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MAT 614 Problem Set 1

Problems.

Problem 1. This problem explicitly computes the contraction of a (-1)-curve in one of the simplest cases: a line on a cubic surface in $\mathbb{P}^3$. Let $[x_0, x_1, x_2, x_3]$ be homogeneous coordinates on projective space $\mathbb{P}^3$. For every quadratic homogeneous polynomial $Q(x_0, x_1, x_2, x_3)$, form the cubic homogeneous polynomial

$$F(x_0, x_1, x_2, x_3) = x_2(x_2^2 - x_0x_1) + x_3Q(x_0, x_1, x_2, x_3).$$

(a) Prove that the zero locus $X = Z(F)$ is a cubic surface in $\mathbb{P}^3$ that contains the line $L = Z(x_2, x_3)$ as well as the conic $C = Z(x_2^2 - x_0x_1, x_3)$.

(b) Assuming that X is a smooth cubic surface, prove that $Q(1, 0, 0, 0)$ and $Q(0, 1, 0, 0)$ are nonzero. Conversely, prove that X is smooth for a general choice of Q.
Hint. Bertini's Theorem.

(c) Assuming that X is a smooth hypersurface, prove that $\mathcal{O}_{\mathbb{P}^3}(1)|_X$ is isomorphic to $\mathcal{O}_X(\underline{L} + \underline{C})$ as invertible sheaves. In particular, $\mathcal{O}_{\mathbb{P}^3}(1)|_X(\underline{L})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^3}(2)|_X(-\underline{C})$. It is this second formulation that is easier to use when computing global sections.

(d) Prove that $\mathcal{O}_X(\underline{L})|_L$ has degree -1, i.e., L is a (-1)-curve.
Hint. The adjunction formula.

(e) Prove that the vector space of global sections of $\mathcal{O}_{\mathbb{P}^3}(2)|_X(-\underline{C})$ has as basis the following 5 elements.

$$y_0 = x_0x_3,\ y_1 = x_1x_3,\ y_2 = x_2x_3,\ y_3 = x_3^2,\ y_4 = x_2^2 - x_0x_1.$$

Hint. This is the same as the vector space $(I_C)_2 := \Gamma(\mathbb{P}^3, \mathcal{I}_C(2))$, where $\mathcal{I}_C$ is the ideal sheaf of C in $\mathbb{P}^3$.

(f) Explicitly verify that the the 5 global sections above generate the invertible sheaf $\mathcal{O}_{\mathbb{P}^2}(2)|_X(-\underline{C})$ at every point of X. Therefore there is a unique morphism $\phi : X \to \mathbb{P}^4$ such that $\phi^*\mathcal{O}_{\mathbb{P}^4}(1)$ equals $\mathcal{O}_{\mathbb{P}^2}(2)|_X(-\underline{C})$ and the pullback of the homogeneous coordinates are the 5 sections above.

(g) Also verify that y_0, y_1, y_2, y_3 are identically zero on the line L. Therefore ϕ contracts L to the point $p = [0, 0, 0, 0, 1]$.

(h) Check that $X' := \text{Image}(\phi)$ is contained in the zero locus of the following two homogeneous, degree 2 polynomials $G(y_0, y_1, y_2, y_3, y_4)$,

$$G_1 = y_3y_4 - y_2^2 + y_0y_1,\ G_2 = y_2y_4 + Q(y_0, y_1, y_2, y_3).$$

(i) Prove that the zero scheme of G_1, G_2 is smooth at p. Also check that ϕ maps $X \setminus L$ isomorphically to $Z(G_1, G_2) \setminus \{p\}$. Conclude that ϕ is a contraction to the smooth surface $Z(G_1, G_2)$.

