MAT131 Fall 2022 Midterm 2 Review Sheet

The topics tested on Midterm 2 will be among the following.

(i) Given a function, its first derivative and its second derivative, use the first derivative test to determine where the function is increasing and where decreasing, and use the second derivative test to determine where the function is concave up and where concave down. Combine this with even/odd, identification of vertical and horizontal asymptotes, and identification of discontinuities to give a rough sketch of the graph of the function.

(ii) Use the rules of differentiation: the sum rule, the product rule, the power rule and the derivatives of exponentials.

(iii) Given the limits $\lim_{h\to 0}\dfrac{\sin(x)}{x} = 1$ and $\lim_{h\to 0}\dfrac{1-\cos(x)}{x} = 0$, use trigonometric identities to find the formulas for the derivatives of $\sin(x)$ and $\cos(x)$ as limits of difference quotients.

(iv) Find the derivatives of other trigonometric functions – $\tan(x)$, $\cot(x)$, $\sec(x)$ and $\csc(x)$ – using the derivatives for $\sin(x)$ and $\cos(x)$ and the rules for differentiation.

(v) Find derivatives using the chain rule.

(vi) Find derivatives using implicit differentiation, including derivatives of inverse functions.

(vii) Find derivatives using logarithmic differentiation.

(viii) Use implicit differentiation of constraint equations between differentiable functions to find the "related rates" among those functions.

(ix) Find the linear approximation to the value of a function, using a known nearby value and the derivative or using differentials. Determine whether your approximation is an overestimate or an underestimate.

(x) Understanding differential notation and the geometric interpretation of differentials. Using differentials to approximate values of functions.

(xi) L'Hôpital's rule. Recognize indeterminate forms. Simplify limits leading to $0/0$ and ∞/∞ indeterminate forms using l'Hôpital's rule. Know how to transform other indeterminate forms into one of these two types.

The midterm **will not** include applied optimization problems. Following are some practice problems. More practice problems are in the textbook as well as on the practice midterms.

Problem 1. In each of the following cases, a function $y = f(x)$ is given as well as the first derivative $f'(x)$ and the second derivative $f''(x)$. In each case do all of the following.

(i) Compute the first and second derivatives of $f(x)$ to verify the given formulas.

(ii) Determine whether $f(x)$ is even, odd or neither.

(iii) Find the equations of all vertical and horizontal asymptotes.

(iv) Determine where $y = f(x)$ is increasing and where $y = f(x)$ is decreasing. Pay careful attention to the change as x crosses a vertical asymptote.

(v) Determine where $y = f(x)$ is concave up and where $y = f(x)$ is concave down. Pay careful attention to the change as x crosses a vertical asymptote.

(vi) Give a rough sketch of the graph of $y = f(x)$ carefully labelling all of the above information on your sketch.

(a)

$$f(x) = x^3 - x,$$
$$f'(x) = 3x^2 - 1,$$

$$f''(x) = 6x.$$

(b)

$$f(x) = x^4 - 2x^2,$$
$$f'(x) = 4x^3 - 4x,$$
$$f''(x) = 12x^2 - 4.$$

(c)

$$f(x) = x^{-1}e^x = \frac{e^x}{x},$$
$$f'(x) = (x-1)x^{-2}e^x = \frac{(x-1)e^x}{x^2},$$
$$f''(x) = (x^2 - 2x + 2)x^{-3}e^x = \frac{(x^2 - 2x + 2)e^x}{x^3}.$$

(d)

$$f(x) = x^{-2}e^x = \frac{e^x}{x^2},$$
$$f'(x) = (x-2)x^{-3}e^x = \frac{(x-2)e^x}{x^3},$$
$$f''(x) = (x^2 - 4x + 6)x^{-4}e^x = \frac{(x^2 - 4x + 6)e^x}{x^4}.$$

