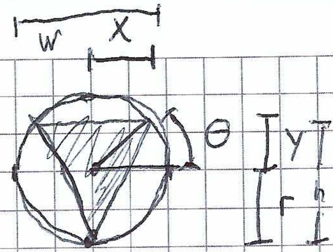



**PROBLEM 1.** Constants: r – radius

Independent variable: θ – angle, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Dependent variables: $x = r\cos\theta$, $y = r\sin\theta$, $w = 2x$ – width

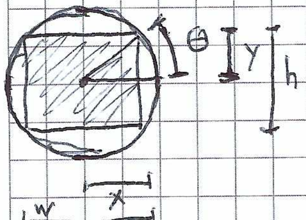
height – $h = r + y$, $A = \frac{hw}{2} = r^2\cos\theta(1+\sin\theta)$ – area

To maximize: $A(\theta) = 2r^2\cos\theta(1+\sin\theta)$. $A(-\frac{\pi}{2}) = 0 = A(\frac{\pi}{2})$.

$A'(\theta) = r^2\left((\cos\theta)^2 - (\sin\theta)^2 - \sin\theta\right) = r^2\left(\cos(2\theta) - \sin\theta\right)$.

Critical points: $\cos(2\theta) = \sin\theta \iff \theta = -\frac{\pi}{2}$ (endpoint) or $\theta = \frac{\pi}{6}$

$A(\frac{\pi}{6}) = r^2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{3}{2} = \frac{3\sqrt{3}}{2} r^2 = \pi r^2 \cdot \left(\frac{3\sqrt{3}}{4\pi}\right)$. **Fraction = $3\sqrt{3}/4\pi$**




**PROBLEM 2.** Constants: r – radius

Indep. variable: θ – angle, $0 \le \theta \le \pi/2$

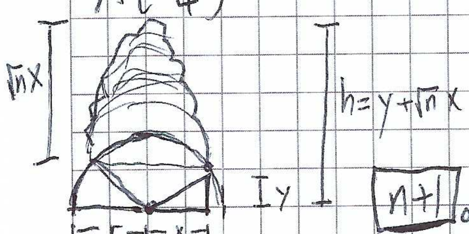
Depend. variables: $x = r\cos\theta$, $y = r\sin\theta$, $w = 2x = 2r\cos\theta$,

$h = 2y = 2r\sin\theta$, $A = hw = 4xy = 4r^2\sin\theta\cos\theta = 2r^2\sin(2\theta)$.

To maximize: $A(\theta) = 2r^2\sin(2\theta)$. $A(0) = 0$, $A(\frac{\pi}{2}) = 0$.

Critical Points: $A'(\theta) = 4r^2\cos(2\theta)$, $\cos(2\theta) = 0 \iff \theta = \pi/4$.

$A(\frac{\pi}{4}) = 2r^2 = \pi r^2 \cdot \left(\frac{2}{\pi}\right)$. **Fraction = $\frac{2}{\pi}$**




**PROBLEM 3.** By way of induction, for fixed positive integer n, assume the maximum height is $\sqrt{n}$ · base radius.

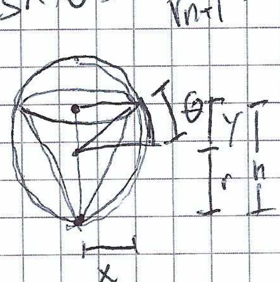
**n+1**: Constants: r – radius. Indep. var.: θ – angle, $0 \le \theta \le \frac{\pi}{2}$

Dep. vars: $x = r\cos\theta$, $y = r\sin\theta$, $\sqrt{n}\,x = \sqrt{n}\,r\cos\theta$, $h = y + \sqrt{n}\,x = r(\sin\theta + \sqrt{n}\cos\theta)$.

To maximize: $h(\theta)$, $h(0) = \sqrt{n}\,r$, $h(\frac{\pi}{2}) = r$

Critical points: $h'(\theta) = r(\cos\theta - \sqrt{n}\sin\theta)$. $h'(\theta) = 0 \iff \tan\theta = \frac{1}{\sqrt{n}}$

$\sin\theta = \frac{1}{\sqrt{n+1}}$, $\cos\theta = \frac{\sqrt{n}}{\sqrt{n+1}}$, $h = r \cdot \left(\frac{1}{\sqrt{n+1}} + \frac{n}{\sqrt{n+1}}\right) = \boxed{\sqrt{n+1} \cdot r}$ Proved by induction.




**PROBLEM 4.** Constants: r – radius. Indep. Vars: θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Dep. Vars: $x = r\cos\theta$, $y = r\sin\theta$, $h = r + y = r(1+\sin\theta)$.

$V = \frac{1}{3}\pi x^2 \cdot h = \frac{1}{3}\pi(r\cos\theta)^2 r(1+\sin\theta) = \frac{\pi r^3}{3}(\cos\theta)^2(1+\sin\theta)$.

To maximize: $V(\theta) = \frac{\pi r^3}{3}(1-(\sin\theta)^2)(1+\sin\theta) = \frac{\pi r^3}{3}(1+\sin\theta)^2(2-(1+\sin\theta))$

Critical points: $V'(\theta) = \frac{\pi r^3}{3}\left(4(1+\sin\theta) - 3(1+\sin\theta)^2\right)\cos\theta$, $\theta = -\frac{\pi}{2}, \frac{\pi}{6}$