MAT 614 Problem Set 1

Homework Policy. Please read through all the problems. I will be happy to discuss the solutions during office hours.

Problems.

Problem 1. For a finite type, separated k-scheme X, recall the alternative definition of the subgroup $\mathrm{Rat}_l(X) \subset Z_l(X)$. It equals the subgroup generated by all pushforward classes, $i_*[W_0] - i_*[W_\infty]$, for all k-schemes W of pure dimension $l+1$, and for all proper morphisms

$$(h, i) : W \to \mathbb{P}^1 \times_k X,$$

such that the first component morphism

$$h : W \to \mathbb{P}^1,$$

is flat over 0 and ∞. In particular, for every irreducible k-scheme V of dimension l, check directly that for the identity morphism,

$$(\mathrm{pr}_{\mathbb{P}^1}, \mathrm{pr}_V) : \mathbb{P}^1 \times_k V \to \mathbb{P}^1 \times_k V,$$

the pushforward class equals 0. This fills one gap from the presentation in lecture.

Problem 2. For every integer l, define $\mathrm{Alg}_l(X) \subset Z_l(X)$ to be the subgroup generated by all pushforward classes, $i_*[W_{t_0}] - i_*[W_{t_\infty}]$, for all proper, irreducible k-curves C, for all pairs of k-points $t_0, t_\infty \in C(k)$, for all k-schemes W of pure dimension $l+1$, and for all proper morphisms

$$(h, i) : W \to C \times_k X,$$

such that the first component morphism

$$h : W \to C,$$

is flat over t_0 and t_∞. The quotient group $Z_l(X)/\mathrm{Alg}_l(X)$ is denoted by $B_l(X)$. Check that $\mathrm{Rat}_l(X)$ is contained in $\mathrm{Alg}_l(X)$, and thus the quotient $Z_l(X) \to A_l(X)$ factors uniquely through the quotient $Z_l(X) \to B_l(X)$. Give an example where $\mathrm{Rat}_l(X)$ is properly contained in $\mathrm{Alg}_l(X)$.

Problem 3. Continuing the previous problem, prove that for every proper morphism of finite type, separated k-schemes,

$$f : X \to Y,$$

the pushforward maps on cycles map $\mathrm{Alg}_l(X)$ to $\mathrm{Alg}_l(Y)$. Thus there is an induced pushforward map of quotient groups,

$$f_* : B_l(X) \to B_l(Y),$$

i.e., $X \mapsto B_l(X)$ is covariant for proper morphisms.

Problem 4. Continuing the previous problems, also check that the flat pullback maps preserve the subgroups $\mathrm{Alg}_l(X) \subset Z_l(X)$. Conclude that $X \mapsto B_l(X)$ is contravariant for flat pullback maps (with the appropriate degree shift).

Problem 5. Let $g : U \hookrightarrow X$ be an open subset of X, and denote the closed complement by $i : E \hookrightarrow X$. As in the case of A_*, check that the flat pullback by g and the proper pushforward by i induce an exact sequence,

$$B_l(E) \xrightarrow{i_*} B_l(X) \xrightarrow{g^*} B_l(U) \to 0.$$

Give an example proving that i_* need not be injective.

Problem 6. Let $n \geq 1$ be an integer. Recall the $\mathbb{Z}$-algebra from Problem 10 on Problem Set 1,

$$A_n^* := (\mathbb{Z}[s,t]/\langle x^{n+1}, x^n + \cdots + s^{n-r}t^r + \cdots + t^n, t^{n+1}\rangle)^{\mathfrak{S}_2}.$$

For every pair of integers $0 \leq a \leq b < n$, denote

$$p_{b,a} = (st)^a(s^{b-a} + \cdots + s^{b-a-r}t^r + \cdots + t^{b-a}).$$

Check that the images $(\overline{p}_{b,a})_{0\leq a\leq b\leq n}$ form a free $\mathbb{Z}$-basis for A_n^*.