(e)

$$f(x) = x^{-3}e^x = \frac{e^x}{x^3},$$
$$f'(x) = (x-3)x^{-4}e^x = \frac{(x-3)e^x}{x^4},$$
$$f''(x) = (x^2 - 6x + 12)x^{-5}e^x = \frac{(x^2 - 6x + 12)e^x}{x^5}.$$

(f)

$$f(x) = x^{-1}\ln(x^2) = \frac{\ln(x^2)}{x}$$
$$f'(x) = x^{-2}(2 - \ln(x^2)) = \frac{2 - \ln(x^2)}{x^2},$$

$$f''(x) = x^{-3}(2\ln(x^2) - 6) = \frac{2\ln(x^2) - 6}{x^3}.$$

(g)

$$f(x) = x^{-2}\ln(x^2) = \frac{\ln(x^2)}{x^2},$$

$$f'(x) = x^{-3}(2 - 2\ln(x^2)) = \frac{2 - 2\ln(x^2)}{x^3},$$

$$f''(x) = x^{-4}(6\ln(x^2) - 10) = \frac{6\ln(x^2) - 10}{x^4}.$$

(h)

$$f(x) = (1 + x^{-2})^{1/2} = \frac{1}{\sqrt{1 + (1/x)^2}},$$

$$f'(x) = -x^{-3}(1 + x^{-2})^{-1/2} = \frac{-1}{x^3\sqrt{1 + (1/x)^2}},$$

$$f''(x) = (3x^2 + 2)x^{-6}(1 + x^{-2})^{-3/2} = \frac{3x^2 + 1}{x^6(\sqrt{1 + (1/x)^2})^3}.$$

Solution to Problem 1 There is a sheet with scanned solutions to this problem at http://www.math.stonybrook.edu/ jstarr/mat131.fall22/exams.html.

Problem 2. What is the differential of $y = x^3$? When $x = 2$, a small change $dx = 0.01$ in x produces what change dy in y?

Solution to Problem 2 Since $y' = 3x^2$,

$$\frac{dy}{dx} = y' = 3x^2, \quad \boxed{dy = 3x^2 dx.}$$

When $x = 2$, this gives $dy = 3(2)^2 dx = 12dx$. Thus a small change $dx = 0.01$ produces a change $dy = 12 \times (0.01)$ or $\boxed{0.12}$ in y. Since $y(2) = 8$, this gives $y(2.01) \approx 8.12$.

Problem 3. The inverse function of

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

is called the *inverse hyperbolic cosine*, $\cosh^{-1}(x)$. For $y = \cosh^{-1}(x)$, find a formula for the derivative of y that is an expression only involving polynomials in x and square roots.

Hint. After you find an answer that involves y, consider what you get by squaring your formula for y' and squaring your formula for x. Can you relate these two formulas?

Solution to Problem 3 Since $y = f^{-1}(x)$, this gives $f(y) = x$. This is the implicit equation

$$\frac{1}{2}(e^y + e^{-y}) = x.$$

Implicit differentiation gives

$$\frac{1}{2}(e^y - e^{-y})y' = 1, \quad y' = \frac{2}{e^y - e^{-y}}.$$

Squaring both sides gives

$$(y')^2 = \frac{4}{e^{2y} - 2 + e^{2y}}.$$

On the other hand, squaring both sides of the original implicit equation gives

$$\frac{e^{2y} + 2 + e^{-2y}}{4} = x^2.$$

Thus

$$\frac{e^{2y} - 2 + e^{-2y}}{4} = \frac{e^{2y} + 2 + e^{-2y}}{4} - 1 = x^2 - 1.$$

Substituting this into the equation for $(y')^2$ gives

$$(y')^2 = \frac{1}{x^2 - 1}.$$

On the interval $[1, \infty)$ where $\cosh^{-1}(x)$ is usually defined, y' is given by the positive square root,

$$\boxed{y' = 1/\sqrt{x^2 - 1}.}$$

Problem 4. The double-angle formula for tangent is

$$\tan(2x) = \frac{2\tan(x)}{1 - (\tan(x))^2}.$$

Compute the derivative of each side of this equation. Which derivative is easier to compute?