Problem 2. This problem explicitly computes the contraction of a (-2)-curve in one of the simplest cases: a line on a quartic surface in $\mathbb{P}^3$. Let $[x_0, x_1, x_2, x_3]$ be homogeneous coordinates on projective space $\mathbb{P}^3$. For every cubic homogeneous polynomial of the form

$$H = x_0^3 + x_1^3 + x_2^3 - 3\lambda x_0x_1x_2,$$

with λ not a cube root of 1, and for every cubic homogeneous polynomial $K(x_0, x_1, x_2, x_3)$, form the quartic homogeneous polynomial

$$F(x_0, x_1, x_2, x_3) = x_2H(x_0, x_1, x_2) + x_3K(x_0, x_1, x_2, x_3).$$

(a) Prove that the zero locus $X = Z(F)$ is a quartic surface in $\mathbb{P}^3$ that contains the line $L = Z(X_2, X_3)$ as well as the smooth plane cubic $C = Z(H, x_3)$.

(b) Assuming that X is a smooth hypersurface, prove that $K(-1, \zeta, 0, 0)$ is nonzero for every ζ a cube root of 1. Conversely, prove that for a general choice of K, the hypersurface X is smooth.
Hint. Bertini's Theorem.

(c) Assuming that X is a smooth hypersurface, prove that $\mathcal{O}_{\mathbb{P}^3}(1)|_X$ is isomorphic to $\mathcal{O}_X(\underline{L} + \underline{C})$ as invertible sheaves. In particular, $\mathcal{O}_{\mathbb{P}^3}(2)|_X(\underline{L})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^3}(3)|_X(-\underline{C})$.

(d) Prove that $\mathcal{O}_X(\underline{L})|_L$ has degree -2, i.e., L is a (-2)-curve.
Hint. The adjunction formula.

(e) Prove that the vector space of global sections of $\mathcal{O}_{\mathbb{P}^3}(2)|_X(-\underline{C})$ has as basis the following 11 elements,

$$y_{2,0,0,0} = x_3(x_0^2), y_{1,1,0,0} = x_3(x_0x_1), y_{1,0,1,0} = x_3(x_0x_2), y_{1,0,0,1} = x_3(x_0x_3), y_{0,2,0,0} = x_3(x_1^2),$$

$$y_{0,1,1,0} = x_3(x_1x_2), y_{0,1,0,1} = x_3(x_1x_3), y_{0,0,2,0} = x_3(x_2^2), y_{0,0,1,1} = x_3(x_2x_3), y_{0,0,0,2} = x_3(x_3^2), z = H(x_0, x_1, x_2),$$

i.e., x_3m for every quadratic monomial m together with H.

(f) Explicitly verify that the 11 global sections above generate the invertible sheaf $\mathcal{O}_{\mathbb{P}^2}(3)|_X(-\underline{C})$ at every point of X. Therefore there is a unique morphism $\phi : X \to \mathbb{P}^{10}$ such that $\phi^*\mathcal{O}_{\mathbb{P}^{10}}(1)$ equals $\mathcal{O}_{\mathbb{P}^2}(3)|_X(-\underline{C})$ and the pullback of the homogeneous coordinates are the 11 sections above.

(g) Also verify that the sections y_i are identically zero on the line L. Therefore ϕ contracts L to the point $p = [0, \ldots, 0, 1]$.

(h) Check that $X' := \text{Image}(\phi)$ is contained in the zero locus of the following cubic homogeneous polynomials $G(y_0, y_1, y_2, y_3, y_4)$,

$$G_{1,0,0,0} = H(y_{2,0,0,0}, y_{1,1,0,0}, y_{1,0,1,0}) - y_{2,0,0,0}y_{1,0,0,1}z,$$

$$G_{0,1,0,0} = H(y_{1,1,0,0}, y_{0,2,0,0}, y_{0,1,1,0}) - y_{0,2,0,0}y_{0,1,0,1}z,$$

$$G_{0,0,1,0} = H(y_{1,0,1,0}, y_{0,1,1,0}, y_{0,0,2,0}) - y_{0,0,2,0}y_{0,0,1,1}z,$$

$$G_{0,0,0,1} = H(y_{1,0,0,1}, y_{0,1,0,1}, y_{0,0,1,1}) - y_{0,0,0,2}^2 z.$$

How many further cubic polynomials are needed to produce a basis of all cubic polynomials vanishing on $\text{Image}(\phi)$? Can you see a way to produce them?