By convention, extend this to all pairs of nonnegative integers (a,b), defining $\overline{p}_{b,a}$ to be 0 if either $b \geq n$ or $a > b$.

Problem 7. Now, let V be a k-vector space of dimension $n+1$, and let

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r \subsetneq \cdots \subsetneq V_n \subsetneq V_{n+1} = V,$$

be a complete flag of k-linear subspaces of V. For every pair of nonnegative integers (a,b), denote

$$\Sigma_{b,a}^o(V_\bullet) := \{[U] \in \mathrm{Grass}(2,V) : \dim(U \cap V_{n-b}) = 1,\ \dim(U \cap V_{n+1-a}) = 2\}.$$

as a locally closed subset of the Grassmannian $\mathrm{Grass}(2,V) = \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}V)$. Denote by $\Sigma_{b,a}(V_\bullet)$ the Zariski closure of $\Sigma_{b,a}^o(V_\bullet)$.

If either $b \geq n$ or $a > b$, prove that $\Sigma_{b,a}^o(V_\bullet)$ is empty. If $0 \leq a \leq b < n$, prove that $\Sigma_{b,a}^o(V_\bullet)$ is isomorphic to an affine space of codimension $a+b$ in $\mathrm{Grass}(\mathbb{P}^1, \mathbb{P}V)$, and the Zariski closure equals

$$\Sigma_{b,a}(V_\bullet) := \{[U] \in \mathrm{Grass}(2,V) : \dim(U \cap V_{n-b}) \geq 1,\ \dim(U \cap V_{n+1-a}) \geq 2\}.$$

Denote the cycle class $[\Sigma_{b,a}(V_\bullet)]$ by $\sigma_{b,a}$. In particular, this is zero if either $b \geq n$ or $a > b$.

Problem 8. Show that there is a well-defined $\mathbb{Z}$-module homomorphism,

$$\Phi : A_n^* \to A_*(\text{Grass}(\mathbb{P}^1, \mathbb{P}V)),$$

sending each element $\overline{p}_{b,a}$ to $\sigma_{b,a}$. Moreover, check that this homomorphism is surjective.

Problem 9. Pieri's Rule. First, show that for every $0 \le a \le b < n$ and for every $0 \le l < n$, there is an identity,

$$\overline{p}_{l,0}\overline{p}_{b,a} = \sum_{i=0} \overline{p}_{b+l-i,a+i}.$$

Denote by J_n the ideal in the ring $\mathbb{Z}[\pi_{b,a}]_{0\le a\le b<n}$ generated by the polynomials,

$$P_{l,a,b} = \pi_{l,0} \cdot \pi_{b,a} - \sum_{i=0}^{l} \pi_{b+l-i,a+i}, \ \ 0 \le l < n, \ \ 0 \le a \le b < n,$$

with the convention that $\pi_{c,d}$ equals 0 if either $c \ge n$ or $d > c$. Denote the quotient ring by,

$$R_n^* = \mathbb{Z}[\pi_{b,a}]_{0\le a\le b<n}/J_n.$$

Show that the elements $\overline{\pi}_{b,a}$ generate R_n^* as a $\mathbb{Z}$-module. Next use the previous paragraph to show that there is a well-defined $\mathbb{Z}$-algebra homomorphism,

$$\overline{p} : R_n^* \to A_n^*, \ \ \overline{\pi}_{b,a} \mapsto \overline{p}_{b,a}.$$

Since the elements $\overline{p}_{b,a}$ form a $\mathbb{Z}$-basis for A_n^*, conclude that $\overline{p}$ is an isomorphism.

For every $1 \le a \le b < n$, prove "Giambelli's formula":

$$\overline{\pi}_{b,a} = \overline{\pi}_{a,0}\overline{\pi}_{b,0} - \overline{\pi}_{a-1,0}\overline{\pi}_{b+1,0}.$$

Thus R_n^* is generated as a $\mathbb{Z}$-algebra by the "special classes" $\overline{\pi}_{b,0}$ for $1 \le b < n$.