Solution to Problem 4 The derivative of $\tan(u)$ is given by

$$d\tan(u) = \sec^2(u)du.$$

Thus, by the chain rule,

$$d\tan(2x) = \sec^2(2x)d(2x) = \sec^2(2x)2dx.$$

This gives

$$\boxed{\frac{d\tan(2x)}{dx} = 2\sec^2(2x).}$$

And the quotient rule gives

$$\frac{d}{dx}\left(\frac{2\tan(x)}{1-(\tan(x))^2}\right) = \frac{(1-\tan^2(x))(2\sec^2(x)) - (2\tan(x))(-2\tan(x)\sec^2(x))}{(1-\tan^2(x))^2}.$$

After simplification this gives

$$\frac{2\sec^2(x)(1+\tan^2(x))}{(1-\tan^2(x))^2}.$$

Using the identity that

$$1+\tan^2(x) = \sec^2(x),$$

this becomes

$$\frac{d}{dx}\left(\frac{2\tan(x)}{1-(\tan(x))^2}\right) = \boxed{2\sec^4(x)/(1-\tan^2(x))^2.}$$

Problem 5. Let a be a positive constant and consider the parametric curve

$$\begin{cases} x(t) &= 2at \\ y(t) &= \frac{2a}{1+t^2} \end{cases}$$

Compute the slope of the tangent line at the point where $t = 1/\sqrt{3}$.

Solution to Problem 5. The domain consists of all real numbers such that the denominator is nonzero, i.e., all real numbers except $x = -3$. In interval notation, this is $\boxed{(-\infty, -3) \cup (-3, +\infty)}$.

The implicit form of the equation for y is

$$(x+3)y-(2x+5)=0, \text{ i.e., } xy+3y-2x-5=0.$$

Solving for x, this gives,

$$x=g(y)=\boxed{(-3y+5)/(y-2)}.$$

Thus the domain of the inverse function is all real numbers except $y=2$. In interval notation, this is $\boxed{(-\infty,2)\cup(2,\infty)}$.

The derivatives are

$$f'(x)=\frac{1}{(x+3)^2}\left(2\cdot(x+3)-(2x+5)\cdot 1\right)=\boxed{1/(x+3)^2},$$

$$g'(y)=\frac{1}{(y-2)^2}\left((-3)(y-2)-(-3y+5)(1)\right)=\boxed{1/(-y+2)^2}.$$

Notice that $x=-1/(-y+2)+3$, so that $(x-3)=1/(-y+2)$. Thus, the derivative $g'(y)$ equals $(x-3)^2$. Therefore, the product of derivatives equals,

$$f'(x)g'(y)=f'(x)(x-3)^2=\frac{1}{(x-3)^2}\cdot(x-3)^2=\boxed{1}.$$

Problem 6. Repeat the previous problem for the differentiable function $y=f(x)=(rx+s)/(tx+u)$, where r, s, t and u are specified real numbers such that $ru-st$ is nonzero.

Solution to Problem 6. This is similar to the previous problem. The domain equals $\boxed{(-\infty,-u/t)\cup(-u/t,+\infty)}$ unless t equals 0, in which case the domain is $\boxed{(-\infty,+\infty)}$.

The inverse function is

$$y=g(x)=\boxed{(-uy+s)/(ty-r)}.$$

The domain of the inverse function equals $\boxed{(-\infty,r/t)\cup(r/t,+\infty)}$ unless t equals 0, in which case the domain is $\boxed{(-\infty,+\infty)}$.

The derivatives are

$$f'(x)=\boxed{(ru-st)/(tx+u)^2},\quad g'(y)=\boxed{(ru-st)/(ty-r)^2}.$$

Since $(tx+u)$ equals $-(ru-st)/(ty-r)$, it follows that $(tx+u)^2/(ru-st)$ equals $(ru-st)/(ty-r)^2$. Therefore, the product of derivatives is,

$$f'(x)g'(y) = f'(x)(ru-st)/(ty-r)^2 = f'(x)(tx+u)^2/(ru-st) = \left((ru-st)(tx+u)^2\right) / \left((ru-st)(tx\right.$$

Problem 7. Compute each of the following derivatives.

