MAT 614 Notes on Infinitesimal Deformation Theory

These are some notes accompanying the discussion in lecture on deformation theory and obstruction theory. The canonical sources are Schlessinger’s thesis, [Sch68], and Artin’s articles on algebraization, [Art69], [Art74]. For the obstruction theory of the Hilbert scheme, the original source is [Art69], and an excellent treatment is also given in [Kol96 I.2].

1 Local Artin Algebras and Complete, Noetherian, Local Algebras.

For every local ring \( R \) in what follows, \( m_R \) denotes the maximal ideal.

Let \( \tilde{A} \) be a complete, regular, local Noetherian ring, let \( E \subset m_{\tilde{A}}^2 \) be an ideal, and denote the quotient by \( A \). Thus, \( A \) is also a complete, local Noetherian ring. Denote the residue field \( A/m_A \) by \( k \).

Denote by \( \mathcal{C} = \mathcal{C}_A \) the category whose objects are \( A \)-algebras \( \hat{A} \) such that

(i) \( \hat{A} \) is a complete, local, Artin ring with \( m_A \cdot \hat{A} \subset m_{\hat{A}} \), and

(ii) the induced homomorphism \( k \rightarrow \hat{A}/m_{\hat{A}} \) is an isomorphism.

The morphisms in \( \mathcal{C}_A \) are homomorphisms of \( A \)-algebras; these are automatically local homomorphisms. Similarly, denote by \( \hat{\mathcal{C}} = \hat{\mathcal{C}}_A \) the category whose objects are \( \Lambda \)-algebras \( \hat{A} \) such that

(i) \( \hat{A} \) is a complete, local, Noetherian ring with \( m_{\Lambda} \cdot \hat{A} \subset m_{\hat{A}} \), and

(ii) the induced homomorphism \( k \rightarrow \hat{A}/m_{\hat{A}} \) is an isomorphism.

The morphisms in \( \hat{\mathcal{C}}_A \) are local homomorphisms of \( \Lambda \)-algebras. Of course \( \mathcal{C}_A \) is a full subcategory of \( \hat{\mathcal{C}}_A \). Moreover, for every \( \hat{A} \) in \( \hat{\mathcal{C}}_A \), for every integer \( N > 0 \), the \( \Lambda \)-algebra \( \hat{A}/m_{\hat{A}}^N \) is an object of \( \mathcal{C}_A \).

For every object \( \hat{R} \) in \( \hat{\mathcal{C}}_A \), denote by \( h_{\hat{\mathcal{C}}, \hat{R}} \) the covariant functor

\[
h_{\hat{\mathcal{C}}, \hat{R}} : \hat{\mathcal{C}}_A \rightarrow \text{Sets}, \quad \hat{S} \mapsto \text{Hom}_{\hat{\mathcal{C}}} (\hat{R}, \hat{S}).
\]
Also denote by $h_{\mathcal{C,R}}$ the restriction of $h_{\mathcal{C},\hat{R}}$ to the full subcategory $\mathcal{C}$. Observe that $h_{\mathcal{C},\hat{R}}(k)$ is a singleton set. A pointed functor on $\mathcal{C}_A$ is a covariant functor

$$F_\mathcal{C}: \mathcal{C}_A \to \text{Sets}$$

such that $F_\mathcal{C}(k)$ is a singleton set, and similarly for a pointed functor $F_{\hat{\mathcal{C}}}$ on $\hat{\mathcal{C}}_A$. Each of the categories $\mathcal{C}_A$ and $\hat{\mathcal{C}}_A$ has a small, reflective subcategory, e.g., the full category of all objects that are quotients of $\Lambda[t_1,t_2,\ldots]$ as complete, local, $\Lambda$-algebras. Every natural transformation between pointed functors on $\mathcal{C}_A$, resp. $\hat{\mathcal{C}}_A$, is uniquely determined by its restriction to this small, reflective subcategory. There is a set of natural transformations between these restriction functors. Therefore, there is a set of natural transformations between two pointed functors (the “realization” of this set depends on the small, reflective subcategory only up to unique bijection). Thus there is a category $\text{Fun}(\mathcal{C}_A)$, resp. $\text{Fun}(\hat{\mathcal{C}}_A)$ of pointed functors on $\mathcal{C}_A$, resp. on $\hat{\mathcal{C}}_A$. Finally, by the Yoneda Lemma there is a fully faithful embedding

$$h: \mathcal{C} \to \text{Fun}(\mathcal{C}_A), \ A \mapsto h_{\mathcal{C},A},$$

and there is a fully faithful embedding

$$h: \hat{\mathcal{C}} \to \text{Fun}(\hat{\mathcal{C}}_A), \ \hat{R} \mapsto h_{\mathcal{C},\hat{R}}.$$  

Moreover, for every object $A$ of $\mathcal{C}_A$ and for every pointed functor $F$ on $\mathcal{C}_A$, there is a bifunctorial bijection

$$\text{Hom}_{\text{Fun}(\mathcal{C})}(h_{\mathcal{C},A}, F) \leftrightarrow F(A)$$

sending a natural transformation $\theta$ to the image under $\theta$ of $\text{Id}_A \in h_{\mathcal{C},A}(A)$. There is a similar bifunctorial bijection for $\hat{\mathcal{C}}_A$.

Associated to the fully faithful embedding of $\mathcal{C}_A$ in $\hat{\mathcal{C}}_A$, there is a restriction functor

$$\bullet|_\mathcal{C}: \text{Fun}(\hat{\mathcal{C}}_A) \to \text{Fun}(\mathcal{C}_A).$$

For a pointed functor $F_{\hat{\mathcal{C}}}$ on $\hat{\mathcal{C}}_A$, denote by $F_{\hat{\mathcal{C}}}|_\mathcal{C}$ the restriction to $\mathcal{C}_A$. Similarly, for a natural transformation $\eta$ of pointed functors on $\hat{\mathcal{C}}_A$, denote by $\eta|_\mathcal{C}$ the associated natural transformation of restriction functors. As we will see below, there is an important right adjoint to the restriction functor.

For the pointed functor $h_{\mathcal{C},\hat{R}}$, for every object $\hat{S}$ of $\hat{\mathcal{C}}_A$, since $\hat{S}$ is complete the following natural map is a bijection

$$h_{\mathcal{C},\hat{R}}(\hat{S}) \to \varprojlim_N h_{\mathcal{C},\hat{R}}(\hat{S}/\mathfrak{m}_S^N).$$

In general, a pointed functor $G_{\hat{\mathcal{C}}}$ on $\hat{\mathcal{C}}_A$ is continuous if for every object $\hat{S}$ of $\hat{\mathcal{C}}_A$ the following natural map is a bijection

$$G_{\hat{\mathcal{C}}}(\hat{S}) \to \varprojlim_N G_{\hat{\mathcal{C}}}|_\mathcal{C}(\hat{S}/\mathfrak{m}_S^N).$$
For every pointed functor $F_C$ on $\mathcal{C}_\Lambda$, there exists a pointed functor $\widehat{F}_C$ on $\widehat{\mathcal{C}}_\Lambda$ defined by

$$\widehat{F}_C(\widehat{S}) := \lim_{\leftarrow N} F_C(\widehat{S}/m_{\widehat{S}}^N).$$

Moreover, for every natural transformation of pointed functors $\theta : F_C \Rightarrow F'_C$, there is an associated natural transformation

$$\widehat{\theta} : \widehat{F}_C \Rightarrow \widehat{F}'_C, \quad \widehat{\theta}(\widehat{S}) = \lim_{\leftarrow N} \theta(\widehat{S}/m_{\widehat{S}}^N).$$

Together these operations define a functor,

$$\bullet : \text{Fun}(\mathcal{C}_\Lambda) \to \text{Fun}(\widehat{\mathcal{C}}_\Lambda).$$

For every pointed functor $F_C$ on $\mathcal{C}_\Lambda$, there is a natural transformation of pointed functors on $\mathcal{C}_\Lambda$,

$$\alpha'_F : F_C \Rightarrow \widehat{F}_C|_C, \quad F_C(A) \to \lim_{\leftarrow N} F_C(A/m_A^N).$$

In fact this is a natural transformation from the identity functor on $\text{Fun}(\mathcal{C}_\Lambda)$ to the composite functor $(\bullet)|_C$,

$$\alpha' : \text{Id}_{\text{Fun}(\mathcal{C})} \Rightarrow (\bullet)|_{\mathcal{C}}.$$

Moreover, because $A \to A/m_A^N$ is an isomorphism for $N$ sufficiently large, each $\alpha'_F$ is a natural equivalence of functors, i.e., $\alpha'$ is a natural equivalence of functors. For this reason we shall usually identify $F_C$ with $\widehat{F}_C|_C$. Moreover, we will denote by $\alpha$ and $\alpha_F$ the inverse natural equivalence,

$$\alpha : (\bullet)|_C \Rightarrow \text{Id}_{\text{Fun}(\mathcal{C})}.$$}

Similarly, for every pointed functor $G$ on $\widehat{\mathcal{C}}_\Lambda$, there is a natural transformation of pointed functors on $\widehat{\mathcal{C}}_\Lambda$,

$$\beta_G : G \Rightarrow \widehat{G}|_C, \quad G(\widehat{S}) \to \lim_{\leftarrow N} G|_C(\widehat{S}/m_{\widehat{S}}^N).$$

This is also natural in $G$, hence defines a natural transformation from the identity functor on $\widehat{\mathcal{C}}$ to the composite functor $(\bullet)|_{\widehat{\mathcal{C}}}$,

$$\beta : \text{Id}_{\text{Fun}(\widehat{\mathcal{C}})} \Rightarrow (\bullet)|_{\widehat{\mathcal{C}}}.$$ By definition, $G$ is continuous if and only if $\beta_G$ is a natural bijection.

For every pointed functor $G$ on $\widehat{\mathcal{C}}$ and for every pointed functor $F$ on $\mathcal{C}$, for every natural transformation of pointed functors on $\mathcal{C}$,

$$\theta : G|_C \Rightarrow F,$$

there is an associated natural transformation of pointed functors on $\widehat{\mathcal{C}}$,

$$\widehat{\theta} \circ \beta_G : F \Rightarrow \widehat{G}|_C \Rightarrow \widehat{F}.$$
Similarly, for every natural transformation of pointed functors on $\mathcal{C}$,

$$\eta : G \Rightarrow \tilde{F}$$

there is an associated natural transformation of pointed functors on $\mathcal{C}$,

$$\alpha_F \circ \eta|_C : G|_C \Rightarrow \tilde{F}|_C \Rightarrow F.$$  

**Lemma 1.1.** The functors $\bullet|_C$ and $\bullet$ together with the natural transformations $\alpha$ and $\beta$ form an adjoint pair, i.e., the bifunctorial set maps

$$\text{Hom}_{\text{Fun}(\mathcal{C})}(G|_C, F) \rightarrow \text{Hom}_{\text{Fun}(\hat{\mathcal{C}})}(G, \hat{F}), \theta \mapsto \hat{\theta} \circ \beta_G;$$

$$\text{Hom}_{\text{Fun}(\hat{\mathcal{C}})}(G, \hat{F}) \rightarrow \text{Hom}_{\text{Fun}(\mathcal{C})}(G|_C, F), \eta \mapsto \alpha_F \circ \eta|_C,$$

are inverse bijections for every $F$ and $G$.