Problem 3. Explain why there is no (-1)-curve on any smooth hypersurface $X \subset \mathbb{P}^3$ of degree $d \geq 4$. For each integer $d > 4$, what is the smallest positive integer e such that there exists a $(-e)$-curve E on some smooth, degree d hypersurface $X \subset \mathbb{P}^3$. Contemplate putting the defining equation of X into a "normal form" as in the previous two problems, and then explicitly constructing the contraction of E.

Problem 4. This problem and the next explain how to produce a smooth curve in $\mathbb{P}^3$ and a smooth surface containing the curve such that the curve has negative self-intersection on the surface, yet there is no contraction of the curve to a projective surface. This first problem produces the curve. The next problem produces the surface and establishes nonexistence of a contraction to a projective surface.

(a) Let $a, b \geq 0$ be integers. Denote by $S_{a,b} := k[x_0, x_1, y_0, y_1]_{a,b}$ the vector space of bihomogeneous polynomials of bidegree (a, b), i.e., the vector space with basis $x_0^i x_1^{a-i} y_0^j y_1^{b-j}$ for $0 \leq i \leq a$ and $0 \leq j \leq b$. Prove that $\dim_k(S_{a,b})$ equals $(a+1)(b+1)$. Therefore the projective space $\mathbb{P}S_{a,b}$ has dimension $ab + a + b$.

(b) Denote by R the projective subvariety in $\mathbb{P}S_{a,b}$ parameterizing polynomials that are reducible, i.e., the union over all nonzero pairs of integers (a', b') with $0 \leq a' \leq a$ and $0 \leq b' \leq b$ of the image of the Segre morphism

$$\sigma_{a',b'} : \mathbb{P}S_{a',b'} \times \mathbb{P}S_{a-a',b-b'} \to \mathbb{P}S_{a,b}.$$

Prove that $\text{Image}(\sigma_{a',b'})$ has codimension $\geq a'(b - b') + (a - a')b'$. Assuming that $a, b \geq 2$, conclude that every irreducible component of R has codimension ≥ 2 (this is false if either a or b is ≤ 1).

(c) By considering $S_{a,b}$ as the space of global sections on $\mathbb{P}^1 \times \mathbb{P}^1$ of the invertible sheaf $\mathcal{O}(a, b) := \text{pr}_1^*\mathcal{O}(a) \otimes \text{pr}_2^*\mathcal{O}(b)$, interpret elements $[F] \in \mathbb{P}S_{a,b}$ as Cartier divisors $Z(F)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Prove that $Z(F)$ is an integral curve if and only if $[F]$ is not in R.

(d) Let Π be a line in $\mathbb{P}S_{a,b}$ that is disjoint from R; since R has codimension ≥ 2, such lines exist. Denote by $B \subset \mathbb{P}^1 \times \mathbb{P}^1$ the base locus of Π, i.e., $Z(\{F | [F] \in \Pi\})$. Prove that B has codimension 2 in $\mathbb{P}^1 \times \mathbb{P}^1$. Also prove that B is a reduced set of points if Π is "general".

(e) Let Π be a line as above such that Π is a reduced set of points. Denote by $\nu : P \to \mathbb{P}^1 \times \mathbb{P}^1$ the blowing up of the ideal sheaf of B. Denote by $\pi : P \to \Pi$ the unique morphism whose (scheme-theoretic) fiber over every point $[F] \in \Pi$ is the strict transform of $Z(F)$ in P. Prove that P is a smooth variety, and prove that every geometric fiber of π is an integral Cartier divisor on P.

(f) Let $\mathcal{L}$ be an invertible sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ such that for *the* generic fiber D_η of π, the restriction $\nu^*\mathcal{L}|_{D_\eta}$ is isomorphic to $\mathcal{O}_{D_\eta}$ as an invertible sheaf. Conclude that $\nu^*\mathcal{L}$ is isomorphic to $\pi^*\mathcal{M}$ for an invertible sheaf $\mathcal{M}$ on Π.
Hint. Consider a rational section s of $\nu^*\mathcal{L}$ that is regular on D_η and generates $\nu^*\mathcal{L}|_{D_\eta}$. What can you say about the Cartier divisor of s on P? Use the fact that all fibers of π are integral.