(a)

$$y = \ln(\ln(x)),\ x > 1$$

Solution to (a)

$$y' = \frac{1}{\ln(x)}\frac{1}{x} = \boxed{1/(x\ln(x)).}$$

(b)

$$y = e^{e^x}$$

Solution to (b)

$$y' = e^{e^x}e^x = \boxed{e^{x+e^x}.}$$

(c)

$$y = \frac{2x}{1+x^2}$$

Solution to (c)

$$y' = \frac{(1+x^2)(2)-(2x)(2x)}{(1+x^2)^2} = \boxed{2(1-x^2)/(1+x^2)^2.}$$

(d)

$$y = \frac{x^3\sqrt{\sin(x)}}{\sqrt{\cos(x)}}$$

Solution to (d) Denote by u the logarithm

$$u = \ln(y) = 3\ln(x) + \frac{1}{2}\ln(\sin(x)) - \frac{1}{2}\ln(\cos(x)).$$

Then

$$\frac{du}{dx} = \frac{3}{x} + \frac{1}{2}\frac{1}{\sin(x)}\cos(x) - \frac{1}{2}\frac{1}{\cos(x)}(-\sin(x)).$$

Simplifying, this becomes

$$\frac{du}{dx} = \frac{6\sin(x)\cos(x)+1}{2x\sin(x)\cos(x)}.$$

And since also

$$du = \frac{1}{y}dy, \quad dy = ydu$$

this gives

$$\frac{dy}{dx} = y\frac{du}{dx} = \boxed{x^2(6\sin(x)\cos(x)+1)/(2\sqrt{\sin(x)\cos^3(x)}).}$$

(e)

$$y = \ln(-x+\sqrt{x^2-1})$$

(Simplify your answer as much as possible.)

Solution to (e)

$$y' = \frac{1}{-x+\sqrt{x^2-1}}\frac{d}{dx}(-x+\sqrt{x^2-1}) = \frac{1}{-x+\sqrt{x^2-1}}\left(-1+\frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{-x+\sqrt{x^2-1}}\frac{x-\sqrt{x}}{\sqrt{x^2}}$$

Simplifying gives

$$y' = \boxed{1/\sqrt{x^2-1}.}$$

Problem 8. Using your known formulas for the derivatives of $\sin(x)$ and $\cos(x)$, find the limit

$$\lim_{h\to 0}\frac{\cos(h)-1}{h}$$

by relating it to the derivative of $\cos(x)$ for some value of x.

Solution to Problem 8 By definition, the derivative of $\cos(x)$ at $x=0$ is

$$\lim_{h\to 0}\frac{\cos(h)-\cos(0)}{h}.$$

Since $\cos(0)=1$, this gives

$$\frac{d\cos(x)}{dx}|_{x=0} = \lim_{h\to 0}\frac{\cos(h)-1}{h}.$$

On the other hand, the formula for the derivative is

$$\frac{d\cos(x)}{dx} = -\sin(x).$$

Since $\sin(0) = 0$, this gives

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = \boxed{0.}$$

Problem 9. The value of $1/(1+\sqrt{10})$ is close to $1/4 = 0.25$. Using a linear approximation or differentials, estimate whether the true value is closer to 0.2 or closer to 0.3.

Solution to Problem 9 Observe 10 is close to 9 and $\sqrt{9} = 3$. Thus let

$$y = \frac{1}{1+\sqrt{x}}.$$

The differential of y is

$$dy = \frac{-1}{2\sqrt{x}(1+\sqrt{x})^2}dx.$$

Thus at $x = 9$ the differential is

$$dy = \frac{-1}{2(3)(1+3)^2}dx = \frac{-1}{96}dx.$$

So the change $dx = 10 - 9 = 1$ gives a change $dy = (-1/96)(1)$ for x close to 9. This gives a linear approximation

$$\frac{1}{1+\sqrt{10}} \approx \frac{1}{4} - \frac{1}{96} = \boxed{23/96}$$

which is closer to $\boxed{0.2}$ than to 0.3.