**Proof.** This will be an exercise on Problem Set 2. \hfill $\Box$

We are mainly interested in the case where $G$ is a representable functor, $h_{\hat{\mathcal{C}}, \hat{R}}$. In this case, by Lemma 1.1, every natural transformation $\theta : h_{\mathcal{C}, \hat{R}} \Rightarrow F$ is equivalent to a natural transformation $\hat{\theta} \circ \beta_{\hat{R}} : h_{\hat{\mathcal{C}}, \hat{R}} \Rightarrow \tilde{F}$. By the Yoneda Lemma, this is equivalent to an element of $\hat{F}(\hat{R})$, i.e., a datum

$$(\theta_N)_{N > 0}, \theta_N \in F(\hat{R}/m^N_{\hat{R}})$$

that is a compatible family in the sense that for every $N > 0$, for the set map $F(\hat{R}/m^{N+1}_{\hat{R}}) \rightarrow F(\hat{R}/m^N_{\hat{R}})$ associated to the canonical surjection, the element $\theta_{N+1}$ maps to $\theta_N$.

Here is the basic definition of this section.

**Definition 1.2.** A pointed functor $F$ on $\mathcal{C}_\Lambda$ is prorepresentable if there exists an object $\hat{R}$ in $\mathcal{C}_\Lambda$ and a natural equivalence of functors on $\mathcal{C}_\Lambda$, $\theta : h_{\mathcal{C}, \hat{R}} \Rightarrow F$.

By the discussion above, $F$ is prorepresentable if and only if there exists a natural equivalence of functors there exists an object $\hat{R}$ in $\mathcal{C}_\Lambda$ and a compatible family $(\theta_N)_{N > 0}$ of elements $\theta_n \in F(\hat{R}/m^N_{\hat{R}})$ such that for every object $A$ of $\mathcal{C}_\Lambda$, for every object $a \in F(A)$, and for one (hence every) integer $N > 0$ such that $m^N_{\hat{R}}$ equals $\{0\}$, there exists a unique local homomorphism of $\Lambda$-algebra, $u_n : \hat{R}/m^N_{\hat{R}} \rightarrow A$ such that $F(u_n)$ maps $\theta_n$ to $a$.

The thesis of Michael Schlessinger, [Sch68], and also work of Rim, characterizes prorepresentable functors, as well as functors admitting a hull, in terms of an efficient list of axioms that can be verified for many functors of interest.
2 Obstruction Theory.

Let $F$ be a pointed functor on $C_\Lambda$. There are many different uses of obstruction theories, hence there are many definitions. The following definition is essentially the definition of Artin, [Art74, Definition (2.6) p. 169].

**Definition 2.1.** An *infinitesimal extension* in $C_\Lambda$ is a surjective homomorphism $q : A' \to A$ such that $m_{A'} \cdot \text{Ker}(q)$ equals $\{0\}$. For infinitesimal extensions $q_A : A' \to A$ and $q_B : B' \to B$, a *morphism* of infinitesimal extensions is a pair $(u, u')$ of morphisms $u : A \to B$ and $u' : A' \to B'$ such that $u \circ q_A$ equals $q_B \circ u'$. The restriction of $u'$ to $\text{Ker}(q_A)$ gives a morphism denoted by $u'_K : \text{Ker}(q_A) \to \text{Ker}(q_B)$.

A *deformation situation* is a pair $(q : A' \to A, a)$ of an infinitesimal extension and an element $a \in F(A)$. For deformation situations $(q_A : A' \to A, a)$ and $(q_B : B' \to B, b)$, a *morphism* of deformation situations is a morphism of infinitesimal extensions, $(u, u')$, such that $F(u)$ maps $a$ to $b$.

With these notions, there is a category $\text{InfDef}_\Lambda$ of infinitesimal extensions as well as a category $\text{Def}_{\Lambda, F}$ of deformation situations. There is a forgetful functor

$$\Phi : \text{Def}_{\Lambda, F} \to \text{InfDef}_\Lambda.$$ 

For a morphism $(u, u')$ of infinitesimal deformations, denote by $u'_K$ the restriction of $u'$ to $\text{Ker}(q_A)$,

$$u'_K : \text{Ker}(q_A) \to \text{Ker}(q_B).$$

This is a $k$-linear map of finite dimensional $k$-vector space, and it is functorial in $(u, u')$. Thus there is a functor from the category of infinitesimal extensions to the category of finite dimensional $k$-vector space,

$$K : \text{InfDef}_\Lambda \to \text{Vec}_k, \ (q : A' \to A) \mapsto \text{Ker}(q), \ (u, u') \mapsto u'_K.$$ 

Also there is the composite functor,

$$K \circ \Phi : \text{Def}_{\Lambda, F} \to \text{Vec}_k.$$ 

Similarly, for every finite dimensional $k$-vector space $\mathcal{O}$, there is a functor

$$\mathcal{O} \otimes_k K : \text{InfDef}_\Lambda \to \text{Vec}_k, \ (q : A' \to A) \mapsto \mathcal{O} \otimes_k \text{Ker}(q), \ (u, u') \mapsto \text{Id}_\mathcal{O} \otimes_k u'_K,$$

and there is also the functor

$$\mathcal{O} \otimes_k (K \circ \Phi) : \text{Def}_{\Lambda, F} \to \text{Vec}_k.$$ 

A *section* $o$ of the functor $\mathcal{O} \otimes_k (K \circ \Phi)$ over $\text{Def}_{\Lambda, F}$ is an assignment to every deformation situation $(q : A' \to A, a)$ of an element $q_{q, a} \in \mathcal{O} \otimes_k \text{Ker}(q)$ such that for every morphism of deformation situations, $(u, u')$, the $k$-linear map

$$\text{Id}_\mathcal{O} \otimes_k u'_K : \mathcal{O} \otimes_k \text{Ker}(q_A) \to \mathcal{O} \otimes_k \text{Ker}(q_B)$$
maps \( o_{q,a} \) to \( o_{q',b} \). Equivalently, introducing the functor
\[
1 : \text{Def}_\Lambda \to \text{Vec}_k, \quad 1(q, a) = k, \quad 1(u, u') = \text{Id}_k,
\]
a section \( o \) is a natural transformation \( o : 1 \Rightarrow \mathcal{O} \otimes_k (K \circ \Phi) \).

**Definition 2.2.** An obstruction theory for \( F \) is a pair \((\mathcal{O}, o)\) of a finite dimensional \( k \)-vector space \( \mathcal{O} \) together with a section \( o \) of \( \mathcal{O} \otimes_k (K \circ \Phi) \) over \( \text{Def}_\Lambda \) such that for every deformation situation \((q : A' \to A, a)\), there exists \( a' \in F(A') \) mapping to \( a \) under \( F(q) \) if and only if \( o(q,a) \) equals 0 as elements of \( \mathcal{O} \otimes_k \text{Ker}(q) \). A morphism of obstruction theories, \((\mathcal{O}, o)\) and \((\mathcal{O}', o')\), is a \( k \)-linear map \( L : \mathcal{O} \to \mathcal{O}' \) such that for every deformation situation \((q : A' \to A, a)\), \( L \otimes_k \text{Id}_{\text{Ker}(f)} \) maps \( o_{q,a} \) to \( o'_{q,a} \).

Every prorepresentable functor \( h_{C,\hat{R}} \) has a canonical associated obstruction theory that is functorial in \( \hat{R} \). The simplest construction I know of passes through non-Noetherian rings. So the obstruction theory that follows is not quite the canonical one, however it is (non-canonically) isomorphic to the canonical obstruction theory. Before describing the obstruction theory, there is some setup. Denote by \( \Lambda_n := \Lambda \left[ [t_1, \ldots, t_n] \right] \) the power series ring over \( \Lambda \), considered as a complete, local, Noetherian \( \Lambda \)-algebra. For every complete, local, Noetherian \( \Lambda \)-algebra \( \hat{R} \) and for every ordered \( n \)-tuple of elements \( \underline{r} = (r_1, \ldots, r_n) \) in \( m_{\hat{R}}^{\oplus n} \), by the universal property of power series algebras there exists a unique local homomorphism of \( \Lambda \)-algebras,
\[
f_{\hat{R},\underline{r}} : \Lambda \left[ [t_1, \ldots, t_n] \right] \to \hat{R}, \quad f(t_i) = r_i.
\]
In fact, the rule \( \underline{r} \mapsto f_{\hat{R},\underline{r}} \) gives a natural equivalence of functors \( \hat{\Phi}^{\oplus n} \to h_{C,\Lambda_n} \), where \( m^{\oplus n} \) is the obvious functor
\[
m^{\oplus n} : \widehat{C}_\Lambda \to \text{Sets}, \quad \hat{R} \mapsto m^{\oplus n}_\hat{R}, \quad (u : \hat{R} \to \hat{S}) \mapsto (u^{\oplus n} : m^{\oplus n}_\hat{R} \to m^{\oplus n}_\hat{S}).
\]
Using Nakayama’s Lemma and completeness, \( f_{\hat{R},\underline{r}} \) is surjective if and only if the images \( \overline{r}_1, \ldots, \overline{r}_n \in m_{\hat{R}}^{\oplus n} \) generate as a \( k \)-vector space. Moreover, the images form a \( k \)-basis if and only if the induced map on \( m/m^2 \) is an isomorphism of \( k \)-vector spaces. In particular, for \( r_1, \ldots, r_n \in m_{\Lambda_n} \), the associated homomorphism
\[
f_{\Lambda_n,\underline{r}} : \Lambda \left[ [t_1, \ldots, t_n] \right] \to \Lambda \left[ [t_1, \ldots, t_n] \right], \quad f(t_i) = r_i
\]
is an isomorphism if and only if \( \overline{r}_1, \ldots, \overline{r}_n \) forms a basis for \( m/m^2 \). Also \( f_{\Lambda_n,\underline{r}} \) is an isomorphism inducing the identity map on \( m/m^2 \) if and only if each \( r_i \) equals \( t_i + s_i \) for \( s_i \in m^2 \). For each ordered \( n \)-tuple \( \underline{s} = (s_1, \ldots, s_n) \) of elements \( s_i \in m^2 \), denote by \( f_{\Lambda_n,\underline{s}+\underline{r}} \) the associated isomorphism of \( \Lambda \left[ [t_1, \ldots, t_n] \right] \) inducing the identity on \( m/m^2 \).

