MAT131 Fall 2021 Final Exam Review Sheet

The final exam will be **cumulative**. Please look at the review sheets for Midterms 1 and 2 to review the material from earlier in the semester. (We may choose to write up a few more problems from earlier in the semester. You can find many more such problems in the textbook and on exams from previous semesters.)

Particular emphasis on the final will be placed on the material which has not been tested on Midterms 1 or 2. Among this new material is the following.

(1) Optimization problems. Given a word problem attempting to maximize or minimize some quantity given a collection of constraints, turn this into a calculus problem for finding an absolute maximum or absolute minimum. Solve this calculus problem.

(2) L'Hôpital's rule. Recognize indeterminate forms. Simplify limits leading to $0/0$ and ∞/∞ indeterminate forms using l'Hôpital's rule. Know how to transform other indeterminate forms into one of these two types.

(3) Riemann sums. Know how to set up a Riemann sum associated to a given integrand and a given interval. Be able to evaluate the limit of Riemann sums to compute the Riemann integral in the case of some simple integrands.

(4) Antiderivatives. Recognize the most common antiderivatives: those arising as the derivatives of x^n, trigonometric functions, exponential functions, logarithmic functions and inverse trigonometric functions.

(5) Know the statement of the Fundamental Theorem of Calculus. Understand how to use this to evaluate definite integrals when you can find

a simple form for the antiderivative. Understand how the fundamental theorem always gives an antiderivative of a continuous function, where the antiderivative is defined in terms of the Riemann integral/definite integral.

(6) Given a limit of sums, recognize when this is a limit of Riemann sums. Be able to use the fundamental theorem of calculus to evaluate this limit of Riemann sums.

(7) Simplify antiderivatives using direct substitution.

(8) Evaluate definite integrals using direct substitution and the fundamental theorem of calculus.

Following are some practice problems. More practice problems are in the textbook as well as on the practice midterm.

Problem 1. Let $u(x)$ and $v(x)$ be continuously differentiable functions defined on an open interval containing $x = a$ such that $u(a)$ and $v(a)$ both equal 0, and such that the limit as x approaches a of $u(x)/v(x)$ is defined and equals $L \neq 0$. Consider the functions $f(x) = 1/v(x)$ and $g(x) = 1/u(x)$. What kind of indeterminate form arises from the limit as x approaches a of $f(x)/g(x)$? What are the derivatives $f'(x)$ and $g'(x)$ as expressions in $u(x)$, $u'(x)$, $v(x)$ and $v'(x)$? What is the limit as x approaches a of $f'(x)/g'(x)$ when expressed in these terms? What is the limit as x approaches a of $f(x)/g(x)$? Does the limit of $f(x)/g(x)$ equal the limit of $f'(x)/g'(x)$?

Problem 2. Let $u(x)$ and $v(x)$ be continuously differentiable functions defined on an open interval containing $x = a$ such that $u(a)$ and $v(a)$ both equal 0, such that $v(x)$ is positive for $x \neq a$, and such that the limit as x approaches a of $u'(x)/v'(x)$ is defined and equals L. Define $h(x) = e^{1/v(x)}$ as a continuously differentiable function on the complement of $x = a$ in the interval. Rewrite $u(x)/v(x)$ as $\ln(h(x)^{u(x)})$. What kind of indeterminate form arises as the limit of x to a of $h(x)^{u(x)}$? Show that the limit as x approaches a of $-(h'(x)/h(x))/(u'(x)/u(x)^2)$ is defined and equals L, and show that the limit as x approaches a of $h(x)^{u(x)}$ exists and equals e^L.

Problem 3. Let $u(x)$ and $v(x)$ be continuously differentiable functions defined on an open interval containing $x = 0$ such that $u(0)$ and $v(0)$ both equal 0, and such that the limit as x approaches 0 of $u'(x)/v'(x)$ is defined and equals L. Define $f(x) = u(1/x^2)$ and $g(x) = v(1/x^2)$. Show that the limit as x approaches $+\infty$ of $f(x)/g(x)$ gives a $0/0$ indeterminate form. Compute $f'(x)$ and $g'(x)$ in terms of $u'(x)$ and $v'(x)$, and describe the limit as x approaches $+\infty$ of $f'(x)/g'(x)$ in terms of a limit L. Conclude that the limit as x approaches $+\infty$ of $f'(x)/g'(x)$ exists and equals the limit as x approaches $+\infty$ of $f(x)/g(x)$.

