1. Statement of the Lemma

Let $\mathcal{C}$ be an Abelian category. In particular, every image in $\mathcal{C}$ equals the coimage. Thus we make no distinction between images and coimages in what follows. One of the fundamental notions of homological algebra is the following.

**Definition 1.1.** A short exact sequence

$$\Sigma_A : 0 \longrightarrow A' \xrightarrow{q_A} A \xrightarrow{p_A} A'' \longrightarrow 0$$

is a pair of morphisms in $\mathcal{C}$

$$\Sigma_A = (q_A : A' \to A, p_A : A \to A)$$

such that all of the following hold:

(i) $q_A$ is a monomorphism,
(ii) $p_A$ is an epimorphism, and
(iii) the image of $q_A$ equals the kernel of $p_A$.

There is a category whose objects are short exact sequences in $\mathcal{C}$. Here is the notion of morphism in this category.

**Definition 1.2.** Let $\Sigma_A = (q_A, p_A)$ and $\Sigma_B = (q_B, p_B)$ be short exact sequences in $\mathcal{C}$. A morphism $\Sigma_f$ from $\Sigma_A$ to $\Sigma_B$.

$$\Sigma_A : 0 \longrightarrow A' \xrightarrow{q_A} A \xrightarrow{p_A} A'' \longrightarrow 0$$

$$\Sigma_B : 0 \longrightarrow B' \xrightarrow{q_B} B \xrightarrow{p_B} B'' \longrightarrow 0$$

is a triple of morphisms in $\mathcal{C}$

$$\Sigma_f = (f' : A' \to B', f : A \to B, f'' : A'' \to B'')$$

such that every square commutes, i.e., both of the following hold:

(i) $q_B \circ f'$ equals $f \circ q_A$, and
(ii) $p_B \circ f$ equals $f'' \circ p_A$. 

*Date: October 12, 2010.*
In the category of short exact sequences the identity morphisms and the compositions are the obvious notions. The category of short exact sequences is an additive category.

Let \( \Sigma f \) be a morphism of short exact sequences as above. Denote the kernels of \( f' \), respectively \( f'', \) by,

\[
\iota' : K'_{\Sigma f} \rightarrow A', \text{ resp. } \iota : K_{\Sigma f} \rightarrow A, \quad \iota'' : K''_{\Sigma f} \rightarrow A''.
\]

Similarly, denote the cokernels of \( f' \), respectively \( f'' \), \( f' \) by,

\[
\sigma' : B' \rightarrow C'_{\Sigma f}, \text{ resp. } \sigma : B \rightarrow C_{\Sigma f}, \quad \sigma'' : B'' \rightarrow C''_{\Sigma f}.
\]

Because \( q_B \circ f' \) equals \( f \circ q_A \), also \( f \circ (q_A \circ \iota') \) equals \( q_B \circ (f' \circ \iota') \), which equals \( q_B \circ 0 = 0 \). Thus, by the universal property of the kernel, there is a unique morphism

\[
q_K : K'_{\Sigma f} \rightarrow K_{\Sigma f}
\]

such that \( \iota \circ q_K \) equals \( q_A \circ \iota' \). For a similar reason, there is a unique morphism

\[
p_K : K_{\Sigma f} \rightarrow K''_{\Sigma f}
\]

such that \( \iota'' \circ p_K \) equals \( p_A \circ \iota \). And by analogous arguments there are unique morphisms

\[
q_C : C'_{\Sigma f} \rightarrow C_{\Sigma f}, \quad p_C : C_{\Sigma f} \rightarrow C''_{\Sigma f}
\]

such that \( q_C \circ \sigma' \) equals \( \sigma \circ q_B \), and \( p_C \circ \sigma \) equals \( \sigma'' \circ p_B \). To summarize, we have that the following diagram is commutative.

