SOLUTIONS

MAT131 Fall 2008 Final Exam

Name: ______________________________ SB ID number: ____________________

Please circle the number of your recitation.

1. TuTh 12:50 – Physics Norton	**6.** MW 10:40 – Physics Lyubich	**11.** TuTh 3:50 – Physics Stimpson
2. MF 2:20 – ESS Malkoun	**7.** MW 6:55 – Physics Dalton	**12.** TuTh 8:20 – Lgt Engr Lab Nam
3. TuTh 8:20 – ESS Rogers	**8.** MW 3:50 – SB Union Dalton	**13.** MF 12:50 – Lgt Engr Lab Malkoun
4. WF 11:45 – Physics Flanagan	**9.** TuTh 5:20 – ESS Norton	**14.** TuTh 3:50 – Physics Findley
5. TuTh 11:20 – ESS Kim	**10.** WF 9:35 – Lgt Engr Lab Flanagan	

Problem 1: ________ /35	**Problem 2:** ________ /25	**Problem 3:** ________ /25
Problem 4: ________ /30	**Problem 5:** ________ /20	**Problem 6:** ________ /40
Problem 7: ________ /25		**TOTAL:** ________ /200

Instructions: *Please write your name at the top of every page of the exam.* The exam is closed book, closed notes, calculators are not allowed, and all cellphones and other electronic devices must be turned off for the duration of the exam. You will have approximately 90 minutes for this exam. The point value of each problem is written next to the problem – use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown.

You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., raise your hand.

Name: ______________________________ Problem 1: ____________ /35

Problem 1(35 points) **(a)**(25 points) For the function $y = f(x)$,

$$f(x) = 2x^3 - 6x$$

do all of the following.

(i) Determine whether $f(x)$ is even, odd or neither.

(ii) Determine where the function is increasing and where the function is decreasing.

(iii) Determine where the function is concave up and where the function is concave down.

(iv) Find the x and y coordinates of every local maximum, and find the x and y coordinates of every local minimum.

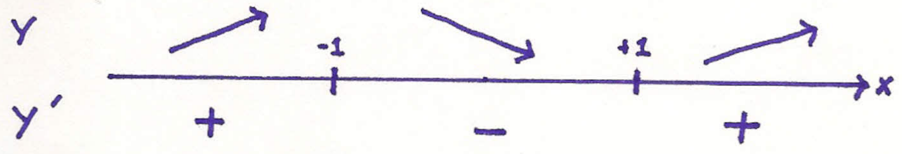
(v) Find the x and y coordinates of every inflection point.

(vi) On the following page sketch the graph of $f(x)$ labelling every local maximum, local minimum and inflection point.

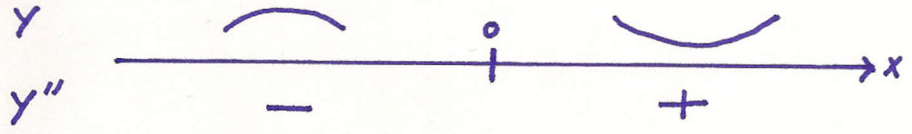
Show all work.

(i) $f(-x)$ equals $-f(x)$, thus $f(x)$ is $\boxed{\text{odd}}$.