For every object \( \hat{R} \) of \( \widehat{C}_\Lambda \), and for every \( \underline{r} = (r_1, \ldots, r_n) \) in \( m_{\hat{R}} \) mapping to a basis \( \overline{r}_1, \ldots, \overline{r}_n \) of \( m_{\hat{R}}/m_{\hat{R}}^2 \), there exists a surjection
\[
f_{\hat{R},\underline{r}} : \Lambda \left[ [t_1, \ldots, t_n] \right] \to \hat{R},
\]
mapping the basis \( (\overline{t}_i)_i \) of \( m/m^2 \) to the basis \( (\overline{r}_i)_i \) of \( m/m^2 \).
**Definition 2.3.** For every object \( \hat{R} \) in \( \mathcal{C}_\Lambda \) and for every lift \( \underline{r} \) to \( m\hat{R} \) of a \( k \)-basis of \( m\hat{R}/m\hat{R}^2 \), define the associated obstruction space to be the finite dimensional "dual" \( k \)-vector space

\[
\mathcal{O}_{\hat{R}, \underline{r}} := \text{Hom}_k(\mathcal{O}(\hat{R}, \underline{r}))/m \cdot \mathcal{O}(\hat{R}, \underline{r}), \quad k.
\]

If \( \underline{r}' \) maps to the same basis, then \( f_{\hat{R}, \underline{r}} \) equals \( f_{\hat{R}, \underline{r}} \circ f_{\hat{R}, \underline{r}+2} \) for some choice of \( s \).

**Lemma 2.4.** For every integer \( e \geq 0 \), the \( \Lambda \)-algebra automorphism \( f_{\Lambda, n \underline{r}} \) maps \( m_{\Lambda, n} \cdot \text{Ker}(f_{\hat{R}, \underline{r}}) \) isomorphically onto \( m_{\Lambda, n} \cdot \text{Ker}(f_{\hat{R}, \underline{r}}) \). In particular, \( f_{\Lambda, n \underline{r}} \) induces a \( k \)-linear isomorphism

\[
f_{\Lambda, n \underline{r}} : \text{Ker}(f_{\hat{R}, \underline{r}})/\left(m_{\Lambda, n} \cdot \text{Ker}(f_{\hat{R}, \underline{r}}) + m_{\Lambda, n} \cap \text{Ker}(f_{\hat{R}, \underline{r}})\right) \to \text{Ker}(f_{\hat{R}, \underline{r}})/\left(m_{\Lambda, n} \cdot \text{Ker}(f_{\hat{R}, \underline{r}}) + m_{\Lambda, n} \cap \text{Ker}(f_{\hat{R}, \underline{r}})\right)
\]

If \( \text{Ker}(f_{\hat{R}, \underline{r}}) \) is contained in \( \text{Ker}(f_{\hat{R}, \underline{r}}) + m_{\Lambda, n} \cdot \text{Ker}(f_{\hat{R}, \underline{r}}) \), then the following ideals in \( \Lambda \left[t_1, \ldots, t_n\right] \) are equal,

\[
\text{Ker}(f_{\hat{R}, \underline{r}}) + m_{\Lambda, n} = \text{Ker}(f_{\hat{R}, \underline{r}}) + m_{\Lambda, n},
\]

\[
m_{\Lambda, n} \cdot \text{Ker}(f_{\hat{R}, \underline{r}}) + m_{\Lambda, n} = m_{\Lambda, n} \cdot \text{Ker}(f_{\hat{R}, \underline{r}}) + m_{\Lambda, n},
\]

and the \( k \)-linear map \( f_{\Lambda, n \underline{r}} \) above is an automorphism.

**Proof.** This will be an exercise on Problem Set 2. \( \square \)

When \( e = 0 \), the morphism \( f_{\Lambda, n \underline{r}} \) induces an isomorphism of obstruction spaces, \( \mathcal{O}_{\hat{R}, \underline{r}} \to \mathcal{O}_{\hat{R}, \underline{r}'} \) (note that the variance is reversed).

Let \( q : A' \to A \) be an infinitesimal extension in \( \mathcal{C}_\Lambda \). Let \( a : \hat{R} \to A \) be a morphism in \( \mathcal{C}_\Lambda \). Then there is an induced morphism

\[
\tilde{a} : \Lambda \left[t_1, \ldots, t_n\right] \to A, \quad \tilde{a} = a \circ f_{\hat{R}, \underline{r}}.
\]

Denote by \( (\tilde{a}_1, \ldots, \tilde{a}_n) \) the images of \( (t_1, \ldots, t_n) \) under \( \tilde{a} \). Since \( q \) is surjective, there exists an ordered \( n \)-tuple of elements \( (\tilde{a}_1', \ldots, \tilde{a}_n') \) mapping to \( (\tilde{a}_1, \ldots, \tilde{a}_n) \) under \( q \). By the universal property of power series algebras, there exists a unique local homomorphism of \( \Lambda \)-algebras,

\[
\tilde{a}' : \Lambda \left[t_1, \ldots, t_n\right] \to A'
\]

mapping \( (t_1, \ldots, t_n) \) to \( (\tilde{a}_1', \ldots, \tilde{a}_n') \). By construction \( q \circ \tilde{a}' \) equals \( \tilde{a} \). Thus \( \tilde{a}' \) maps \( \text{Ker}(f_{\hat{R}, \underline{r}}) \) to \( \text{Ker}(q) \). In particular \( \tilde{a}' \) maps \( \text{Ker}(f_{\hat{R}, \underline{r}}) \) to \( \text{Ker}(q) \). Denote this induced \( \Lambda \left[t_1, \ldots, t_n\right] \)-module homomorphism by

\[
o_{q, a, \underline{r}, \tilde{a}}^{\text{pre}} : \text{Ker}(f_{\hat{R}, \underline{r}}) \to \text{Ker}(q).
\]

Because \( q : A' \to A \) is an infinitesimal extension, \( m_{A'} \cdot \text{Ker}(q) \) equals \( \{0\} \). Because \( \tilde{a}' \) is a local homomorphism, the image of \( m \cdot \text{Ker}(f_{\hat{R}, \underline{r}}) \) is contained in \( m_{A'} \cdot \text{Ker}(q) \), which is \( \{0\} \). Thus the map above factors uniquely through a \( k \)-linear map

\[
o_{q, a, \underline{r}, \tilde{a}} : \text{Ker}(f_{\hat{R}, \underline{r}})/m_{\Lambda, n} \cdot \text{Ker}(f_{\hat{R}, \underline{r}}) \to \text{Ker}(q).
\]

Because the \( k \)-vector spaces involved are finite dimensional, this is equivalent to an element in \( \mathcal{O}_{\hat{R}, \underline{r}} \otimes_k \text{Ker}(q) \).
Proposition 2.5. Each element \( o_{q,a,\mathbb{L}} \in \mathcal{O}_{\mathbb{R}_\mathbb{L}} \otimes_k \text{Ker}(q) \) is independent of the choice of lift \( \tilde{a}' \) of \( a \). Moreover, the pair \((\mathcal{O}_{\mathbb{R}_\mathbb{L}}, o)\) is an obstruction theory for the Yoneda functor \( h_{\mathbb{C},\mathbb{R}} \).

Proof. Every other lift is \( \tilde{a}''_i = \tilde{a}'_i + \kappa_i \) for elements \( \kappa_i \in \text{Ker}(q) \). In particular, every quadratic monomial is
\[
\tilde{a}''_i \cdot \tilde{a}''_j = \tilde{a}'_i \cdot \tilde{a}'_j + (\tilde{a}'_i \cdot \kappa_j + \tilde{a}'_j \cdot \kappa_i + \kappa_i \cdot \kappa_j).
\]
Since \( m_{A'} \cdot \text{Ker}(q) \) equals \( \{0\} \), each element in the parentheses equals 0 in \( A' \). Therefore the restricted map
\[
\tilde{a}_2 : m_{\Lambda_n}^2 \rightarrow A'
\]
is independent of the choice of the lift \( \tilde{a}' \). Denote this map by \( \tilde{a}_2 \). Since \( \text{Ker}(f_{\mathbb{R}_\mathbb{L}}) \) is contained in \( m_{\Lambda_n}^2 \), it follows that \( o_{q,a,\mathbb{L}} \) is independent of the choice of \( \tilde{a}' \).

If there exists a lift \( a' : \tilde{R} \rightarrow A' \) of \( a \), then we can define \( \tilde{a}' = a' \circ f_{\mathbb{R}_\mathbb{L}} \). This maps \( \text{Ker}(f_{\mathbb{R}_\mathbb{L}}) \) to \( \{0\} \), so that \( o_{q,a,u,r} \) equals 0. Conversely, if \( o_{q,a,\mathbb{L}} \) equals 0, then \( \tilde{a}' \) maps \( \text{Ker}(f_{\mathbb{R}_\mathbb{L}}) \) to \( \{0\} \), hence factors through a morphism \( a' : \tilde{R} \rightarrow A' \) that lifts \( a \). Therefore \( o_{q,a,\mathbb{L}} \) equals 0 if and only if there exists a lift \( a' \) of \( a \).

For a morphism \((u,u')\) of deformation situations \((q_A : A' \rightarrow A,a) \rightarrow (q_B : B' \rightarrow B,b)\), for every lift \( \tilde{a}' \) of \( \tilde{a} \), the composition \( \tilde{b} := u' \circ \tilde{a}' \) is a lift of \( b \). Using these two lifts, it follows directly that \( \text{Id}_\mathcal{O} \otimes_k u_K' \) maps \( o_{q_A,a,\mathbb{L}} \) to \( o_{q_B,b,\mathbb{L}} \). Therefore \( o \) is a section of \( \mathcal{O}_{\mathbb{R}_\mathbb{L}} \otimes (K \circ \Phi) \) over the category of deformation situations. Therefore \((\mathcal{O}_{\mathbb{R}_\mathbb{L}}, o)\) is an obstruction theory for \( h_{\mathbb{C},\mathbb{R}} \).

One can eliminate the dependence of this obstruction theory on \( _\mathbb{L} \), and also make the obstruction theory functorial in \( \mathbb{R} \), but this requires a detour through non-Noetherian rings. For every \( \mathbb{R} \), denote by \( M = M_{\mathbb{R}} \) the free \( \Lambda \)-module on the underlying set of \( m_{\mathbb{R}} \). One can begin with a canonical choice of a free \( \Lambda \)-module \( M \) together with a surjection \( M \rightarrow m_{\mathbb{R}} \), e.g., the free module on the underlying set of \( m_{\mathbb{R}} \). Then one can form the (infinitely generated) symmetric algebra \( \Lambda[M] \). One can complete this with respect to the maximal ideal \( \mathfrak{m} \cdot \Lambda[M] + M \cdot \Lambda[M] \) to obtain a complete, local \( \Lambda \)-algebra \( \Lambda[M] \) (that is very non-Noetherian). This comes with a canonical surjection \( f : \Lambda[M] \rightarrow \mathbb{R} \). One can define \( \mathcal{O}_{\mathbb{R}} \) to be the kernel of the map \( \text{Ker}(f)/m \cdot \text{Ker}(f) \rightarrow m/m^2 \). For every choice of \( f_{\mathbb{R}_\mathbb{L}} : \Lambda[t_1, \ldots , t_n] \rightarrow \mathbb{R} \), there is a canonical associated local homomorphisms of \( \Lambda \)-algebras, \( \Lambda[t_1, \ldots , t_n] \rightarrow \Lambda[M] \) that induces an isomorphism of \( k \)-vector spaces
\[
i_{\mathbb{L}} : \text{Ker}(f_{\mathbb{R}_\mathbb{L}})/m \cdot \text{Ker}(f_{\mathbb{R}_\mathbb{L}}) \rightarrow \mathcal{O}_{\mathbb{R}}.
\]
Then for every deformation situation \((q : A' \rightarrow A,a)\), this isomorphism maps the element \( o_{q,a,\mathbb{L}} \) above to an element \( o_{q,a} \in \mathcal{O}_{\mathbb{R}} \otimes_k \text{Ker}(q) \). It is not hard to check that \( o_{q,a} \) is independent of the choice of \( \mathbb{L} \), essentially because the dependence of the isomorphism \( i_{\mathbb{L}} \) on the choice of \( \mathbb{L} \) is “inverse” to the dependence of the element \( o_{q,a} \) on \( \mathbb{L} \).