Problem 10. For the curve with implicit equation

$$y + \frac{1}{y} = 4x + 2x^2$$

find the slope of the tangent line to the curve at the point $(x, y) = (1/2, 2)$.

Solution to Problem 10 Differentiating both sides gives

$$\frac{y^2-1}{y^2}y' = 4+4x.$$

When y is not 0 or ± 1, we can divide and get

$$y' = \frac{4(x+1)y^2}{y^2-1}.$$

Plugging in $(x, y) = (1/2, 2)$ gives

$$y' = \boxed{8.}$$

Problem 11. Find the derivative y' of the function y in each of the following cases.

(a)

$$y = \sin(\sqrt{x^2+1})$$

Solution to (a)

$$\frac{dy}{dx} = \cos(\sqrt{x^2+1})\left(\frac{1}{2}(x^2+1)^{-1/2}\right)(2x) = \boxed{x\cos(\sqrt{x^2+1})/\sqrt{x^2+1}.}$$

(b)

$$y = (\ln(x^2))^2$$

Solution to (b) First of all,

$$y = (2\ln(x))^2 = 4(\ln(x))^2.$$

Thus,

$$\frac{dy}{dx} = 4(2\ln(x))\left(\frac{1}{x}\right) = \boxed{8\ln(x)/x.}$$

(c)

$$y = (\sqrt{x})^{\cos(x)}$$

Solution to (c) Setting $u = \ln(y)$ gives

$$u = \ln(y) = \ln(\sqrt{x}^{\cos(x)}) = \cos(x)\ln(\sqrt{x}) = \frac{1}{2}\cos(x)\ln(x).$$

The derivative of this is

$$\frac{du}{dx} = \frac{1}{2}(-\sin(x))\ln(x) + \frac{1}{2}\cos(x)\left(\frac{1}{x}\right) = \frac{\cos(x) - x\sin(x)\ln(x)}{2x}.$$

Since also $u' = y'/y$, this gives $y' = yu'$ or

$$\frac{dy}{dx} = y\frac{du}{dx} = (\sqrt{x})^{\cos(x)}\frac{\cos(x) - x\sin(x)\ln(x)}{2x} = \boxed{(\cos(x) - x\sin(x)\ln(x))(\sqrt{x})^{\cos(x)}/2x.}$$

(d)

$$y = \frac{x^2 e^x}{x^2 + 2x + 1}$$

Solution to (d) Setting $u = \ln(y)$ gives

$$u = \ln(y) = \ln(x^2e^x/(x^2+2x+1)) = 2\ln(x)+x-\ln(x^2+2x+1) = 2\ln(x)+x-2\ln(x+1)$$

The derivative of this is

$$\frac{du}{dx} = \frac{2}{x} + 1 - 2\frac{1}{x+1} = \frac{x^2 + x + 2}{x(x+1)}.$$

Since also $u' = y'/y$, this gives $y' = yu'$ or

$$\frac{dy}{dx} = y\frac{du}{dx} = \frac{x^2e^x}{(x+1)^2}\frac{x^2 + x + 2}{x(x+1)} = \boxed{x(x^2 + x + 2)e^x/(x+1)^3.}$$