(ii) $f'(x)$ equals $6x^2 - 6 = 6(x+1)(x-1)$

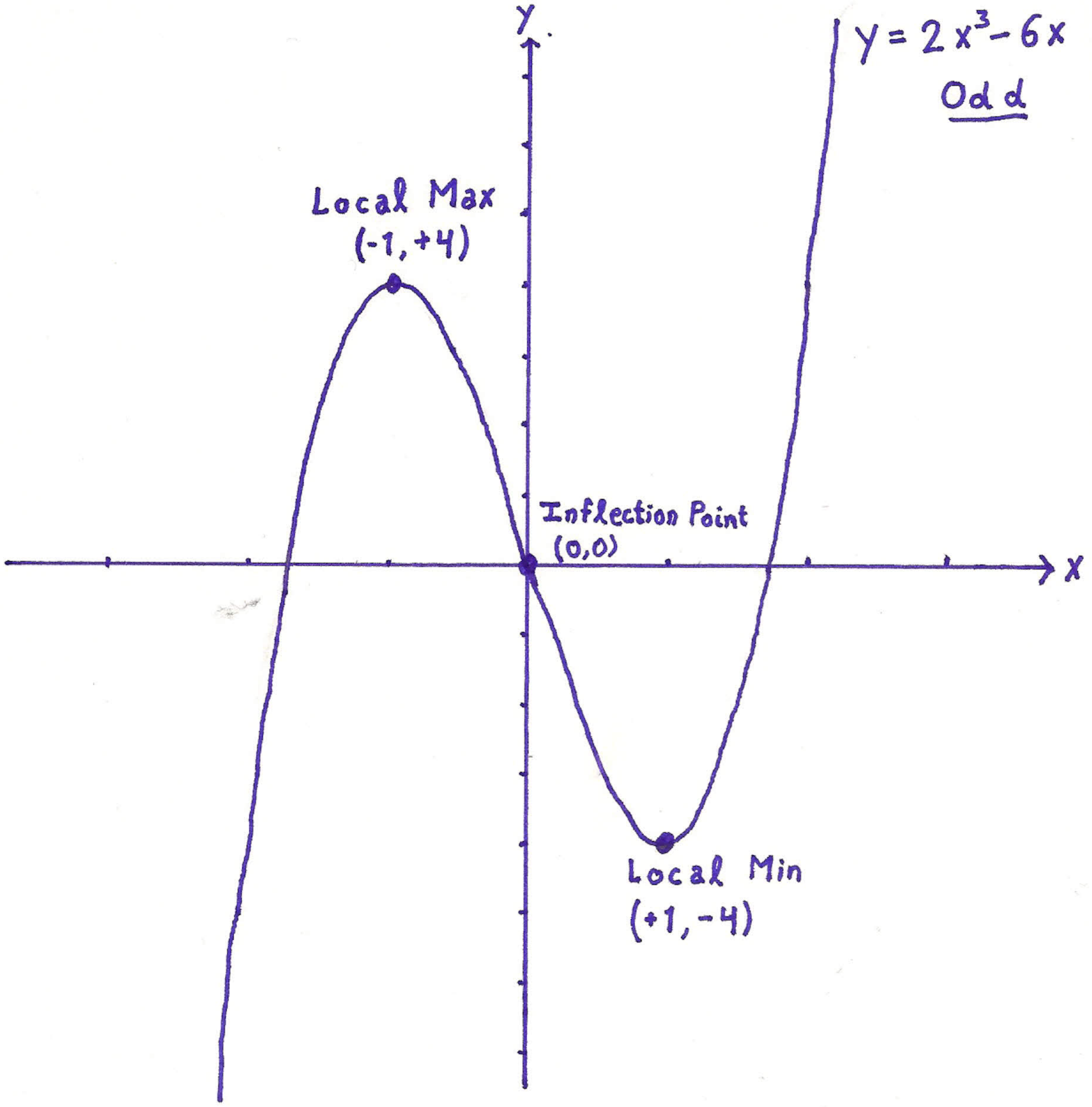


(iii) $f''(x)$ equals $12x$



(iv) Local max @ $(x,y) = \boxed{(-1,+4)}$, Local min @ $(x,y) = \boxed{(+1,-4)}$.

(v) Inflection point @ $(x,y) = \boxed{(0,0)}$


Y
$y = 2x^3 - 6x$
Odd
Local Max
(-1, +4)
Inflection Point
(0,0)
X
Local Min
(+1, -4)

(b)(10 points) Still for the function $y = f(x)$,

$$f(x) = 2x^3 - 6x$$

find the absolute maximum value on the closed interval $[-2, 3]$ and find the absolute minimum value on the closed interval $[-2, 3]$. Show all work and circle your final answers.

<u>Endpoints</u>. $x = -2$ and $x = 3$

<u>Critical points in $[-2,3]$</u>. $x = -1$ and $x = +1$.