The canonical obstruction theory on a representable functor is minimal in the following sense.

Proposition 2.6. For every obstruction theory \((\mathcal{O}, o)\) of \( h_{\mathbb{C},\mathbb{R}} \), there exists an injective, \( k \)-linear map \( L : \mathcal{O}_{\mathbb{R}_\mathbb{L}} \rightarrow \mathcal{O} \) that is a morphism of obstruction theories.
Proof. First of all, notice that $\mathcal{O}_{\mathcal{R}_L}$ equals $\{0\}$ if and only if $f_{\mathcal{R}_L}$ is an isomorphism. In this case the zero map, $L = 0$, is the unique $k$-linear map, and it is trivially a map of obstruction theories. Thus, without loss of generality, assume that $\mathcal{O}_{\mathcal{R}_L}$ is nonzero.

By the Artin-Rees Lemma, there exists an integer $E$ such that $m^E_{\Lambda,n} \cap \ker(f_{\mathcal{R}_L})$ is contained in $m^E_{\Lambda,n} \cdot \ker(f_{\mathcal{R}_L})$ as ideals in $\Lambda[\{t_1, \ldots, t_n\}]$. Consider the following natural surjection of local, Artin $\Lambda$-algebras,

$$Q : \Lambda[\{t_1, \ldots, t_n\}] / \left( m^E_{\Lambda,n} \cdot \ker(f_{\mathcal{R}_L}) + m^E_{\Lambda,n} \right) \to \Lambda[\{t_1, \ldots, t_n\}] / \left( \ker(f_{\mathcal{R}_L}) + m^E_{\Lambda,n} \right).$$

Denote by $\tilde{\alpha}$, resp. $\tilde{\alpha}'$, the natural surjection from $\Lambda[\{t_1, \ldots, t_n\}]$ to the target of $Q$, resp. the source of $Q$. Since $f_{\mathcal{R}_L}$ is a surjection, there exists a unique surjection

$$\alpha : \mathcal{R} \to \Lambda[\{t_1, \ldots, t_n\}] / \left( \ker(f_{\mathcal{R}_L}) + m^E_{\Lambda,n} \right)$$

such that $\alpha \circ f_{\mathcal{R}_L}$ equals $\tilde{\alpha}$. The kernel of $Q$ equals

$$\left( \ker(f_{\mathcal{R}_L}) + m^E_{\Lambda,n} \right) / \left( m^E_{\Lambda,n} \cdot \ker(f_{\mathcal{R}_L}) + m^E_{\Lambda,n} \right) \cong \left( \ker(f_{\mathcal{R}_L}) \right) / \left( m^E_{\Lambda,n} \cdot \ker(f_{\mathcal{R}_L}) + m^E_{\Lambda,n} \cap \ker(f_{\mathcal{R}_L}) \right).$$

By construction of $E$, this is canonically isomorphic to $\ker(f_{\mathcal{R}_L}) / m^E_{\Lambda,n} \cdot \ker(f_{\mathcal{R}_L})$. Thus the obstruction element $o_{Q,\alpha}$ is an element in

$$\mathcal{O} \otimes_{k} \left( \ker(f_{\mathcal{R}_L}) / m^E_{\Lambda,n} \cdot \ker(f_{\mathcal{R}_L}) \right) = \text{Hom}_{k}(\mathcal{O}_{\mathcal{R}_L}, \mathcal{O}).$$

For every nonzero element

$$\phi : \ker(f_{\mathcal{R}_L}) / m^E_{\Lambda,n} \cdot \ker(f_{\mathcal{R}_L}) \to k$$

in $\mathcal{O}_{\mathcal{R}_L}$, denote by $I_\phi$ the unique ideal in $\Lambda[\{t_1, \ldots, t_n\}]$ containing $m^E_{\Lambda,n} \cdot \ker(f_{\mathcal{R}_L}) + m^E_{\Lambda,n}$, contained in $\ker(f_{\mathcal{R}_L}) + m^E_{\Lambda,n}$, and whose image in $\ker(f_{\mathcal{R}_L}) / m^E_{\Lambda,n} \cdot \ker(f_{\mathcal{R}_L})$ equals $\ker(\phi)$. Denote by $\tilde{\alpha}_\phi'$ the following natural surjection,

$$\tilde{\alpha}_\phi' : \Lambda[\{t_1, \ldots, t_n\}] \to \Lambda[\{t_1, \ldots, t_n\}] / I_\phi.$$

Since $I_\phi$ is contained in $\ker(f_{\mathcal{R}_L}) + m^E_{\Lambda,n}$, there exists a unique surjection

$$q_\phi : \Lambda[\{t_1, \ldots, t_n\}] / I_\phi \to \Lambda[\{t_1, \ldots, t_n\}] / \left( m^E_{\Lambda,n} \cdot \ker(f_{\mathcal{R}_L}) + m^E_{\Lambda,n} \right)$$

such that $q_\phi \circ \tilde{\alpha}_\phi'$ equals $\tilde{\alpha}$. Similarly, since $I_\phi$ contains $m^E_{\Lambda,n} \cdot \ker(f_{\mathcal{R}_L}) + m^E_{\Lambda,n}$, there exists a unique surjection

$$u'_\phi : \Lambda[\{t_1, \ldots, t_n\}] / \left( m^E_{\Lambda,n} \cdot \ker(f_{\mathcal{R}_L}) + m^E_{\Lambda,n} \right) \to \Lambda[\{t_1, \ldots, t_n\}] / I_\phi,$$
such that \( u'_\phi \circ \tilde{\alpha}' \) equals \( \tilde{\alpha}' \). Finally, the morphism \( \phi \) determines an isomorphism of \( \text{Ker}(q_\phi) \) with \( k \) such that the following diagram commutes,

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Ker}(f)/m \cdot \text{Ker}(f) \\
\phi & & \downarrow u'_\phi \\
0 & \longrightarrow & \Lambda [t_1, \ldots, t_n]/(m \cdot \text{Ker}(f) + m^E) \\
& & \downarrow u \\
\Lambda [t_1, \ldots, t_n]/(\text{Ker}(f) + m^E) & \longrightarrow & \Lambda [t_1, \ldots, t_n]/(\text{Ker}(f) + m^E).
\end{array}
\]

In particular, \( (u'_\phi, \text{Id}) \) is a morphism of deformation situations. Since \( o \) is a section of \( \mathcal{O} \otimes_k K \), the associated \( k \)-linear map

\[
\text{Id}_\mathcal{O} \otimes \phi : \mathcal{O} \otimes_k \left( \text{Ker}(f_{R,L})/m_{\Lambda_n} \cdot \text{Ker}(f_{R,L}) \right) \rightarrow \mathcal{O} \otimes_k
\]

maps \( o_{Q,\alpha} \) to \( o_{q_\phi,\alpha} \). Finally, \( o_{R,L,q_\phi,\alpha} \) is the element \( \phi \) as an element in \( \mathcal{O}_{R,L} \otimes_k k = \mathcal{O}_{R,L} \). By hypothesis this is nonzero. Thus there exists no lift \( \alpha'_\phi \) such that \( q_\phi \circ \alpha'_\phi \) equals \( \alpha \). Therefore, since \( (\mathcal{O}, o) \) is an obstruction theory, also \( o_{q_\phi,\alpha} \) is nonzero. On the other hand, \( o_{q_\phi,\alpha} \) equals \( L(\phi) \) in \( \mathcal{O} \). Therefore \( L(\phi) \) is nonzero for every nonzero \( \phi \) in \( \mathcal{O}_{R,L} \), i.e., \( L \) is an injective \( k \)-linear map.

For every deformation situation \( (q : A' \rightarrow A, a) \), there exists an integer \( e \geq E \) such that \( m^e_{A'} \) equals \{0\}. Then there exists a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Ker}(f)/m \cdot \text{Ker}(f) \\
\downarrow u'_K & & \downarrow u' \\
0 & \longrightarrow & \Lambda [t_1, \ldots, t_n]/(m \cdot \text{Ker}(f) + m^E) \\
\downarrow q & & \downarrow u \\
\Lambda [t_1, \ldots, t_n]/(\text{Ker}(f) + m^E) & \longrightarrow & A, \\
\end{array}
\]

where \( u \) is the unique homomorphism such that \( u \circ \alpha \) equals \( a \), and where \( u' \) exists for the same reason that \( \tilde{\alpha}' \) exists in the proof of Proposition 2.5. Because \( o \) is a section of \( \mathcal{O} \otimes_k K \), the \( k \)-linear map

\[
\text{Id} \otimes_k u'_K : \mathcal{O} \otimes_k \left( \text{Ker}(f_{R,L})/m_{\Lambda_n} \cdot \text{Ker}(f_{R,L}) \right) \rightarrow \mathcal{O} \otimes_k \text{Ker}(q)
\]

maps \( o_{Q,\alpha} \) to \( o_{q,\alpha} \). Also, by the definition of \( L \), \( o_{Q,\alpha} \) equals the image under \( L \otimes \text{Id} \) of the identity map

\[
\text{Id} \in \text{Hom}_k \left( \text{Ker}(f_{R,L})/m_{\Lambda_n} \cdot \text{Ker}(f_{R,L}), \text{Ker}(f_{R,L})/m_{\Lambda_n} \cdot \text{Ker}(f_{R,L}) \right).
\]

Thus \( o_{q,\alpha} \) equals the image of \( \text{Id} \) under the \( k \)-linear map

\[
(\text{Id} \otimes_k u'_K) \circ (L \otimes_k \text{Id}) = L \otimes_k u'_K.
\]

But of course this also equals \( (L \otimes_k \text{Id}) \circ (\text{Id} \otimes_k u'_K) \). The image of \( \text{Id} \) under \( \text{Id} \otimes_k u'_K \) is the definition of \( o_{R,L,q,\alpha} \). Therefore, \( L \otimes \text{Id} \) maps \( o_{R,L,q,\alpha} \) to \( o_{q,\alpha} \) for every deformation situation \( (q, a) \). In other words, \( L \) is a morphism of obstruction theories.
The main application is the following.