(g) Let U be the open subscheme $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus B$, i.e., the maximal open subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\nu : \nu^{-1}(U) \to U$ is an isomorphism. By restricting the isomorphism from (f) to $\nu^{-1}(U) \cong U$, conclude that $\mathcal{L}$ is isomorphic, on U, to $\mathcal{O}(m \cdot a, m \cdot b)$ for some integer m. Finally, since B has codimension 2, use Hartog's phenomenon / Property S2 to conclude that $\mathcal{L}$ is isomorphic to $\mathcal{O}(m \cdot a, m \cdot b)$ on all of $\mathbb{P}^1 \times \mathbb{P}^1$. By considering the intersection number with D_η, conclude that the integer m equals 0, i.e., the following restriction map of Picard groups is injective,

$$\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathrm{Pic}(D_\eta).$$

(h) Assume now that k equals the algebraic closure of a Fermat field, $k = \overline{\mathbb{F}}_p$. For every k-point $[F]$ in Π, prove that the restriction map of Picard groups

$$\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathrm{Pic}(Z(F))$$

is *not* injective. Hence it really is necessary to pass to the generic point of Π.

Problem 5. Let $a, b \geq 2$ be integers as above such that also $a > b$, e.g., $(3, 2)$ to be definite. Let $D_\eta \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a generic element of $\mathbb{P}S_{a,b}$ so that the restriction map on Picard groups is injective, e.g., for $(3, 2)$ the vanishing set of

$$F = x_0x_1(x_1 - x_0)y_1^2 - (x_1 - sx_0)(x_1 - tx_0)(x_1 - ux_0)y_0^2$$

will do, where s, t, u are elements of k that are algebraically independent over the prime subfield.

(a) Embed $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ by the usual Segre map, $\sigma([x_0, x_1], [y_0, y_1]) = [x_0y_0, x_0y_1, x_1, y_0, x_1y_1]$. Using Problem 4, prove that for every pair of integers $(m, n) \neq (0, 0)$, the invertible sheaf $\sigma^*\mathcal{O}(m)|_{D_\eta} \otimes \omega_{D_\eta}^{\otimes n}$ is nontrivial.

(b) Let $q \geq 1$ be a positive integer, and let $p \geq 2q + 1$ be an integer. Let P be a smooth variety of dimension p and let $Q \subset P$ be a smooth, closed subscheme of dimension q. Let $\mathcal{L}$ be an invertible sheaf on P, and let $V \subset \Gamma(P, \mathcal{I}_Q \otimes \mathcal{L})$ be a finite-dimensional vector space of sections of $\mathcal{L}$ that all vanish on Q; $\mathcal{I}_Q$ is the ideal sheaf of Q. Assume that for every $x \in C$, the composite map

$$V \to \mathfrak{m}_{P,x} \otimes \mathcal{L}_x \to \mathfrak{m}_{P,x}/(\mathfrak{m}_{P,x}^2 + \mathfrak{m}_{C,x}) \otimes \mathcal{L}_x$$

is surjective; this is automatic if P is projective, $\mathcal{L}$ is sufficiently ample, and V is the vector space of all sections of $\mathcal{I}_Q \otimes \mathcal{L}$. Repeat the usual proof of Bertini's theorem to conclude that for the generic section s of V, the divisor $Z(s)$ is smooth at *every* point of Q. In particular, for a smooth curve C in $\mathbb{P}^3$, for all integers $d \gg 0$, a general degree d surface in $\mathbb{P}^3$ containing C is smooth at every point of C. Since the base locus of the linear system $|\mathcal{I}_C(d)|$ equals C for $d \gg 0$, the usual Bertini's theorem then implies that the general degree d surface containing C is everywhere smooth.