<u>Table of values</u>.

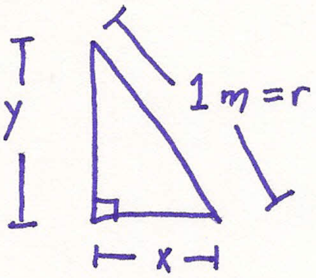
x	$f(x)$	
-2	-4	Absolute min
-1	$+4$	
$+1$	-4	Absolute min
3	36	← Absolute max

$\boxed{\text{Absolute maximum } y = 36}$

$\boxed{\text{Absolute minimum } y = -4}$

Name: ______________________________ **Problem 2:** ____________ /25

Problem 2(25 points) Find the largest area of a right triangle whose hypotenuse has fixed length 1 meter. Although you may use geometric intuition to guess the answer, you must show your calculus work to justify your answer (correct guesses without calculus justification will receive little credit).



Constants. $r = 1\ m$

Variables. x, y

To maximize. $A = \text{area} = \frac{1}{2}xy$

Constraint. $x^2 + y^2 = r^2 = 1m^2$, so

$y = (r^2 - x^2)^{\frac{1}{2}}$

$A(x) = \frac{1}{2}x(r^2 - x^2)^{\frac{1}{2}}$, $0 \leq x \leq r$

Critical points. $\frac{dA}{dx} = \frac{1}{2}(r^2-x^2)^{\frac{1}{2}} + \frac{1}{2}x \cdot \frac{1}{2}(r^2-x^2)^{-\frac{1}{2}}(-2x)$

$$= \frac{1}{2(r^2-x^2)^{\frac{1}{2}}}\left((r^2-x^2) - x^2\right) = \frac{r^2-2x^2}{2(r^2-x^2)^{\frac{1}{2}}}$$

$\frac{dA}{dx} = 0$ if $r^2 - 2x^2 = 0$, $x = \pm\frac{1}{\sqrt{2}}r$. Critical point in $[0, r]$: $x = \frac{+1}{\sqrt{2}}r$.

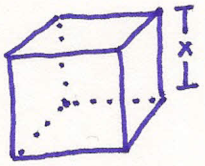
Compare.

x	$A(x)$
0	0
r	0
$\frac{1}{\sqrt{2}}r$	$\frac{r^2}{4}$ ← Abs. max.

$$\boxed{\text{The maximal area is } A\left(\frac{r}{\sqrt{2}}\right) = \frac{r^2}{4} = \frac{1}{4}\ m^2.}$$

Name: ______________________________ Problem 3: ____________ /25

Problem 3(25 points) A cube is increasing in volume. At the moment when the volume of the cube equals 8 cubic inches, the surface area is increasing at a rate of 2 square inches per minute. At what rate is the volume of the cube increasing at this moment? Show all work and circle your final answer.



Constants. None

Indep. var. t = time

Dep. vars. $x(t)$, $A(t)$ = total surface area, $V(t)$ = volume

Known rate. $\frac{dA}{dt}(t_0) = 2\ \frac{in^2}{min}$, Other knowns. $V(t_0) = 8\ in^3$

Rate to find. $\frac{dV}{dt}(t_0)$. Constraints. $A = 6\cdot(\text{Area of 1 face}) = 6x^2$

$V = x^3$

$\frac{dA}{dt} = 12x\frac{dx}{dt}$, $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$. Soln. $V(t_0) = 8\ in^3$, so $x(t_0) = 2\ in$.

$\frac{dA}{dt}(t_0) = 12(2in)\cdot\frac{dx}{dt}(t_0)$, so $\frac{dx}{dt}(t_0) = \frac{1}{24\ in}\frac{dA}{dt}(t_0) = \frac{2}{24}\frac{in}{min} = \frac{1}{12}\frac{in}{min}$

So $\frac{dV}{dt}(t_0) = 3(2in)^2\cdot\frac{1}{12}\frac{in}{min} = \frac{12}{12}\frac{in^3}{min} = 1\ \frac{in^3}{min}$.