(e)

$$x^2 + 4y^2 = 5$$

Solution to (e) Implicitly differentiating both sides with respect to x gives

$$2x + 8y\frac{dy}{dx} = 0.$$

Solving for the derivative gives

$$\frac{dy}{dx} = \boxed{-x/4y.}$$

Solving for y as well gives an answer depending only on x,

$$\frac{dy}{dx} = \boxed{-x/(2\sqrt{5 - x^2}).}$$

(f)

$$y = \frac{(x^2+1)^3}{x^2-1}$$

Solution to (f) Rewrite this as

$$y = \frac{(x^2+1)^3}{(x+1)(x-1)}.$$

Setting $u = \ln(y)$ gives

$$u = \ln(y) = \ln\left(\frac{(x^2+1)^3}{(x+1)(x-1)}\right) = 3\ln(x^2+1) - \ln(x+1) - \ln(x-1).$$

The derivative of this is

$$\frac{du}{dx} = 3\frac{1}{x^2+1}(2x) - \frac{1}{x+1} - \frac{1}{x-1} = \frac{4x(x^2-2)}{(x^2+1)(x+1)(x-1)}$$

Since also $u' = y'/y$, this gives $y' = yu'$ or

$$\frac{dy}{dx} = y\frac{du}{dx} = \frac{(x^2+1)^3}{(x+1)(x-1)}\frac{4x(x^2-2)}{(x^2+1)(x+1)(x-1)} = \boxed{4x(x^2-2)(x^2+1)^2/((x+1)(x-1)).}$$

Problem 12 In each of the following two cases, find the linearization of $f(x)$ near the point $x = a$.

(a)

$$f(x) = x^{2/5}, \quad a = 32.$$

Solution to (a) The derivative is

$$f'(x) = \frac{2}{5}x^{-3/5}.$$

Plugging in $x = 32$ so that $x^{1/5} = 2$, this gives

$$f'(32) = \frac{2}{5}2^{-3} = \frac{1}{20}.$$

So the linearization is

$$f(x) \approx f(a) + f'(a)(x-a) = \boxed{2 + (1/20)(x-32).}$$

(b)

$$f(x) = \frac{1}{\sqrt{1+x^2}}, \quad a = \sqrt{3}.$$

Solution to (b) The derivative is

$$f'(x) = \frac{-1}{2}(1+x^2)^{-3/2}(2x) = \frac{-x}{(1+x^2)^{3/2}}.$$

Plugging in $a = \sqrt{3}$ so that $f(\sqrt{3}) = 1/\sqrt{1+3} = 1/2$, the derivative is

$$f'(\sqrt{3}) = \frac{-\sqrt{3}}{(1/2)^3} = -8\sqrt{3}.$$

So the linearization is

$$f(x) \approx f(a) + f'(a)(x-a) = \boxed{(1/2) - 8\sqrt{3}(x-\sqrt{3}).}$$

Problem 13 Using differentials or an appropriate linear approximation, approximate the following number.

$$\frac{1}{\sqrt{25.1}}$$

Solution to Problem 13 Let $f(x) = x^{-1/2}$ and let $a = 25$ so that $f(a) = 1/5$. The differential is

$$dy = df(x) = \frac{-1}{2}x^{-3/2}dx.$$

When $x = 25$ this gives

$$dy = \frac{-1}{2}5^{-3}dx = -\frac{1}{250}dx.$$

Thus the linear approximation is

$$f(25.1) \approx 1/5 - \frac{1}{250}(0.1) = \boxed{0.1996} = \boxed{499/2500.}$$

Problem 14 A calculus instructor of height 170 cm is moving at speed 5.2 km/hr away from a street lamp of height 3m. What is the speed of the shadow of the instructor's head?

Solution to Problem 14 There is a sheet with scanned solutions to this problem at http://www.math.stonybrook.edu/ jstarr/mat131.fall22/exams.html.

Problem 15 A long, straight piece of wood of length 7 meters rests with its foot on the ground and its midpoint on the top of a 3 meter tall fence. The foot of the plank begins to slide straight away from the fence with the plank still touching the top of the fence. At the moment when the foot of the plank is 4 meters from the base of the fence, the distance between the top of the plank and the ground is decreasing at 0.3 meters per second. At what speed is the foot of the plank moving away from the base of the fence?