Problem 4. The double-angle formula for cosine is as follows,

$$\cos(2x) = (\cos(x))^2 - (\sin(x))^2.$$

Compute the limit as x approaches $\pi/4$ of $(\cos(x)^2 - \sin(x)^2)/\cos(2x)$ both using the double-angle formula and using LH́ôpital's Rule.

Problem 5. There exists a differentiable function $y = f(x)$ with domain $(0, \infty)$ and range $(-1, 0)$ that satisfies the following implicit equation,

$$y - y^{-1} = 2x.$$

Compute the derivative of $f(x)$ using implicit differentiation, and then compute the limit as x approaches 0 of $x/(y - 1)$.

Problem 6. In each of the following cases, sketch the graph of the given function. On your graph state whether the function is even, odd or neither. State whether or not the function is periodic, and state the period if it is periodic. Label all discontinuities and the type. Label all vertical and horizontal asymptotes.

Label *both* the x and y-coordinates of all local maxima and minima and state where the function is increasing and where decreasing. Label *both* the x and y-coordinates of all inflection points and state where the function is concave up and where concave down.

(a)

$$y = x^2 - 4.$$

(b)

$$y = \frac{1}{x^2 - 4}$$

(c)

$$y = \frac{1}{x + 1} - 2 + \frac{1}{x - 1}.$$

(d)

$$y = \sin(x)$$

(e)

$$y = \tan(x)$$

(f)

$$y = \sin(x)\cos(x)$$

(g)

$$y = e^{-x^2/2}$$

(h)

$$y = \ln((x+1)^2/(x-1)^2).$$

(i)

$$y = \frac{e^x}{1+e^x}.$$

(j)

$$y = e^{-1/x^2}.$$

Problem 7 Let S be the square in the xy-plane centered at the origin, of edge length $\sqrt{2}$, and with diagonal edges of slopes $+1$ respectively -1. The equation for this square is

$$|x| + |y| = 1.$$

Let R be a square in the xy-plane centered at the origin whose horizontal and vertical edges are parallel to the x-axis and y-axis respectively. Among squares R which intersect S, what is the maximum possible area of the region lying outside the inner square and inside the outer square?

Problem 8 Compute the maximum volume of a right circular cone whose surface area (just of the cone, not of the "bottom" disk of the cone) is a fixed constant A. The surface area of a right circular cone is $A = \pi r s$ where r is the radius of the bottom disk and s is the *slant height*, i.e., the distance from the vertex of the cone to a point on the bottom circle of the cone. What is the ratio of radius to height for such a cone?

Problem 9 You will build a box in a corner of a room using the floor and two walls as sides of the box. To do this, remove a square from one corner of a square sheet of metal of edge length 10 feet. Fold the edges of the sheet meeting the missing square to form two sides of a box. The remaining square of the sheet forms the top of the box. Slide these three sides into the corner to form a box with square top. What is the maximum volume of this box?

Problem 10 Compute each of the following limits.

(a)

$$\lim_{x \to 0^+} \frac{1 - \cos(x)}{\sin^2(x)}$$

(b)

$$\lim_{x \to 0^+} \ln(1 - \cos(x)) - 2\ln(\sin(x))$$

(c)

$$\lim_{x \to \infty} \frac{1 + x}{x^2}$$

(d)

$$\lim_{x \to \pi/2} \tan(x) - \sec(x)$$

(e)

$$\lim_{x \to \infty} x \ln\left(\frac{x}{x+1}\right).$$

Problem 11 By thinking about areas compute an antiderivative of $\sqrt{1 - x^2}$. (Hint: Sketch the region whose area is the definite integral of this function from 0 to x.)

Problem 12 Consider the Riemann integral

$$\int_1^4 (2x + 1)dx.$$

Partition the interval into n subintervals of equal length. Compute the Riemann sum S_n for this partition using right endpoints. Write down the value of this Riemann sum. Directly compute the limit as n goes to infinity to find the Riemann integral. Double-check your answer against the Fundamental Theorem of Calculus.

Problem 13 Consider the Riemann integral

$$\int_0^1 3^x dx.$$

Partition the interval into n subintervals of equal length. Compute the Riemann sum S_n for this partition using left endpoints. Write down the value of this Riemann sum. Directly compute the limit as n goes to infinity to find the Riemann integral (you may use L'Hôpital's rule to evaluate the limit). Double-check your answer against the Fundamental Theorem of Calculus.