The volume is increasing at 1 cubic inch per minute.

Name: ________________________ **Problem 4:** ____________ /30

Problem 4(30 points) In each of the following cases, compute the value of the limit. Show all work and circle your final answers.

(a)(5 points)

$$\lim_{x\to 1} \frac{x^2 - x}{x^3 - x} = \lim_{x\to 1} \frac{x(x-1)}{x(x-1)(x+1)} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

OR $$= \lim_{x\to 1} \frac{2x-1}{3x^2-1} = \frac{2-1}{3-1} = \boxed{\frac{1}{2}}$$

(b)(5 points)

$$\lim_{x\to 0} \frac{\tan(2x)}{\sin(3x)} = \lim_{x\to 0} \frac{2\tan(x)}{1-\tan^2(x)} \Big/ \sin(x)\left(3\cos^2(x) - \sin^2(x)\right) = \lim_{x\to 0} \frac{2}{(1-\tan^2(x))\cos(x)(3\cos^2(x) - \sin^2(x))}$$

$$= \frac{2}{(1-0)\cdot 1\cdot(3\cdot 1-0)} = \boxed{\frac{2}{3}}$$

OR $$= \lim_{x\to 0} \frac{2\sec^2(2x)}{3\cos(3x)} = \frac{2\cdot 1^2}{3\cdot 1} = \boxed{\frac{2}{3}}$$

Name: ______________________ Problem 4, continued

(c) (10 points)

$$\lim_{x\to\infty}\left(x-\frac{x^3}{x^2+1}\right) = \lim_{x\to\infty} \frac{x\,(x^2+1)}{(x^2+1)} - \frac{x^3}{(x^2+1)} = \lim_{x\to\infty} \frac{(x^3+x)-x^3}{x^2+1}$$

$$= \lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{x\cdot 1}{x^2(1+\frac{1}{x^2})}$$

$$= \lim_{x\to\infty} \frac{1}{x}\cdot\frac{1}{1+\frac{1}{x}} = \frac{1}{\infty}\cdot\frac{1}{1} = \boxed{0}$$

OR $$= \lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{1}{2x} = \frac{1}{2\cdot\infty} = \boxed{0}.$$

(d) (10 points)

$$\lim_{x\to\infty}(x^2+1)^{1/x}$$, $y=(x^2+1)^{\frac{1}{x}}$, $u=\ln(y)=\frac{\ln(x^2+1)}{x}$.

$$\lim_{x\to\infty} u = \lim_{x\to\infty} \frac{\ln(x^2+1)}{x} = \lim_{x\to\infty} \frac{\frac{1}{x^2+1}\cdot(2x)}{1} = \lim_{x\to\infty} \frac{2x}{x^2+1}$$

$$= \lim_{x\to\infty} \frac{2}{2x} = \frac{1}{\infty} = \underline{0}$$

$$\lim_{x\to\infty} (x^2+1)^{\frac{1}{x}} = \lim_{x\to\infty} e^u = e^0 = \boxed{1}$$

Name: ________________________________ Problem 5: ______________ /20

Problem 5(20 points) Find the equation of the tangent line to the curve

$$x^2 + y^3 = 5$$

at the point $(2, 1)$.

$2x + 3y^2 \cdot y' = 0.$ $\quad y' = -\frac{2x}{3y^2}$

So slope @ $(2,1)$ is $\frac{-2\cdot(2)}{3(1)^2} = \frac{-4}{3}$

$(y-1) = -\frac{4}{3}(x-2),$ $\quad \boxed{y = -\frac{4}{3}x + \frac{11}{3}}$

Name: ______________________________ Problem 6: ____________ /40

Problem 6(40 points) In each of the following cases, evaluate the given integral.