Problem 10. For generic choices of complete flags $V_\bullet$ and $W_\bullet$ in V, for integers $0 \le a \le b < n$ and $0 \le l < n$, check that $[\Sigma_{l,0}(V_\bullet) \cap \Sigma_{b,a}(W_\bullet)]$ equals $\sum_{i=0}^{l} \sigma_{b+l-i,a+i}$. This strongly suggests that there is a natural "intersection product" on $A_*(\text{Grass}(\mathbb{P}^1, \mathbb{P}V))$ such that Φ is an isomorphism of rings. In the following exercises, assume this.

Problem 11. Inside A_3^* check the following identities,

$$\sigma_{1,0}^2 = \sigma_{2,0} + \sigma_{1,1}, \ \sigma_{1,0}\sigma_{2,0} = \sigma_{1,0}\sigma_{1,1} = \sigma_{2,1}, \ \sigma_{1,0}\sigma_{2,1} = \sigma_{2,2}, \ \sigma_{2,0}\sigma_{2,0} = \sigma_{1,1}\sigma_{1,1} = \sigma_{2,1}, \ \sigma_{2,0}\sigma_{1,1} = 0.$$

In particular, check that $\sigma_{1,0}^4$ equals 1.

Problem 12. Let $X \subset \mathbb{P}^3$ be a smooth, degree d hypersurface, and assume that the characteristic of p is prime to $d(d-1)$. Denote,

$$\text{Tan}(X) := \{[L] \in \text{Grass}(\mathbb{P}^1, \mathbb{P}^3) : \exists p \in L, \ 2\underline{p} \subset L \cap X\},$$

i.e., X is tangent to L at some point p in L. Use the method of test families to prove the identity,

$$[\mathrm{Tan}(X)] = d(d-1)\sigma_{1,0}.$$

Problem 13 Let $C \subset \mathbb{P}^3$ be a smooth, linearly nondegenerate curve with degree d and with genus g. Denote,

$$\mathrm{Inc}(C) := \{[L] \in \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^3) : L \cap C \neq \emptyset\},$$

i.e., L intersects C. Use the method of test families to prove the identity,

$$[\mathrm{Inc}(C)] = d\sigma_{1,0}.$$

In particular, for smooth, linearly nondegenerate curves $C_1, C_2 \subset \mathbb{P}^3$ of degrees d_1, d_2, for a generic projective linear equivalence $g : \mathbb{P}^3 \to \mathbb{P}^3$, conclude that

$$[\mathrm{Inc}(C_1) \cap \mathrm{Inc}(gC_2)] = d_1 d_2 \sigma_{2,0} + d_1 d_2 \sigma_{1,1}.$$

Problem 14. Let $C \subset \mathbb{P}^3$ be a smooth, linearly nondegenerate curve with degree d and with genus g. Denote,

$$\mathrm{Sec}^2(C) := \{[L] \in \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^3) : \exists \underline{p} + \underline{q} \in \mathrm{Sym}^2(C),\ \underline{p} + \underline{q} \subset L \cap C\},$$

i.e., L intersects C in a divisor of degree at least 2 in C. Use the method of test families to prove the identity,

$$[\mathrm{Sec}^2(C)] = \left(\frac{(d-1)(d-2)}{2} - g\right)\sigma_{2,0} + \frac{d(d-1)}{2}\sigma_{1,1}.$$

Now let g_{t_t} be a one-parameter family of projective equivalences of $\mathbb{P}^3$ such that g_0 is the identity. Prove that, as t specializes to 0, the "flat limit" of $\mathrm{Inc}(C) \cap \mathrm{Inc}(g_t C)$ contains $\mathrm{Sec}^2(C)$ with multiplicity 2. How do you account for the discrepancy,

$$[\mathrm{Inc}(C) \cap \mathrm{Inc}(g_t C)] - 2[\mathrm{Sec}^2(C)] = (3d + g - 2)\sigma_{2,0} + d\sigma_{1,1}?$$

In particular, note that the flat limit depends on the family $(g_t)_t$. Precisely how does the family enter? This illustrates that some care must be exercised when computing intersections by specialization and "conservation of number".