**Corollary 2.7.** Let \((\mathcal{O}, o)\) be an obstruction theory for \(F = h_{\mathcal{C}, \hat{R}}\). Then there exists an isomorphism

\[ f_{\hat{R}_{\mathcal{E}}} : \Lambda \left[ t_1, \ldots, t_n \right] / \langle p_1, \ldots, p_m \rangle \to \hat{R}, \quad g_i \in m_{\Lambda}^2 \]

where \(n\) equals \(\dim_k F(k[\epsilon]/(\epsilon^2))\) and where \(m\) equals \(\dim_k \mathcal{O}\). In particular, \(\text{Krull-dim}(\hat{R}/m_{\Lambda} \hat{R})\) is \(\geq n - m\). If \(\text{Krull-dim}(\hat{R}/m_{\Lambda} \hat{R})\) equals \(n - m\), then \(\hat{R}\) is \(\Lambda\)-flat.

**Proof.** Let \(f_{\hat{R}_{\mathcal{E}}}\) be as above. Since there exists a \(k\)-linear injection of \(\mathcal{O}_{\hat{R}_{\mathcal{E}}}\) into \(\mathcal{O}\), it follows that \(m\) is greater than or equal to the dimension of \(\text{Ker}(f_{\hat{R}_{\mathcal{E}}})/m_{\Lambda} \cdot \text{Ker}(f_{\hat{R}_{\mathcal{E}}})\). Therefore there exist elements \(g_1, \ldots, g_m \in \text{Ker}(f_{\hat{R}_{\mathcal{E}}})\) whose images are a \(k\)-spanning set modulo \(m \cdot \text{Ker}(f_{\hat{R}_{\mathcal{E}}})\). By Nakayama’s Lemma, the elements \(g_1, \ldots, g_m\) generate the ideal \(\text{Ker}(f_{\hat{R}_{\mathcal{E}}}f)\). Thus \(f_{\hat{R}_{\mathcal{E}}}\) is the isomorphism as above.

By the above, the image ideal \(\overline{\text{Ker}(f)}\) in

\[ \Lambda \left[ t_1, \ldots, t_n \right] / m_{\Lambda} \cdot \Lambda \left[ t_1, \ldots, t_n \right] = k \left[ t_1, \ldots, t_n \right] \]

is generated by the elements \(\overline{g}_1, \ldots, \overline{g}_m\). By the Krull Hauptidealsatz, for every minimal prime \(p \subset k \left[ t_1, \ldots, t_n \right]\) over \(\overline{\text{Ker}(f)}\), \(p\) has height \(\leq m\), i.e., the quotient domain has Krull dimension \(\geq n - m\). \(\text{Ker}(f_{\hat{R}_{\mathcal{E}}}) + m_{\Lambda}\). Finally, if \(\hat{R}/m_{\Lambda} \hat{R}\) has dimension equals to \(n - m\), then \(\overline{g}_1, \ldots, \overline{g}_m\) is a regular sequence in \(k \left[ t_1, \ldots, t_n \right]\) by [Mat89, Theorem 17.4]. Then, by [Mat89, Corollary 22.5’, p. 177], also \(g_1, \ldots, g_m\) is a regular sequence in \(\Lambda \left[ t_1, \ldots, t_n \right]\), and the quotient ring is \(\Lambda\)-flat. Since this quotient ring is \(\Lambda\)-isomorphic to \(\hat{R}\), also \(\hat{R}\) is \(\Lambda\)-flat. \(\square\)

## 3 An Obstruction Theory for the Hilbert Scheme.

Let \(X_{\Lambda}\) be a separated, flat, finitely presented scheme over \(\text{Spec} \, \Lambda\). For every \(\Lambda\)-algebra \(\hat{R}\), denote by \(X_{\hat{R}}\) the base change \(\text{Spec} \, \hat{R} \times_{\text{Spec} \, \Lambda} X_{\Lambda}\). Let \(Z_k\) be a closed subscheme of \(X_k\). Denote by

\[ e_{Z_k/X_k} : \mathcal{I}_{Z_k/X_k} \to \mathcal{O}_{X_k} \]

the ideal sheaf of \(Z_k\) inside \(\mathcal{O}_{X_k}\).

**Definition 3.1.** The *Hilbert functor*, \(\text{Hilb}_{X_{\Lambda}/\Lambda, Z_k}\), is the functor \(\mathcal{C}_\Lambda \to \text{Sets}\) sending each object \(A\) of \(\mathcal{C}_\Lambda\) to the set of \(A\)-flat, closed subschemes \(Z_A \subset X_A\) such that \(\text{Spec} \, k \times_{\text{Spec} \, \Lambda} Z_A\) equals \(Z_k\) as a closed subscheme of \(\text{Spec} \, k \times_{\text{Spec} \, \Lambda} X_A\) equals \(X_k\). For every morphism \(q : A' \to A\) in \(\mathcal{C}_\Lambda\), the induced map

\[ \text{Hilb}_{X_{\Lambda}/\Lambda, Z_k}(A') \to \text{Hilb}_{X_{\Lambda}/\Lambda, Z_k}(A) \]

sends each \(A'\)-flat closed subscheme \(Z_{A'}\) to the base change \(\text{Spec} \, A \times_{\text{Spec} \, A'} Z_{A'}\) as a closed subscheme of \(\text{Spec} \, A \times_{\text{Spec} \, A'} X_{A'}\) equals \(X_A\).
The standard obstruction group of \( \text{Hilb}_{X/k, Z_k} \) is

\[
\text{Ext}^1_{O_{X_k}}(I_{Z_k/X_k}, O_{X_k}/I_{Z_k/X_k}).
\]

The normal sheaf \( Z_{Z_k/X_k} \) is the \( O_{Z_k} \)-module,

\[
N_{Z_k/X_k} = \text{Hom}_{O_{X_k}}(I_{Z_k/X_k}, O_{X_k}/I_{Z_k/X_k}).
\]

The global subgroup is

\[
H^1(X_k, \text{Hom}_{O_{X_k}}(I_{Z_k/X_k}, O_{X_k}/I_{Z_k/X_k})) = H^1(Z_k, N_{Z_k/X_k}).
\]

By construction \( \text{Hilb}_{X/k, Z_k} \) is a pointed functor. For every closed subscheme \( Z_A \) of \( X_A \) (not necessarily flat), denote the ideal sheaf of \( Z_A \) by

\[
e_{Z_A/X_A} : I_{Z_A/X_A} \to O_{X_A}.
\]

**Lemma 3.2.** The closed subscheme \( Z_A \) is \( A \)-flat if and only if both \( I_{Z_A/X_A} \) is flat and the following \( O_{X_k} \)-module homomorphism is injective,

\[
e_{Z_A/X_A} \otimes_A \text{Id}_k : I_{Z_A/X_A} \otimes_A k \to O_{X_A} \otimes_A k,
\]

i.e., the natural surjection \( I_{Z_A/X_A} \otimes_A k \to I_{Z_k/X_k} \) is an isomorphism.

**Proof.** One direction is straightforward. Assume that \( Z_A \) is flat, i.e., \( O_{X_A}/I_{Z_A/X_A} \) is \( A \)-flat. Then from the short exact sequence,

\[
0 \longrightarrow I_{Z_A/X_A} \longrightarrow O_{X_A} \longrightarrow O_{X_A}/I_{Z_A/X_A} \longrightarrow 0,
\]

for every \( A \)-module \( M \) there is a long exact sequence of Tor sheaves,

\[
\text{Tor}^1_A(O_{X_A}/I_{Z_A/X_A}, M) \longrightarrow \text{Tor}^1_A(I_{Z_A/X_A}, M) \longrightarrow \text{Tor}^1_A(O_{X_A}, M).
\]

Since \( O_{X_A}/I_{Z_A/X_A} \) and \( O_{X_A} \) are \( A \)-flat, the outer terms are zero, hence also the middle term is zero. Therefore \( I_{Z_A} \) is \( A \)-flat. Similarly, since \( \text{Tor}^1_A(O_{X_A}/I_{Z_A/X_A}, k) \) is zero, the short exact sequence above gives rise to a short exact sequence,

\[
0 \longrightarrow I_{Z_A/X_A} \otimes_A k \longrightarrow O_{X_A} \otimes_A k \longrightarrow O_{X_A}/I_{Z_A/X_A} \otimes_A k \longrightarrow 0.
\]

Thus \( e_{Z_A/X_A} \otimes_A \text{Id}_k \) is injective.

The opposite direction follows from the Local Flatness Criterion, [Mat89, Theorem 22.3]. The details are left to the reader.
Let \( q : A' \rightarrow A \) be an infinitesimal extension and let \( Z_A \subset X_A \) be an element of \( \text{Hilb}_{X_A/A,Z_k} \). Denote by 
\[
q_X : \mathcal{O}_{X_A'} \rightarrow \mathcal{O}_{X_A}
\]
the surjective map of structure sheaves associated to \( q \). There is a short exact sequence of \( \mathcal{O}_{X_{A'}} \)-modules,
\[
\begin{array}{c}
0 \longrightarrow \text{Ker}(q) \otimes_k \mathcal{O}_{X_k} \longrightarrow q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \longrightarrow \mathcal{I}_{Z_A/X_A} \longrightarrow 0.
\end{array}
\]
Multiplying by \( m_{A'} \) induces a commutative diagram,
\[
\begin{array}{ccccccccc}
m_{A'} \otimes A' \text{Ker}(q) \otimes_k \mathcal{O}_{X_k} & \longrightarrow & m_{A'} \otimes A' q_X^{-1}(\mathcal{I}_{Z_A/X_A}) & \longrightarrow & m_{A'} \otimes A \mathcal{I}_{Z_A/X_A} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ker}(q) \otimes_k \mathcal{O}_{X_k} & \longrightarrow & q_X^{-1}(\mathcal{I}_{Z_A/X_A}) & \longrightarrow & \mathcal{I}_{Z_A/X_A} & \longrightarrow & 0.
\end{array}
\]
Since \( m_{A'} \cdot \text{Ker}(q) \) equals \( \{0\} \), the first vertical map is zero, hence the second vertical map factors through the surjection to \( m_{A'} \otimes A' \mathcal{I}_{Z_A/X_A} \). This has two consequences. Using the commutative diagram,
\[
\begin{array}{ccccccccc}
\text{Ker}(q) \otimes A' \mathcal{I}_{Z_A/X_A} & \longrightarrow & m_{A'} \otimes A' \mathcal{I}_{Z_A/X_A} & \longrightarrow & m_A \otimes A \mathcal{I}_{Z_A/X_A} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ker}(q) \otimes_k \mathcal{O}_{X_k} & \longrightarrow & q_X^{-1}(\mathcal{I}_{Z_A/X_A}) & \longrightarrow & \mathcal{I}_{Z_A/X_A} & \longrightarrow & 0,
\end{array}
\]
and the injectivity of the last vertical map, the intersection of \( \text{Ker}(q) \otimes_k \mathcal{O}_{X_k} \) and \( m_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \) equals the image of the first vertical map, i.e., \( \text{Ker}(q) \otimes_k \mathcal{I}_{Z_k/X_k} \). Thus the quotient of \( \text{Ker}(q) \otimes_k \mathcal{O}_{X_k} \) by this intersection is \( \text{Ker}(q) \otimes_k (\mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}) \). Also, by Lemma 3.2 the quotient of \( q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \) by both \( \text{Ker}(q) \otimes_k \mathcal{O}_{X_k} \) and \( m_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \), i.e., the quotient \( \mathcal{I}_{Z_A/X_A}/m_{A'} \cdot \mathcal{I}_{Z_A/X_A} \), equals \( \mathcal{I}_{Z_k/X_k} \). Thus we have a short exact sequence of \( \mathcal{O}_{X_k} \)-modules,
\[
0 \longrightarrow \text{Ker}(q) \otimes_k (\mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}) \longrightarrow q_X^{-1}(\mathcal{I}_{Z_A/X_A})/m_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \longrightarrow q \mathcal{I}_{Z_k/X_k} \longrightarrow 0.
\]