(c) Let $C \subset \mathbb{P}^3$ be a smooth curve. Let $\Pi \subset \Gamma(\mathbb{P}^3, \mathcal{I}_C(d))$ be a pencil of degree d surfaces containing C whose general member is smooth and whose base locus B equals $C \cup C'$ for an irreducible curve C'. Denote by $\nu : P \to \mathbb{P}^3$ the blowing up of B. Denote by $\pi : P \to \Pi$ the unique morphism whose fiber over each point $[F] \in \Pi$ equals the strict transform of $Z(F)$. For the fiber D_η of π over *the* generic point η of Π, prove that $\mathrm{Pic}(D_\eta)$ is generated by $\mathcal{O}_{\mathbb{P}^3}(1)|_{D_\eta}$ and $\mathcal{O}_{D_\eta}(\underline{C})$.
(Big) Hint. Extend the invertible sheaf to an open subset of P whose complement has codimension 2; restrict to the complement of the exceptional divisor; finally use the fact that this invertible sheaf, considered as a sheaf on an open subset of $\mathbb{P}^3$, extends to an invertible sheaf on all of $\mathbb{P}^3$, necessarily $\mathcal{O}_{\mathbb{P}^3}(m)$ for some integer m. Thus the restriction of the original invertible sheaf on D_η to the complement of B equals $\mathcal{O}_{\mathbb{P}^3}(m)|_{D_\eta}$. Finally, use that $\mathcal{O}_{D_\eta}(\underline{C}+\underline{C}')$ equals $\mathcal{O}_{\mathbb{P}^3}(d)|_{D_\eta}$ to write the invertible sheaf $\mathcal{O}_{\mathbb{P}^3}(m)|_{D_\eta}(a\underline{C} + a'\underline{C'})$ as $\mathcal{O}_{\mathbb{P}^3}(m + a'd)|_{D_\eta}((a - a')\underline{C})$
Nota bene. For the *geometric* generic fiber $D_{\overline{\eta}}$, the Picard group certainly can be larger than this (although there are *Noether-Lefschetz theorems* for d sufficiently large and Π sufficiently general). Thus it is crucial to work with the generic fiber rather than the geometric generic fiber.

(d) For C as in (a), prove that $\mathcal{O}_{D_\eta}$ is the unique invertible sheaf on D_η whose restriction to C is trivial. In particular, there is no invertible sheaf on D_η whose restriction to C is trivial and that has positive intersection number with the hyperplane class.

(e) Now consider the geometric generic fiber $D_{\overline{\eta}}$, i.e., the base change of D_η by the separable closure of the residue field $\kappa(\eta) = k(\Pi)$. By way of contradiction, assume that there exists an invertible sheaf $\mathcal{M}$ whose restriction to C is trivial and that has positive intersection number with the hyperplane class. Prove that $\mathcal{M}$ exists already after a base change from $\kappa(\eta)$ to a finite Galois extension. Next consider the tensor product of the finitely many Galois conjugates of $\mathcal{M}$. Prove that this is also an invertible sheaf whose restriction to C is trivial and that has positive intersection number with the hyperplane class, yet now it is Galois invariant. Since we can choose C to have k-points, this Galois invariant invertible sheaf is the base change of an invertible sheaf on D_η. This contradicts (d), hence there is no such invertible sheaf $\mathcal{M}$ on $D_{\overline{\eta}}$.

(f) Conclude that there is no morphism $c : D_{\overline{\eta}} \to D'$ with D' projective that contracts precisely C. In fact, conclude that there is not even a contraction such that D' is a scheme; if U is an open affine neighborhood of the singleton Image(C), then $D' \setminus U$ is a Cartier divisor whose pullback by c contradicts (e). Therefore in Castelnuovo's theorem, it was crucial that the contracted curve had genus 0. (There does exist a contraction of C such that D' is an *algebraic space*.)