Problem 15. Let $C \subset \mathbb{P}^3$ be a smooth, linearly nondegenerate curve with degree d and with genus g. Denote,

$$\mathrm{Tan}(C) := \{[L] \in \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^3) : \exists p \in C,\ 2\underline{p} \subset L \cap C\},$$

i.e., L is tangent to C at some point p. Use the method of test families to prove the identity,

$$[\mathrm{Tan}(C)] = (2d + 2g - 2)\sigma_{2,1}.$$

Problem 16. Double-check the identity from lecture for the "Plücker degree" of $\text{Grass}(\mathbb{P}^1, \mathbb{P}^n)$, i.e., check

$$\sigma_{1,0}^{2n-2} = \frac{1}{n}\binom{2n-2}{n-1}.$$

Problem 17. Give a second computation of the identity from the first lecture, i.e., for generic complete flags $V_\bullet^a$, $V_\bullet^b$, $V_\bullet^c$ and $V_\bullet^d$ of a vector space V of dimension $n+1 = 2m$, we have

$$[\Sigma_{m-1,0}(Va_\bullet) \cap \Sigma_{m-1,0}(V_\bullet^b) \cap \Sigma_{m-1,0}(V_\bullet^c) \cap \Sigma_{m-1,0}(V_\bullet^d)] = \sigma_{m-1,0}^4 = m\sigma_{2m-2,2m-2}.$$

Problem 18. Let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface. Denote,

$$\text{Fano}_1(Q) := \{[L] \in \text{Grass}(\mathbb{P}^1, \mathbb{P}^n) : L \subset Q\},$$

i.e., the line L is contained in the hypersurface Q. Use the method of test families to prove the identity,

$$[\text{Fano}_1(Q)] = (2\overline{s} + 0\overline{t})(1\overline{s} + 1\overline{t})(0\overline{s} + 2\overline{t}) = 4\sigma_{2,1}.$$

Problem 19. This problem is considerably harder without further techniques, but worth seeing now. Let $X \subset \mathbb{P}^n$ be a smooth cubic hypersurface. Denote,

$$\text{Fano}_1(X) := \{[L] \in \text{Grass}(\mathbb{P}^1, \mathbb{P}^n) : L \subset X\},$$

i.e., the line L is contained in the hypersurface X. Use the method of test families to prove the identity,

$$[\text{Fano}_1(X)] = (3\overline{s} + 0\overline{t})(2\overline{s} + 1\overline{t})(1\overline{s} + 2\overline{t})(0\overline{s} + 3\overline{t}) = 18\sigma_{3,1} + 45\sigma_{2,2}.$$

Problem 20. For a degree d hypersurface $X \subset \mathbb{P}^n$, denote

$$\text{Fano}_1(X) := \{[L] \in \text{Grass}(\mathbb{P}^1, \mathbb{P}^n) : L \subset X\},$$

i.e., the line L is contained in the hypersurface X. For a sufficiently general degree d hypersurface X, we will eventually prove that $\text{Fano}_1(X)$ is smooth of the "expected" codimension $d+1$, and has class,

$$[\text{Fano}_1(X)] = (d\overline{s} + 0\overline{t}((d-1)\overline{s} + 1\overline{t}) \cdots ((d-r)\overline{s} + r\overline{t}) \cdots (1\overline{s} + (d-1)\overline{t})(0\overline{s} + d\overline{t}).$$

However, for every integer $d \geq 4$, for some choice of n, find an example where the dimension of $\text{Fano}_1(X)$ is strictly larger than the "expected" dimension. Can you find any such example where $d \leq n$? This is related to the (open) Debarre - de Jong Conjecture.