Solution to Problem 15 There is a sheet with scanned solutions to this problem at http://www.math.stonybrook.edu/ jstarr/mat131.fall22/exams.html.

Problem 16 In a flat ceiling two hooks are fastened 21 cm apart. A length of 27 cm of inextensible wire is suspended between the hooks. A heavy weight is hung from the wire near the first hook and slides along the wire toward a point equidistant from both hooks, pulling the wire taut at each moment. At the moment when the weight is 10 cm from the first hook, it is moving away from the first hook at a speed of 1 cm/sec.

(a) With what speed is the weight moving towards the second hook at this moment?

(b) What is distance between the weight and the ceiling at this moment?

(c) With what speed is the weight moving away from the ceiling (i.e., what is the rate of change of the distance between the weight and the ceiling)?

Solution to Problem 16 There is a sheet with scanned solutions to this problem at http://www.math.stonybrook.edu/ jstarr/mat131.fall22/exams.html.

Problem 17 A car approaches a large hotel at night, driving along a semi-circular driveway which becomes tangent to the front wall of the hotel at the entrance. The headlights of the car illuminate a spot on the front wall. If the radius of the driveway is 10 meters, and if the car is moving at a speed of 10 km/hour, with what speed is the spot moving when the angle between the entrance, the center of the circle and the car is $\pi/3$ radians, i.e., 60 degrees?

Solution to Problem 17 There is a sheet with scanned solutions to this problem at http://www.math.stonybrook.edu/ jstarr/mat131.fall22/exams.html.

Problem 18 A cube of ice rests on a hot plate. It melts in such a way that its shape at every moment is a cube and the rate of decrease of the volume is a constant multiple of the area of the face resting on the hot plate. If after 5 minutes the volume of the cube is one eighth of its initial volume, how much longer is required before the cube melts entirely?

Solution to Problem 18 There is a sheet with scanned solutions to this problem at http://www.math.stonybrook.edu/ jstarr/mat131.fall22/exams.html.

Problem 19. Let $u(x)$ and $v(x)$ be continuously differentiable functions defined on an open interval containing $x = a$ such that $u(a)$ and $v(a)$ both equal 0, and such that the limit as x approaches a of $u(x)/v(x)$ is defined and equals $L \neq 0$. Consider the functions $f(x) = 1/v(x)$ and $g(x) = 1/u(x)$. What kind of indeterminate form arises from the limit as x approaches a of $f(x)/g(x)$? What are the derivatives $f'(x)$ and $g'(x)$ as expressions in $u(x)$, $u'(x)$, $v(x)$ and $v'(x)$? What is the limit as x approaches a of $f'(x)/g'(x)$ when expressed in these terms? What is the limit as x approaches a of $f(x)/g(x)$? Does the limit of $f(x)/g(x)$ equal the limit of $f'(x)/g'(x)$?

Solution to Problem 19 The ∞/∞ indeterminate form of u/v equals the $0/0$ indeterminate form of f/g, so we can apply L'Hôpital's rule to compute the limit as the limit of f'/g' (if it exists). The quotient rule gives $f' = -v'/v^2$ and $g' = -u'/u^2$. Thus, the fraction of derivatives f'/g' equals $(u/v)^2/(u'/v')$. By hypothesis, u'/v' limits to L. By L'Hôpital's rule, also u/v limits to L. Thus, $(u/v)^2/(u'/v')$ also limits to L^2/L. In other words, f'/g' also limits to $\boxed{L}$.

Problem 20. Let $u(x)$ and $v(x)$ be continuously differentiable functions defined on an open interval containing $x = a$ such that $u(a)$ and $v(a)$ both equal 0, such that $v(x)$ is positive for $x \neq a$, and such that the limit as x approaches a of $u'(x)/v'(x)$ is defined and equals L. Define $h(x) = e^{1/v(x)}$ as a continuously differentiable function on the complement of $x = a$ in the interval. Rewrite $u(x)/v(x)$ as $\ln(h(x)^{u(x)})$. What kind of indeterminate form arises as the limit of x to a of $h(x)^{u(x)}$? Show that the limit as x approaches a of $-(h'(x)/h(x))/(u'(x)/u(x)^2)$ is defined and equals L, and show that the limit as x approaches a of $h(x)^{u(x)}$ exists and equals e^L.