Problem 14 Compute each of the following definite and indefinite integrals.

(a)

$$\int x^{-1/2} dx$$

(b)

$$\int (x + \sin(x))dx$$

(c)

$$\int \frac{1}{\sqrt{1-x^2}} dx$$

(d)

$$\int \sec(\theta)\tan(\theta)d\theta$$

(e)

$$\int_{-2}^{2} x^4 dx.$$

(f)

$$\int_{-\pi/2}^{\pi/2} \cos(2x) - \cos(x)dx$$

(g)

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx$$

(h)

$$\int_{\pi/4}^{\pi/3} \sec^2(\theta)d\theta$$

Problem 15 Evaluate each of the following limits.

(a)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} 1.$$

(b)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(1-\frac{2i}{n})$$

(c)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n-2i}{n^2}.$$

(d)

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{2^{i/n}}{n}$$

(e)

$$\lim_{m\to\infty} \sum_{i=1}^{m} \frac{m}{m^2+i^2}$$

Problem 16 Compute each of the following indefinite and definite integrals.

(a)

$$\int_a^b f'(cx)dx$$

(b)

$$\int x\sin(x^2)dx$$

(c)

$$\int_0^{\pi/4} \tan(x)dx$$

(d)

$$\int_0^{\pi/2} \sin^3(x)dx$$

(e)

$$\int \frac{\ln(x)}{x}dx$$

(f)

$$\int_1^2 \frac{e^{\ln(x)}}{x}dx$$

(g)

$$\int_0^{x^2} \frac{f'(\sqrt{t})}{\sqrt{t}}dt$$

Challenge Problem 17 For a real number $b > 1$, partition the interval $[1, b]$ into N subintervals $[x_{i-1}, x_i]$ for $i = 1, \ldots, N$ that have **unequal** lengths $\Delta x_i = x_i - x_{i-1}$ by the rule $x_i = q^i$, for $q = \sqrt[N]{b}$. For a positive real number r, for the function $y = x^r$, approximate the Riemann integral,

$$\int_{x=1}^{b} x^r dx$$

by the Riemann sum L_N for this partition using left endpoints and by the Riemann sum R_N for this partition using right endpoints. Use L'Hopital's rule to evaluate the limit,

$$\lim_{q \to 1} \frac{q^{r+1} - 1}{q - 1}.$$

Combine this with the formula for a geometric sum to directly compute the limits as N tends to ∞ of L_N and R_N. Use this to directly compute that the Riemann integral equals $(b^{r+1} - 1)/(r + 1)$.

Challenge Problem 18 Let p, q, r, s be real numbers such that p, r and $ps - qr$ is nonzero. For the following antiderivative problem,

$$\int \frac{1}{(px + q)(rx + s)}dx,$$

use the following direct substitution to compute the antiderivative,

$$u = \frac{px+q}{rx+s}.$$

Remember to back-substitute to get an answer that is a function of only x for your antiderivative. What happens if p and r are nonzero real numbers, but $ps - qr$ equals 0?

Challenge Problem 19 Let r and s be real numbers with r and s both nonzero. For the following atiderivative,

$$\int \frac{1}{x(rx^2+s)}dx,$$

use the following direct substitution to compute the antiderivative,

$$v = x^2.$$

Remember to back-substitute to get an answer that is a function of only x for your antiderivative. What happens if r is nonzero, but s equals 0?

Challenge Problem 20 What is the largest possible volume of a box with square top and bottom that is sitting on top of a flat floor and contained under a hemisphere resting on the floor with radius r? What is the ratio of height of the box to edge length of the base square for the optimal volume?

Challenge Problem 21 In this problem you can use the limit proved in lecture, that for every positive real number q,

$$\lim_{h\to 0} \frac{q^h - q^0}{h} = \left.\frac{dq^x}{dx}\right|_{x=0} = \ln(q).$$

For every real number r and for every positive real number b, use the limit above to directly compute the limit of the difference quotient in the derivative,

$$\left.\frac{dx^r}{dx}\right|_{x=b} = \lim_{x\to b} \frac{x^r - b^r}{x-b},$$

by substituting $x = q^h b$ for a positive real $q \neq 1$ and a real number h that limits to 0. Simplify the difference quotient, and express your answer in terms of a fraction of limits as above. Use this to prove that the derivative of x^r equals rx^{r-1} for all real numbers r and all $x > 0$.