ such that $|f|^2$ is not integrable. For every integer $n > 1$, find an example such that for all integers $1 \leq m < n$, $|f|^m$ is integrable yet $|f|^n$ is not integrable.

Problem 18 Let $(X, \mathcal{M}_X, \mu_X)$ be a measure space, let $(Y, \mathcal{M}_Y)$ be a set with a σ-algebra, and let $f : (X, \mathcal{M}) \to (Y, \mathcal{M}_Y)$ be a measurable function. Define $f_*\mu : \mathcal{M}_Y \to [0, \infty]$ by $f_*\mu(E) := \mu(f^{-1}(E))$. Prove that $f_*\mu$ is a measure function. Also prove that $f_*\mu$ is finite if and only if μ is finite. Finally, for a measurable function $g : (Y, \mathcal{M}_Y) \to [0, \infty]$, prove that $\int_Y g df_*\mu_X$ equals $\int_X g \circ f d\mu_X$.

Problem 19 Let $(X, \mathcal{M}_X, \mu_X)$ be a measure space. Let $f, g : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be measurable functions. If f equals g almost everywhere, prove that $f_*\mu$ equals $g_*\mu$.

Problem 20 For $([0,1], \mathcal{B}_{[0,1]}, m)$, give an example of measurable functions $f, g : [0,1] \to [0,1]$ such that f_*m equals g_*m, yet f equals g nowhere.

Problem 21 Let $(X, \mathcal{M}_X, \mu_X)$ be a finite measure space. Let $f, g : (X, \mathcal{M}_X) \to \mathbb{R}$ be measurable functions. Assume that for every $E \in \mathcal{M}_X$, for the restrictions $f_E, g_E : E \to \mathbb{R}$, the measures $(f_E)_*\mu_X$ and $(g_E)_*\mu_X$ are equal. Prove that f equals g almost everywhere. **Hint.** The Borel algebra on $\mathbb{R}$ is generated by sets of the form $F = [a/2^n, (a+1)/2^n)$ with a an integer. Now consider the composition of f, resp. g with χ_F, and show that there exists a set E, unique up to a null set, such that for every E', $\int_{E'} \chi_F \circ f d\mu_X$ equals $\int_X \chi_F \circ f d\mu_X$ if and only if $\mu_X(E \setminus E')$ equals 0. Use this to show that the sequences in Theorem 2.10 for f and g are equal almost everywhere.

Problem 22 Let $(X, \mathcal{M}_X, \mu_X)$ be a finite measure space. Let (f_n) be a sequence of measurable functions $f_n : (X, \mathcal{M}_X) \to \mathbb{R}$ which converge in measure to a measurable function f. Assume that for every $c \in \mathbb{R}$, the measure $f_*\mu_X(\{c\})$ equals 0. For every $E \in \mathcal{M}_X$ and for every set $I = (a, b) \subset \mathbb{R}$, prove that $(f_n)_{E,*}\mu_X(I)$ converges to $f_{E,*}\mu_X(I)$. **Hint.** Use Theorem 2.30 and then compose with χ_I.

Problem 23 Give an example of a sequence of measurable functions which converges to 0 in measure, yet which converges pointwise nowhere.

Problem 24 Give an example of a sequence of integrable functions (f_n) which converges in measure to 0, yet which does not converge to 0 in L^1.

Problem 25 Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $g : (X, \mathcal{M}) \to [0, \infty]$ be an integrable function. Let (f_n) be a sequence of integrable functions $f_n : (X, \mathcal{M}) \to (\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ such that every $|f_n|$ is bounded above by g. If (f_n) converges in measure to a function $f : X \to \mathbb{C}$, then f is integrable and (f_n) converges to f in L^1.

Problem 26 Find a sequence of measurable functions (f_n) on $([0, 1], \mathcal{B}_{[0,1]}, m)$ which converges almost everywhere to a measurable function f, yet which does not converge uniformly on any set of measure 1.

Problem 27 Exercises 2.45-2.52 on pp. 68-69.