Solution to Problem 20 The ∞^0 indeterminate form h^u equals the exponential of the $0/0$ indeterminate form u/v. By the same method as above, the original limit equals the exponential of the limit of $(\ln(h))'/(1/u)' = (h'/h)/(-u'/u^2)$. This simplifies to $(u/v)^2/(u'/v')$ just as above. Thus the original limit equals $\boxed{e^L}$.

Problem 21. Let $u(x)$ and $v(x)$ be continuously differentiable functions defined on an open interval containing $x = 0$ such that $u(0)$ and $v(0)$ both equal 0, and such that the limit as x approaches 0 of $u'(x)/v'(x)$ is defined and equals L. Define $f(x) = u(1/x^2)$ and $g(x) = v(1/x^2)$. Show that the limit as x approaches $+\infty$ of $f(x)/g(x)$ gives a 0/0 indeterminate form. Compute $f'(x)$ and $g'(x)$ in terms of $u'(x)$ and $v'(x)$, and describe the limit as x approaches $+\infty$ of $f'(x)/g'(x)$ in terms of a limit L. Conclude that the limit as x approaches $+\infty$ of $f'(x)/g'(x)$ exists and equals the limit as x approaches $+\infty$ of $f(x)/g(x)$.

Solution to Problem 21 Denote $y = 1/x$. By the chain rule, $f'(x)$ equals $u'(y)y'(x) = -u'(y)/x^2$. Similarly, $g'(x)$ equals $-v'(y)/x^2$. Thus the fraction $f'(x)/g'(x)$ simplifies to $u'(y)/v'(y)$ for x nonzero. Thus, the limit as x approaches ∞ of $f'(x)/g'(x)$ also equals the limit as y approaches 0^+ of $u'(y)/v'(y)$. By L'Hôpital's rule, this limit equals the limit as y approaches 0^+ of $u(y)/v(y)$. Therefore, the limit as x approaches ∞ of $f(x)/g(x)$ equals the limit of $f'(x)/g'(x)$, and this equals L.

Problem 22. The double-angle formula for cosine is as follows,

$$\cos(2x) = (\cos(x))^2 - (\sin(x))^2.$$

Compute the limit as x approaches $\pi/4$ of $(\cos(x)^2 - \sin(x)^2)/\cos(2x)$ both using the double-angle formula and using LH́ôpital's Rule.

Solution to Problem 22 Of course the fraction is identically 1 by the double-angle formulas (except for odd integer multiples of $\pi/4$ where the fraction is undefined). So the limit of the fraction equals 1.

By L'Hôpital's rule, the limit equals the fraction with numerator $-4\sin(x)\cos(x)$ and the denominator equals $-2\sin(2x)$. The numerator limits to $-4/(\sqrt{2})^2 = -2$, and the denominator limits to -2. Thus, the limit of the fraction is $-2/-2$, i.e., the limit equals $\boxed{1}$.

Problem 23 Compute each of the following limits.

(a)

$$\lim_{x\to 0^+} \frac{1-\cos(x)}{\sin^2(x)}$$

(b)

$$\lim_{x\to 0^+} \ln(1-\cos(x)) - 2\ln(\sin(x))$$

(c)

$$\lim_{x\to\infty} \frac{1+x}{x^2}$$

(d)

$$\lim_{x\to\pi/2} \tan(x) - \sec(x)$$

(e)

$$\lim_{x\to\infty} x \ln\left(\frac{x}{x+1}\right).$$

Solution to Problem 23 Using L'Hôpital's rule, the first limit equals $1/2$, the second limit equals $-\ln(2)$, the third and fourth limits equals 0, and the final limit equals -1.