**Definition 3.3.** For a deformation situation \( (q : A' \rightarrow A, Z_A) \) for \( \text{Hilb}_{X_A/A,Z_k} \), the standard obstruction class is the class \( o_{q,Z_A} \) of the short exact sequence
\[
o_{q,Z_A} : 0 \longrightarrow \text{Ker}(q) \otimes_k (\mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}) \longrightarrow q_X^{-1}(\mathcal{I}_{Z_A/X_A})/m_{A'} \cdot q_X^{-1}(\mathcal{I}_{Z_A/X_A}) \longrightarrow q \mathcal{I}_{Z_k/X_k} \longrightarrow 0.
\]
in the Yoneda Ext group \( \text{Ext}^1_{\mathcal{O}_{X_k}(\mathcal{I}_{Z_k/X_k},\mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k})} \otimes_k \text{Ker}(q) \).

This definition is justified by the following.

**Proposition 3.4.** The element \( o_{q,Z_A} \) equals \( 0 \) if and only if there exists an \( A' \)-flat closed subscheme \( Z_{A'} \) of \( X_{A'} \) such that \( Z_{A'} \times_{\text{Spec } A'} \text{Spec } A \) equals \( Z_A \) as closed subschemes of \( X_{A'} \times_{\text{Spec } A'} \text{Spec } A = X_A \).
Proof. For every $O_{X_{A'}}$-submodule $I_{Z_{A'}/X_{A'}}$ of $q_X^{-1}(I_{Z_A/X_A})$ that surjects onto $I_{Z_A/X_A}$, the subsheaf $m_{A'} \cdot I_{Z_{A'}/X_{A'}}$ of $q_X^{-1}(I_{Z_A/X_A})$ equals $m_{A'} \cdot q_X^{-1}(I_{Z_A/X_A})$. This implies that $I_{Z_{A'}/X_{A'}}$ contains $m_{A'} \cdot q_X^{-1}(I_{Z_A/X_A})$. This also implies that the natural map

$$I_{Z_{A'}/X_{A'}}/m_{A'} \cdot q_X^{-1}(I_{Z_A/X_A}) \to I_{Z_{A'}/X_{A'}} \otimes_{A'} k$$

is an isomorphism. If $Z_{A'}$ is $A'$-flat, then by Lemma 3.2 the induced map

$$I_{Z_{A'}/X_{A'}} \otimes_{A'} k \to I_{Z_k/X_k}$$

is an isomorphism. Thus the $O_{X_k}$-submodule $I_{Z_{A'}/X_{A'}/m_{A'} \cdot q_X^{-1}(I_{Z_A/X_A})}$ of $q_X^{-1}(I_{Z_A/X_A})$ is a splitting of $o_{q,Z_{A'}}$, i.e., $o_{q,Z_{A'}}$ equals 0 in the Yoneda Ext group.

Conversely, given a submodule of $q_X^{-1}(I_{Z_A/X_A})/m_{A'} \cdot q_X^{-1}(I_{Z_A/X_A})$ splitting $o_{q,Z_{A'}}$, define $I_{Z_{A'}/X_{A'}}$ to be the inverse image of the submodule in $q_X^{-1}(I_{Z_A/X_A})$. Then $I_{Z_{A'}/X_{A'}}$ surjects onto $I_{Z_A/X_A}$. Since $I_{Z_{A'}/X_{A'}/m_{A'} \cdot q_X^{-1}(I_{Z_A/X_A})}$ has trivial intersection with $\text{Ker}(q_Z)$, it follows that the intersection of $I_{Z_{A'}/X_{A'}}$ and $\text{Ker}(q_X)$ equals $\text{Ker}(q) \otimes_k I_{Z_k/X_k}$, which also equals $\text{Ker}(q_X) \cdot I_{Z_{A'}/X_{A'}}$. Thus we have a short exact sequence

$$0 \longrightarrow \text{Ker}(q) \otimes_k I_{Z_k/X_k} \longrightarrow I_{Z_{A'}/X_{A'}} \longrightarrow I_{Z_A/X_A} \longrightarrow 0.$$

By the previous paragraph, the natural map $I_{Z_{A'}/X_{A'}/X_{A'}/m_{A'} \cdot q_X^{-1}(I_{Z_A/X_A})} \to I_{Z_k/X_k}$ is an isomorphism. Thus the short exact sequence above proves that the map

$$\text{Ker}(q) \otimes_{A'} I_{Z_{A'}/X_{A'}} \to I_{Z_{A'}/X_{A'}}$$

is injective. Since $I_{Z_A/X_A}$ is $A$-flat, by the Local Flatness Criterion, [Mat89, Theorem 22.3], also

$$m_A \otimes_A I_{Z_A/X_A} \to I_{Z_A/X_A}$$

is injective. Putting these together with the short exact sequence above,

$$m_{A'} \otimes_{A'} I_{Z_{A'}/X_{A'}} \to I_{Z_{A'}/X_{A'}}$$

is injective. Thus, by the Local Flatness Criterion once more, $I_{Z_{A'}/X_{A'}}$ is $A'$-flat. Finally, since the natural map

$$I_{Z_{A'}/X_{A'}/X_{A'}} \otimes_{A'} k \to I_{Z_k/X_k}$$

is an isomorphism, Lemma 3.2 implies that $Z_{A'}$ is $A'$-flat. Therefore $o_{q,Z_{A'}}$ equals 0 if and only if there exists an $A'$-flat closed subscheme $Z_{A'}$ of $X_{A'}$ with $Z_{A'} \times_{\text{Spec} A'} \text{Spec} A$ equal to $Z_A$ as closed subschemes of $X_{A'} \times_{\text{Spec} A'} \text{Spec} A = X_A$. \qed

Finally, it is left to the reader to verify that the elements $o_{q,Z_{A'}}$ are functorial for morphisms of deformation situations.

The “standard” obstruction group above is often larger than strictly necessary. Because of Corollary 2.7, it is crucial to identify the smallest possible obstruction group. One of the basic reductions has to do with the case that $I_{Z_k/X_k}$ is everywhere locally generated by a regular sequence. Thus, for now let $X_A$ be an affine, flat, finitely presented scheme over $\text{Spec} A$. Let $b = (b_1, \ldots, b_r)$ be a regular sequence in $B_k = H^0(X_A, O_{X_A})$. For every $A$ in $C_A$, denote by $B_A$ the $A$-algebra $H^0(X_A, O_{X_A})$.  

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**Proposition 3.5.** Every ideal in $\text{Hilb}_{X_\Lambda/A,Z_k}$ is generated by a regular sequence $b_A = (b_{A,1}, \ldots, b_{A,r})$ in $B_{A}$ that maps to $b$. Conversely, every sequence $b_A$ in $B_{A}$ that maps to $b$ is regular and generates an ideal in $\text{Hilb}_{X_\Lambda/A,Z_k}$.

**Proof.** For a regular sequence $b_A$ that maps to $b$, the corresponding ideal is in $\text{Hilb}_{X_\Lambda/A,Z_k}$ by the Local Flatness Criterion, cf. [Mat89, Corollary, Theorem 22.6, p. 177]. Conversely, for every ideal $I_{Z_A/X_A}$ in $\text{Hilb}_{X_\Lambda/A,Z_k}$, since the map to $I_{Z_k/X_k}$ is surjective on global sections, there exists a sequence of global sections $b_A$ of $I_{Z_A/X_A}$ that maps to $b$. Denote by $I_{Z_A/X_A}$, the sub-ideal sheaf of $I_{Z_A/X_A}$ generated by $b_A$. Consider the commutative diagram,

$$
\begin{array}{cccc}
I_{Z_A/X_A} \otimes_A k & \longrightarrow & I_{Z_A/X_A} \otimes_A k \\
\downarrow & & \downarrow \\
I_{Z_k/X_k} & \longrightarrow & I_{Z_k/X_k}
\end{array}
$$

By Lemma 3.2, both vertical arrows are isomorphisms. Hence the top horizontal arrow is surjective. Thus, by Nakayama’s Lemma, the map $I_{Z_A/X_A} \rightarrow I_{Z_k/X_k}$ is surjective, i.e., $I_{Z_A/X_A}$ is generated by $b_A$.

**Proof.** Since $X_{A'}$ is affine, the surjective homomorphism of sheaves of algebras $q_X : \mathcal{O}_{X_{A'}} \rightarrow \mathcal{O}_{X_A}$ induces a surjection $B_{A'} \rightarrow B_{A}$. Thus every sequence $b_A$ in $B_{A}$ lifts to a sequence $b_{A'}$ in $B_{A'}$. Therefore, by Proposition 3.5, every element $I_{Z_A/X_A}$ in $\text{Hilb}_{X_\Lambda/A,Z_k}(A)$ lifts to an element $I_{Z_{A'}/X_{A'}}$ in $\text{Hilb}_{X_\Lambda/A,Z_k}(A')$.

**Corollary 3.6.** With hypotheses as above, for every deformation situation $(q : A' \rightarrow A, Z_A)$ there exists a lift of $Z_A$ to $Z_{A'}$ in $\text{Hilb}_{X_\Lambda/A,Z_k}(A')$. In particular, every standard obstruction class $o_{q,Z_A}$ equals 0.

**Proof.** Since $X_{A'}$ is affine, the surjective homomorphism of sheaves of algebras $q_X : \mathcal{O}_{X_{A'}} \rightarrow \mathcal{O}_{X_A}$ induces a surjection $B_{A'} \rightarrow B_{A}$. Thus every sequence $b_A$ in $B_{A}$ lifts to a sequence $b_{A'}$ in $B_{A'}$. Therefore, by Proposition 3.5, every element $I_{Z_A/X_A}$ in $\text{Hilb}_{X_\Lambda/A,Z_k}(A)$ lifts to an element $I_{Z_{A'}/X_{A'}}$ in $\text{Hilb}_{X_\Lambda/A,Z_k}(A')$.