(a)(5 points)

$$\int 7^{2x}dx = \int 49^x dx = \boxed{\frac{49^x}{\ln(49)} + C} \quad \text{OR}$$

$u=2x$
$du=2dx$

$$\int 7^u \frac{du}{2} = \frac{1}{2}\frac{7^u}{\ln(7)} + C = \boxed{\frac{7^{2x}}{2\ln(7)} + C}$$

(b)(5 points)

$$\int_0^1 \sqrt[3]{x^2} - \sqrt{x^3}dx = \int_0^1 x^{\frac{2}{3}} - x^{\frac{3}{2}}dx = \left(\frac{3x^{5/3}}{5} - \frac{2x^{5/2}}{5}\right|_0^1 = \frac{3}{5} - \frac{2}{5} = \boxed{\frac{1}{5}}$$

(c)(10 points)

$$\int_{\pi/4}^{\pi/3} \frac{\sin(\theta)}{\cos(\theta)}d\theta$$

$u = \cos(\theta)$
$du = -\sin(\theta)d\theta$

$\theta = \frac{\pi}{3},\ u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
$\theta = \frac{\pi}{4},\ u = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$$\int_{u=\frac{1}{\sqrt{2}}}^{u=\frac{1}{2}} -u^{-1}du = \left(-\ln u\right|_{\frac{1}{\sqrt{2}}}^{\frac{1}{2}} = \ln(2) - \ln(2^{\frac{1}{2}}) = \boxed{\frac{1}{2}\ln(2)}$$

Name: ______________________________ **Problem 6, continued**

(d)(10 points) The inverse function of $\tan(x)$ is often written as $\arctan(x)$.

$$\int_1^{\sqrt{3}} \frac{\arctan(x)}{1+x^2}dx$$

$u = \arctan(x)$ | $x = \sqrt{3},\ u = \arctan(\sqrt{3}) = \frac{\pi}{3}$

$du = \frac{1}{1+x^2}dx$ | $x = 1,\ u = \arctan(1) = \frac{\pi}{4}$

$$\int_{u=\frac{\pi}{4}}^{\frac{\pi}{3}} u\,du = \left(\frac{1}{2}u^2\right|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1}{2}\frac{\pi^2}{9} - \frac{1}{2}\frac{\pi^2}{16} = \boxed{\frac{7\pi^2}{288}}$$

(e)(10 points)

$$\int \frac{\sin(e^{2x})}{e^{-2x}}dx = \int \sin(e^{2x})e^{2x}dx$$

$u = e^{2x}$

$du = 2e^{2x}dx$, $\int \sin(u)\cdot\frac{1}{2}\cdot du = -\frac{1}{2}\cos(u) + C$

$$= \boxed{-\frac{1}{2}\cos(e^{2x}) + C}$$

Name: ______________________________ Problem 7: ____________ /25

Problem 7(25 points) Consider the limit,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+2ni+i^2}.$$

(a)(15 points) Find a Riemann integral, also known as a definite integral, such that the limit above is the limit of Riemann sums for this integral. Show all work and circle your integral.

$a=0,\ b=1,\ \Delta x=\frac{b-a}{n}=\frac{1}{n},\quad x_i=a+i\Delta x=0+\frac{i}{n}=\frac{i}{n},\quad i=nx_i$

$$\sum_{i=1}^{n}\frac{n}{n^2+2ni+i^2}=\sum_{i=1}^{n}\frac{n}{n^2+2n(nx_i)+(nx_i)^2}\cdot(n\Delta x)=\sum_{i=1}^{n}\frac{n^2}{n^2(x_i^2+2x_i+1)}\Delta x$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+2ni+i^2}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{x_i^2+2x_i+1}\Delta x=\boxed{\int_0^1\frac{1}{x^2+2x+1}\,dx}$$

(b)(10 points) Compute the limit by evaluating the integral. Show all work and circle your final answer.

$$\int_0^1\frac{1}{x^2+2x+1}\,dx=\int_0^1\frac{1}{(x+1)^2}\,dx=-(x+1)^{-1}\Big|_0^1=-\frac{1}{1+1}-\left(\frac{-1}{1+0}\right)$$

$$=\frac{1}{1}-\frac{1}{2}=\frac{1}{2}$$

$$\boxed{\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+2ni+i^2}=\frac{1}{2}}$$