Problem 6. For a ring R and an R-module M, a sequence of elements $(r_1, \dots, r_n) \in R$ is M-**regular** if r_1 is a nonzerodivisor on M, r_2 is a nonzerodivisor on M/r_1M, r_3 is a nonzerodivisor on $M/(r_1M + r_2M)$, etc. This is equivalent to acyclicity of the following *Koszul complex* $M \otimes_R K_R(r_1, ..., r_n)$, i.e., exactness at all places except the rightmost term,

$$0 \longrightarrow M \otimes_R \bigwedge\nolimits_R^n(R^{\oplus n}) \xrightarrow{\mathrm{Id}_M \otimes d_n} M \otimes_R \bigwedge\nolimits_R^{n-1}(R^{\oplus n}) \longrightarrow \dots$$
$$\dots \longrightarrow M \otimes_R \bigwedge\nolimits_R^1(R^{\oplus n}) \xrightarrow{\mathrm{Id}_M \otimes d_1} M \otimes_R \bigwedge\nolimits_R^0(R^{\oplus n}) \longrightarrow 0.$$

Here $d_1 : R^{\oplus n} \to R$ is the morphism $(a_1, \dots, a_n) \mapsto a_1r_1 + \dots + a_nr_n$. And for every $q > 1$, $d_q : \bigwedge_R^q(R^{\oplus n}) \to \bigwedge_R^{q-1}(R^{\oplus n})$ is defined by induction using the (differential graded) Leibniz rule:

$$d_q(f \wedge g) = d_1(f) \wedge g + (-1)f \wedge d_{q-1}(g)$$

for every $f \in \bigwedge_R^1(R^{\oplus n})$ and for every $g \in \bigwedge_R^{q-1}(R^{\oplus n})$. If R is a local ring containing the field k, for every $r_1, \dots, r_n \in \mathfrak{m}_R$, prove that $(r_1, dots, r_n)$ is R-regular if and only if the following associated local homomorphism is flat,

$$\phi : k[x_1, \dots, x_n]_{\langle x_1, \dots, x_n \rangle} \to R, \quad x_i \mapsto r_i.$$

Hint. Use the *local flatness criterion*.

Problem 7. A Noetherian local ring R is *Cohen-Macaulay* if there exists an R-regular sequence $r_1, \dots, r_n \in \mathfrak{m}_R$ such that $R/\langle r_1, \dots, r_n \rangle$ has finite length, i.e., $(r_1, \dots, r_n)$ is a *system of parameters*. A locally Noetherian scheme X is *Cohen-Macaulay* if for every $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a Cohen-Macaulay local ring. For a field k, prove that a finite type affine k-scheme X, resp., projective k-scheme X, is Cohen-Macaulay if and only if there exists a finite, flat morphism $f : X \to \mathbb{A}_k^n$, resp. a finite, flat morphism $f : X \to \mathbb{P}_k^n$, for some integer n. In this case, *every* finite morphism $f : X \to \mathbb{A}_k^n$, resp. $f : X \to \mathbb{P}_k^n$, is flat.

Problem 8. J.-P. Serre's criterion for normality implies (as a special case) that a quasi-projective k-scheme X is normal if and only if both

(R1) there exists a closed subset $C \subset X$ of codimension ≥ 2 such that $X \setminus U$ is regular (equiv. smooth if k is algebraically closed), and

(S2) for every point x of X, for a general 2-dimensional linear section S of X containing x, the surface S is Cohen-Macaulay.

In particular, a 2-dimensional quasi-projective scheme is normal if and only if it is Cohen-Macaulay and satisfies Condition (R1). The non-normal surface $S = Z(xz, xw, yz, yw) \subset \mathbb{A}_k^4$ satisfied (R1). Find a finite morphism $f : S \to \mathbb{A}_k^2$ that is not flat, thus verifying that S is not Cohen-Macaulay.

Problem 9. Let Y be a normal surface, let y be a closed point of Y, and let $\nu : X \to Y$ be the blowing up. Assume that X is smooth and that the exceptional set is a $(-e)$-curve $E \subset X$. Let $c : X \to X'$ be the contraction of E in X to a point $z \in X'$. Prove that there exists a unique morphism $f : X' \to Y$ such that $c \circ f$ equals ν, and prove that this unique morphism is an isomorphism.

Problem 10. Give an example of a birational morphism of quasi-projective, normal varieties $f : X \to Y$ such that the exceptional set $E \subset X$ has a component of codimension ≥ 1 and Y is not $\mathbb{Q}$-factorial.