Of course, typically we are interested in proper $\Lambda$-schemes $X_\Lambda$, not affine $\Lambda$-schemes. So now assume that $X_\Lambda$ is a separated, flat, finitely presented scheme over Spec $\Lambda$ that is not necessarily affine. For every pair of coherent sheaves $\mathcal{E}, \mathcal{F}$ on $X_k$, the local-to-global spectral sequence for Ext gives an exact sequence

$$
0 \rightarrow H^1(X_k, \text{Hom}_{\mathcal{O}_{X_k}}(\mathcal{E}, \mathcal{F})) \rightarrow \text{Ext}^1_{\mathcal{O}_{X_k}}(\mathcal{E}, \mathcal{F}) \rightarrow H^0(X_k, \text{Ext}^1_{\mathcal{O}_{X_k}}(\mathcal{E}, \mathcal{F})) \rightarrow H^2(X_k, \text{Hom}_{\mathcal{O}_{X_k}}(\mathcal{E}, \mathcal{F})).
$$

In particular, this gives an exact sequence,

$$
0 \rightarrow H^1(Z_k, \mathcal{N}_{Z_k/X_k}) \rightarrow \text{Ext}^1_{\mathcal{O}_{X_k}}(I_{Z_k/X_k}, \mathcal{O}_{X_k}/I_{Z_k/X_k}) \rightarrow H^0(X_k, \text{Ext}^1_{\mathcal{O}_{X_k}}(I_{Z_k/X_k}, \mathcal{O}_{X_k}/I_{Z_k/X_k}))
$$

The first term is the global group and the third term is the local group.

**Corollary 3.7.** Let $Z_k$ be a closed subscheme of $X_k$ whose ideal sheaf $I_{Z_k/X_k}$ is generated by regular sequences on the opens in some open affine covering. Then the normal sheaf $\mathcal{N}_{Z_k/X_k}$ is a locally free $\mathcal{O}_{Z_k}$-module of finite rank, and every obstruction class $o_{Z_k/Z_A}$ is contained in the global subgroup,

$$
H^1(Z_k, \mathcal{N}_{Z_k/X_k}) \otimes_k \text{Ker}(q).
$$

Thus the global subgroup is the obstruction group of an obstruction theory for $\text{Hilb}_{X_\Lambda/A,Z_k}$.
Proof. Since $I_{Z_k/X_k}$ is locally generated by a regular sequence, say $(b_1, \ldots, b_r)$, then the $O_{X_k}/I_{Z_k/X_k}$-module $I_{Z_k/X_k}/I_{Z_k/X_k}^2$ is locally freely generated by the images of $b_1, \ldots, b_r$. Thus the dual sheaf is also locally free of finite rank.

The restriction of $o_{q,Z_k}$ to each of the opens in this open covering equals 0 by Corollary 3.6. Thus the image of $o_{q,Z_k}$ in the local group $H^0(X_k, \text{Ext}^1_{O_{X_k}}(E,F)) \otimes_k \text{Ker}(q)$ is a global section that is zero when restricted to the opens of an open covering. Therefore this is the zero global section. So $o_{q,Z_k}$ is contained in the global subgroup.

If $X_k$ is smooth over $k$, then by [Mat89, Theorem 21.2], the ideal sheaf of $Z_k$ is everywhere locally generated by a regular sequence if and only if $Z_k$ is locally a complete intersection scheme. In particular, if $Z_k$ is also smooth over $k$, then the ideal sheaf is everywhere locally generated by a regular sequence. When the ideal sheaf is everywhere locally generated by a regular sequence, then the Zariski tangent space of the fiber of Hilb_{X_k/Λ, Z_k} over Spec $k$ is $H^0(Z_k, \mathcal{N}_{Z_k/X_k})$ and the global subgroup is $H^1(Z_k, \mathcal{N}_{Z_k/X_k})$. Combined with Corollary 2.7, this gives a lower bound on the dimension of any pro-representing object. The main pro-representability result is the following.

Theorem 3.8. [Art69, Corollary 6.2] For $X_Λ$ a separated, flat, finitely presented scheme over Spec $Λ$, and for $Z_k$ a closed subscheme of $X_k$ that is proper over Spec $k$, the functor Hilb_{X_Λ/Λ, Z_k} is pro-representable by an object $\hat{R}$ in $\hat{C}_Λ$.

In fact [Art69, Corollary 6.2] proves much more. Although pro-representability is a straightforward application of Schlessinger’s thesis, this application is not contained in [Sch68].

Corollary 3.9. Assume that $Z_k$ is proper, and assume that everywhere locally $I_{Z_k/X_k}$ is generated by a regular sequences. Then every irreducible component of $\hat{R}/\mathfrak{m}_Λ\hat{R}$ has Krull dimension $\ge h^0(Z_k, \mathcal{N}_{Z_k/X_k}) - h^1(Z_k, \mathcal{N}_{Z_k/X_k})$. When this is equality, then $\hat{R}$ is $Λ$-flat. In particular, if $Z_k$ is a curve (or whenever $h^q(Z_k, \mathcal{N}_{Z_k/X_k})$ equals 0 for all $q > 1$), then the Krull dimension is $\ge \chi(Z_k, \mathcal{N}_{Z_k/X_k})$, and equality implies that $\hat{R}$ is $Λ$-flat.

Proof. This follows immediately from Corollaries 2.7 and 3.7.

A special case is when $X_k$ is smooth and $Z_k$ is a geometrically reduced curve that is locally a complete intersection. Then the adjunction formula gives,

$$\det(\mathcal{N}_{Z_k/X_k}) = \det(T_{X_k/k})|_{Z_k} \otimes_{O_{Z_k}} \omega_{Z_k/k},$$

where $T_{X_k/k}$ is the tangent sheaf $\text{Hom}_{O_{X_k}}(\Omega_{X_k/k}, O_{X_k})$ and where $\omega_{Z_k/k}$ is the dualizing invertible sheaf on $Z_k$. Denote by $p_a(Z_k)$ the arithmetic genus $1 - \chi(Z_k, O_{Z_k})$ of $Z_k$. Then we can apply Riemann-Roch to compute the Euler characteristic $\chi(Z_k, \mathcal{N}_{Z_k/X_k})$ above.
Corollary 3.10. Assume that $X_k$ is smooth of pure dimension $\dim(X_k)$ over $k$, and assume that $Z_k$ is a proper, reduced curve that is locally a complete intersection. Then every irreducible component of $whR/\mathfrak{m}_\Lambda R$ has Krull dimension

$$
\geq \deg_Z \det(T_{X_k/k}|_{Z_k}) + (1 - p_a(Z_k))(\dim(X_k) - 3).
$$

If this is equality, then $\widehat{R}$ is $\Lambda$-flat.

Proof. This follows by computing $\chi(Z_k, N_{Z_k/X_k})$ using Riemann-Roch and the adjunction isomorphism above.

4 Variants of the Hilbert Scheme.

There are two variants of this obstruction theory for the Hilbert scheme. First, let $W_\Lambda \subset X_\Lambda$ be a closed subscheme that is $\Lambda$-flat. Let $Z_k$ be a closed subscheme of $X_k$ that contains $W_k$, i.e., such that $\mathcal{I}_{Z_k/X_k}$ is contained in $\mathcal{I}_{W_k/X_k}$.

**Definition 4.1.** The Hilbert functor relative to $W_\Lambda$, $\text{Hilb}_{X_\Lambda/\Lambda,W_\Lambda,Z_k}$, is the subfunctor of $\text{Hilb}_{X_\Lambda/\Lambda,Z_k}$ whose $A$-points parameterize closed subschemes $Z_A$ in $\text{Hilb}_{X_\Lambda/\Lambda,Z_k}(A)$ that contain $W_A$, i.e., such that $\mathcal{I}_{Z_A/X_A}$ is contained in $\mathcal{I}_{W_A/X_A}$. The standard obstruction group of $\text{Hilb}_{X_\Lambda/\Lambda,W_\Lambda,Z_k}$ is

$$
\text{Ext}^1_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}).
$$

The relative normal sheaf is the $\mathcal{O}_{Z_k}$-module,

$$
N_{Z_k/X_k,w_k} = \text{Hom}_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}).
$$

The global subgroup is

$$
\text{H}^1(Z_k, N_{Z_k/X_k,w_k}).
$$

Given a deformation situation $(q : A' \to A, Z_A)$ for $\text{Hilb}_{X_\Lambda/\Lambda,W_\Lambda,Z_k}$, using flatness as in the absolute case, there is a commutative diagram of short exact sequences,

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Ker}(q) \otimes_k (\mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}) \\
& & \downarrow \\
0 & \longrightarrow & \text{Ker}(q) \otimes_k (\mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}) \\
& & \downarrow \\
0 & \longrightarrow & \text{Ker}(q) \otimes_k (\mathcal{O}_{X_k}/\mathcal{I}_{Z_k/X_k}) \longrightarrow [q^{-1}_X(\mathcal{I}_{Z_A/X_A}) \cap \mathcal{I}_{W_A'/X_A}]/m_A' \cdot q^{-1}_X(\mathcal{I}_{Z_A/X_A}) \longrightarrow \mathcal{I}_{Z_k/X_k} \\
& & \downarrow \\
& & \mathcal{I}_{Z_k/X_k}
\end{array}
$$

**Definition 4.2.** For a deformation situation $(q : A' \to A, Z_A)$ for $\text{Hilb}_{X_\Lambda/\Lambda,W_\Lambda,Z_k}$, the standard obstruction class is the class $\omega_{A,q,Z_A}$ of the short exact sequence

$$
0 \to \text{Ker}(q) \otimes_k (\mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}) \to [q^{-1}_X(\mathcal{I}_{Z_A/X_A}) \cap \mathcal{I}_{W_A'/X_A}]/m_A' \cdot q^{-1}_X(\mathcal{I}_{Z_A/X_A}) \to \mathcal{I}_{Z_k/X_k} \to 0
$$

in the Yoneda Ext group $\text{Ext}^1_{\mathcal{O}_{X_k}}(\mathcal{I}_{Z_k/X_k}, \mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}) \otimes_k \text{Ker}(q)$. 

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Proposition 4.3. The standard obstruction element $o_{W_A,q,Z_A}$ equals 0 if and only if there exists $Z_A'$ in $\text{Hilb}_{X_A/\Lambda, W_A, Z_k}(A')$ mapping to $Z_A$ in $\text{Hilb}_{X_A/\Lambda, W_A, Z_k}(A)$.

Proof. The proof is very similar to the proof of Proposition 3.4. □

The local-to-global spectral sequence for Ext gives an exact sequence

$$0 \to H^1(Z_k, N_{Z_k/X_k,W_k}) \to \text{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_{Z_k/X_k}, \mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}) \to H^0(X_k, \text{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_{Z_k/X_k}, \mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}))$$

The first term is the global group and the third term is the local group.

Corollary 4.4. Let $Z_k$ be a closed subscheme of $X_k$ and containing $W_k$ whose ideal sheaf $\mathcal{I}_{Z_k/X_k}$ is generated by regular sequences on the opens in some open affine covering. Then the normal sheaf $N_{Z_k/X_k,W_k}$ is the tensor product of $\mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}$ with a locally free $\mathcal{O}_{Z_k}$-module of finite rank, and every obstruction class $o_{q,Z_A}$ is contained in the global subgroup,

$$H^1(Z_k, N_{Z_k/X_k,W_k}) \otimes_k \text{Ker}(q).$$

Thus the global subgroup is the obstruction group of an obstruction theory for $\text{Hilb}_{X_A/\Lambda, W_A, Z_k}$.