\[
\begin{array}{ccccccccc}
K'_{\Sigma f} & \xrightarrow{q_K} & K_{\Sigma f} & \xrightarrow{p_K} & K''_{\Sigma f} \\
\downarrow \iota' & & \downarrow \iota & & \downarrow \iota'' \\
\Sigma_A : 0 & \rightarrow & A' & \rightarrow & A & \rightarrow & A'' & \rightarrow & 0 \\
\downarrow \Sigma_f & & \downarrow f & & \downarrow f'' \\
\Sigma_B : 0 & \rightarrow & B' & \rightarrow & B & \rightarrow & B'' & \rightarrow & 0 \\
\downarrow \sigma' & & \downarrow \sigma & & \downarrow \sigma'' \\
C'_{\Sigma f} & \xrightarrow{q_C} & C_{\Sigma f} & \xrightarrow{p_C} & C''_{\Sigma f}
\end{array}
\]

By hypothesis, both \( f'' \circ p_A \) and \( p_B \circ f \) are equal. Denote by \( t \) this common morphism

\[
t : A \rightarrow B''.
\]

Denote the kernel of \( t \) by

\[
j : K_t \rightarrow A.
\]

Now \( f'' \circ (p_A \circ j) \) equals \( t \circ j \), which is 0. By the universal property of the kernel of \( f'' \), there is a unique morphism

\[
\overline{p}_A : K_t \rightarrow K''_{\Sigma f}
\]

such that \( \iota'' \circ \overline{p}_A \) equals \( p_A \circ j \). Similarly, \( p_B \circ (f \circ j) \) equals \( t \circ j \), which is 0. By the universal property of the kernel of \( p_B \), there is a unique morphism

\[
\overline{f} : K_t \rightarrow B'
\]

such that \( q_B \circ \overline{f} \) equals \( f \circ j \).
Lemma 1.3 (The Snake Lemma). For a morphism $\Sigma_f$ of commutative diagrams as above, all of the following hold.

(i) The morphism $q_K$ is a monomorphism, and the morphism $p_C$ is an epimorphism.

(ii) The image of $q_K$ equals the kernel of $p_K$, and the kernel of $p_C$ equals the image of $q_C$.

(iii) There is a unique morphism $\delta_{\Sigma_f} : K_{\Sigma_f}'' \to C_{\Sigma_f}'$ such that $\delta_{\Sigma_f} \circ p_A$ equals $s' \circ f$ as morphisms $K_t \to C_{\Sigma_f}'$.

(iv) The image of $p_K$ equals the kernel of $\delta_{\Sigma_f}$, and the kernel of $q_C$ equals the image of $\delta_{\Sigma_f}$.

In summary, the following long sequence is exact,

$$0 \longrightarrow K_{\Sigma_f}' \overset{q_K}{\longrightarrow} K_{\Sigma_f} \overset{p_K}{\longrightarrow} K_{\Sigma_f}'' \overset{\delta_{\Sigma_f}}{\longrightarrow} \ldots$$

$$\ldots \overset{\delta_{\Sigma_f}}{\longrightarrow} C_{\Sigma_f}' \overset{q_C}{\longrightarrow} C_{\Sigma_f} \overset{p_C}{\longrightarrow} C_{\Sigma_f}'' \longrightarrow 0.$$ 

This entire situation is often summarized with the following large diagram.

\[
\begin{array}{ccccccc}
0 & \longrightarrow & K_{\Sigma_f}' & \overset{q_K}{\longrightarrow} & K_{\Sigma_f} & \overset{p_K}{\longrightarrow} & K_{\Sigma_f}'' & \overset{\delta_{\Sigma_f}}{\longrightarrow} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \overset{q_A}{\longrightarrow} & A & \overset{p_A}{\longrightarrow} & A'' & \longrightarrow & 0 \\
\Sigma_A & \downarrow & f' & \downarrow & f & \downarrow & f'' & & \\
\Sigma_f & \downarrow & & & & & & \\
0 & \longrightarrow & B' & \overset{q_B}{\longrightarrow} & B & \overset{p_B}{\longrightarrow} & B'' & \longrightarrow & 0 \\
\Sigma_B & \downarrow & s' & \downarrow & s & \downarrow & s'' & & \\
\ldots & \overset{\delta_{\Sigma_f}}{\longrightarrow} & C_{\Sigma_f}' & \overset{q_C}{\longrightarrow} & C_{\Sigma_f} & \overset{p_C}{\longrightarrow} & C_{\Sigma_f}'' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]