Proof. The proof is very similar to the proof of Corollary 3.7. □

A special case is when $W_k$ is an effective Cartier divisor in $Z_k$.

Lemma 4.5. If $W_k$ is an effective Cartier divisor in $Z_k$, then for every $Z_A \in \text{Hilb}_{X_A/\Lambda, W_A, Z_k}(A)$, the ideal sheaf $\mathcal{I}_{W_A/X_A}/\mathcal{I}_{Z_A/X_A}$ is an invertible $\mathcal{O}_{Z_A}$-module, denoted $\mathcal{O}_{Z_A}(-W_A)$. In particular, $N_{Z_k/X_k,W_k}$ is canonically isomorphic to $N_{Z_k/X_k}(-W_k)$.

Proof. By definition, $W_k$ is an effective Cartier divisor in $Z_k$ precisely when the ideal sheaf $\mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}$ is an invertible sheaf. In this case, the cup product map

$$H_{\mathcal{O}_{Z_k}}(\mathcal{I}_{Z_k/X_k}/\mathcal{I}_{Z_k/X_k}, \mathcal{O}_{Z_k}) \otimes_{\mathcal{O}_{Z_k}} (\mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}) \to H_{\mathcal{O}_{Z_k}}(\mathcal{I}_{Z_k/X_k}/\mathcal{I}_{Z_k/X_k}, \mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k})$$

is an isomorphism. Thus the cup product map gives an isomorphism $N_{Z_k/X_k}(-W_k) \to N_{Z_k/X_k,W_k}$. For any $Z_A$, to prove that the ideal sheaf $\mathcal{I}_{W_A/X_A}/\mathcal{I}_{Z_A/X_A}$ is an invertible $\mathcal{O}_{Z_A}$-module, it is equivalent to prove that it is a flat $\mathcal{O}_{Z_A}$-module. Since flatness is a local property, it suffices to prove that the stalk at each point of $Z_A$ is flat over the stalk of the structure sheaf. Applying the Local Flatness Criterion, [Mat89, Theorem 22.3], to this stalk, where the nilpotent ideal is the ideal generated by $m_{A'}$, it suffices to prove that $\mathcal{I}_{W_k/X_k}/\mathcal{I}_{Z_k/X_k}$ is an invertible $\mathcal{O}_{Z_k}$-module and that the map

$$m_A \otimes A \mathcal{I}_{W_A/X_A}/\mathcal{I}_{Z_A/X_A} \to \mathcal{I}_{W_A/X_A}/\mathcal{I}_{Z_A/X_A}$$

is injective. The first condition is the hypothesis that $W_k$ is an effective Cartier divisor in $Z_k$. The second condition is $A$-flatness of the ideal sheaf $\mathcal{I}_{W_A/X_A}/\mathcal{I}_{Z_A/X_A}$ of $W_A$ in $Z_A$. Since both $Z_A$ and $W_A$ are flat, this ideal sheaf is flat, cf. Lemma 3.2. □
Corollary 4.6. Assume that $X_k$ is smooth of pure dimension $\text{dim}(X_k)$ over $k$, assume that $Z_k$ is a proper, geometrically reduced curve that is locally a complete intersection, and assume that $W_k$ is an effective Cartier divisor in $Z_k$. Then $\text{Hilb}_{X_k/A,W_k,Z_k}$ is pro-representable by an object $\hat{R}$ in $\mathcal{C}_A$. Every irreducible component of $\text{wh}R/\mathfrak{m}_A\hat{R}$ has Krull dimension

$$\geq \deg_{Z_k} \det(T_{X_k/k}|_{Z_k}) + (1 - p_a(Z_k))(\text{dim}(X_k) - 3) - \text{length}(W_k)(\text{dim}(X) - 2).$$

If this is equality, then $\hat{R}$ is $A$-flat.

Proof. The proof is very similar to the proof of Corollary 3.10.

There is one more variation. Let $C_A$ and $Y_A$ be separated, flat, finitely presented schemes over $\text{Spec } A$. Denote by $X_A$ the fiber product $C_A \times_{\text{Spec } A} Y_A$. Let $Z_A \subset X_A$ be an $A$-flat closed subscheme. Denote by

$$\text{pr}_{Z_A,C_A} : Z_A \rightarrow C_A$$

the restriction to $Z_A$ of the projection morphism $\text{pr}_A : X_A \rightarrow C_A$.

Proposition 4.7. If $\text{pr}_{Z_k,C_k}$ is an isomorphism of $k$-schemes, then also $\text{pr}_{Z_A,C_A}$ is an isomorphism of $A$-schemes. In this case, there exists a unique $A$-morphism $u_A : C_A \rightarrow Y_A$ such that $Z_A$ equals the graph of $u_A$. Conversely, for every $A$-morphism $u_A : C_A \rightarrow Y_A$, the graph $Z_A$ of $u_A$ is an $A$-flat closed subscheme of $X_A$ such that $\text{pr}_{Z_A,C_A}$ is an isomorphism of $A$-schemes.

Proof. Assume that the $k$-morphism $\text{pr}_{Z_k,C_k}$ is an isomorphism. Then for every open affine subset of $C_A$, the inverse image in $Z_A$ is an open subset whose intersection with $Z_k$ is affine. By Chevalley’s theorem, cf. [Har77, Exercise III.3.1], the open in $Z_A$ is an affine scheme. Thus, without loss of generality, assume that both $C_A$ and $Z_A$ are affine schemes. Then $\text{pr}_{Z_A,C_A}$ is an isomorphism if and only if the associated ring homomorphism $\text{pr}_{Z_A,C_A}^#$ is an isomorphism.

We prove that $\text{pr}_{Z_A,C_A}^#$ is an isomorphism by induction on the smallest integer $e \geq 1$ such that $\mathfrak{m}_A^e$ equals $\{0\}$. If $e$ equals 1, then $A$ equals $k$ and the result is tautological. Thus, by way of induction, assume that $e > 1$, and assume that the result is proved for smaller $e$. For the infinitesimal extension $q : A \rightarrow B = A/\mathfrak{m}_A^{e-1}$, by the induction hypothesis the morphism $\text{pr}_{Z_B,C_B}^#$ is an isomorphism. Since $\mathcal{O}_{Z_A}$ and $\mathcal{O}_{C_A}$ are $A$-flat, there is a commutative diagram of short exact sequences

$$
\begin{array}{cccc}
0 & \rightarrow & \text{Ker}(q) \otimes_k \mathcal{O}_{Z_k} & \rightarrow & \mathcal{O}_{Z_A} & \rightarrow & \mathcal{O}_{Z_B} & \rightarrow & 0 \\
\downarrow \text{Id}_K \otimes_{\text{pr}_{Z_k,C_k}^#} & & \downarrow \text{pr}_{Z_A,C_A}^# & & \downarrow \text{pr}_{Z_B,C_B}^# & & \\
0 & \rightarrow & \text{Ker}(q) \otimes_k \mathcal{O}_{C_k} & \rightarrow & \mathcal{O}_{C_A} & \rightarrow & \mathcal{O}_{C_A'} & \rightarrow & 0
\end{array}
$$

By hypothesis, the first and third vertical arrows are isomorphisms. Therefore, by the Snake Lemma, also the middle vertical arrow, $\text{pr}_{Z_A,C_A}^#$ is an isomorphism. This proves that $\text{pr}_{Z_A,C_A}$ is an isomorphism by induction on $e$.

Since $\text{pr}_{Z_A,C_A}$ is an isomorphism, there exists a unique $A$-morphism $u_A : C_A \rightarrow Y_A$ such that $u_A \circ \text{pr}_{Z_A,C_A}$ equals the projection $\text{pr}_{Z_A,Y_A} : Z_A \rightarrow Y_A$. Then $Z_A$ equals the graph of $u_A$ for the
unique $A$-morphism $u_A$. Finally, for every $A$-morphism $u_A : C_A \to Y_A$, the graph morphism $\Gamma_{u_A} : C_A \to C_A \times_{\text{Spec } A} Y_A$ is a closed immersion since $Y_A$ is separated over $\text{Spec } A$. Thus, denoting by $Z_A$ the closed image, the morphism $\Gamma_{u_A} : C_A \to Z_A$ is an isomorphism. Since the composition $\text{pr}_{Z_A,C_A} \circ \Gamma_{u_A} : C_A \to C_A$ is an $A$-isomorphism – in fact the identity morphism – it follows that $\text{pr}_{Z_A,C_A}$ is also an $A$-isomorphism. In particular, $Z_A$ is $A$-flat.

In the same way, let $W_A$ be a $\Lambda$-flat closed subscheme of $X_A$ such that $\text{pr}_{W_A,C_A} : W_A \to C_A$ is a closed immersion. Then $\text{pr}_{W_A,C_A} : W_A \to C_A$ is a closed immersion of $\Lambda$-schemes. Denoting by $D_A$ the closed image, then there exists a unique $\Lambda$-morphism $u_{D_A} : D_A \to Y_A$ such that $W_A$ equals the graph of $u_{D_A}$.

Because of Proposition 4.7, for a $k$-morphism $u_k$ with $u_k|_{D_k}$ equal to $u_{D_k}$, for the graph $Z_k$ of $u_k$, the pointed functor Hilb$_{X_A/\Lambda,W_A,Z_k}$ is equivalent to the following Hom functor.

**Definition 4.8.** The Hom functor, $\text{Hom}_A(C_A,Y_A;u_{D_A},u_k)$ is the pointed functor $C_A \to \text{Sets}$ sending $A$ to the set of $A$-morphisms $u_A : C_A \to Y_A$ such that $u_A|_{D_A}$ equals $u_{D_A}$ and such that the base change of $u_A$ to $k$ equals $u_k$.

The obstruction theory for Hilb$_{X_A/\Lambda,W_A,Z_k}$, and every result about this obstruction theory immediately gives an analogue for the Hom functor. Here is the application we will use most often.

**Corollary 4.9.** Assume that $Y_k$ is smooth of pure dimension $\dim(Y_k)$ over $k$, assume that $C_k$ is a proper, geometrically reduced curve that is locally a complete intersection, and assume assume that $D_k$ is an effective Cartier divisor in $C_k$. Then $\text{Hom}_A(C_A,Y_A;u_{D_A},u_k)$ is pro-representable by an object $\hat{R}$ in $\hat{C}_A$. Every irreducible component of $\text{whR}/\text{mA}\hat{R}$ has Krull dimension $\geq \deg_{C_k} u_k^* \text{det}(T_{Y_k/k}) + (1 - p_a(Z_k) - \text{length}(D_k))(\dim(X_k) - 1)$.

If this is equality, then $\hat{R}$ is $\Lambda$-flat.

**Proof.** This follows immediately from Corollary 4.6 under the identification of $\mathcal{N}_{Z_k/X_k}$ with the pullback of $T_{Y_k/k}$. $\square$

**References**


