1. Relative differentials. $f : X \to Y$ have
$\Delta_f : X \longrightarrow X \times_Y X$, locally closed imm.
$\hookrightarrow \mathcal{U}$ open

Let $\mathcal{I}$ = ideal sheaf. Then $\Omega_{X/Y} := \Delta_f^* \mathcal{I}$.

<u>Example:</u> $A \xrightarrow{\varphi} B$, $X = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = Y$.
$B \otimes_A B \xrightarrow{m} B$, $I = \operatorname{Ker}(m)$, is gen'd by
$J = \langle b \otimes 1 - 1 \otimes b \mid b \rangle$: Every $b_1 \otimes_A b_2 \equiv b_1 b_2 \otimes_A 1$ mod J
$b_1 b_2 \otimes 1 = b_1 \otimes 1 \cdot (b_2 \otimes 1 - 1 \otimes b_2)$.

$d_{B/A} : B \to I/I^2 =: \Omega_{B/A}$ by $b \mapsto b \otimes 1 - 1 \otimes b$.

$$d(b_1 b_2) = b_1 b_2 \otimes 1 - 1 \otimes b_1 b_2 = b_1 (b_2 \otimes 1 - 1 \otimes b_2) + b_2 (b_1 \otimes 1 - 1 \otimes b_1) - \underbrace{(b_1 \otimes 1 - 1 \otimes b_1)(b_2 \otimes 1 - 1 \otimes b_2)}_{\in I^2}$$

$\equiv b_1 \, db_2 + b_2 \, db_1 \mod I^2$.

Also $d(a) = 0$ for all $a \in A$.

A-derivation $\partial : B \to M$.
<u>Universal</u> A-derivation $d : B \to \Omega_{B/A}$.

Universal $f^{-1}\mathcal{O}_Y$-derivation: $d : \mathcal{O}_X \to \Omega_{X/Y}$.

Compatibilities: • Base change $\begin{array}{ccc} X' & \xrightarrow{g_X} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g_Y} & Y \end{array}$, $\Omega_{X'/Y'} \cong g_X^* \Omega_{X/Y}$
• Composition $X \xrightarrow{f} Y \xrightarrow{g} Z$
$f^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$.

$i^*\mathcal{I}_{X/Y} \to i^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0$

- Immersions : $\Omega_{X/Y} = 0$,
- Open immersions $i^*\Omega_{X/Y} \to \Omega_{U/Y}$ is an iso.
- $\mathbb{A}^n_Y \xrightarrow{\pi} Y$: $\pi_* \mathcal{O} = \mathcal{O}_Y[x_1, \dots, x_n]$

$d : \mathcal{O}_Y[x_1, \dots, x_n] \to \mathcal{O}_Y\langle dx_1, \dots, dx_n\rangle$

- $\mathcal{E}$ locally free on Y, $\mathbb{A} \xrightarrow{\pi} Y$, $\phi : \pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{A}}$ univ.

$\pi_* \mathcal{O}_{\mathbb{A}} = \mathrm{Sym}^\bullet_{\mathcal{O}_Y} \mathcal{E}$, $d : \mathrm{Sym}^\bullet_{\mathcal{O}_Y}\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_Y} \mathrm{Sym}^\bullet \mathcal{E}$

$d|_{S_1} : \underset{\substack{\| \\ \mathcal{E}}}{S_1} \to \underset{\substack{\| \\ \mathcal{E}\otimes_{\mathcal{O}_Y}\mathcal{O}_Y \cong \mathcal{E}}}{\mathcal{E} \otimes S_0}$ is id. ~~$d|_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}\otimes$~~

Highfalutin way to say $\Omega_{R[x_1,\dots,x_n]/R} \cong R\langle dx_1, \dots, dx_n\rangle$.

But also a morphism of graded modules

$\Omega_{\mathbb{A}/Y} = \left(\mathcal{E} \otimes_{\mathcal{O}_Y} S[-1]\right)^\sim = \pi^*\mathcal{E} \otimes \widetilde{S[-1]}. = \rho^*(\mathcal{E} \otimes \mathcal{O}(-1))$

$$0 \to \rho^*\Omega^1_{\mathbb{P}/Y} \to \underset{\substack{\| \\ \rho^*(\mathcal{E}\otimes\mathcal{O}_{\mathbb{P}}(-1))}}{\Omega_{U/Y}} \xrightarrow{\phi} \underset{\substack{\| \\ \rho^*\mathcal{O}_{\mathbb{P}}}}{\mathcal{O}_U} \to 0$$

$\therefore$ Euler sequence $0 \to \Omega^1_{\mathbb{P}/Y} \to \mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{O}_{\mathbb{P}}(-1) \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}} \to 0$

$\Rightarrow \omega^\circ_{\mathbb{P}/Y} = \Lambda^{top}\Omega^1_{\mathbb{P}/Y}$ =: canonical bundle

Enriques-Severi-Zariski : X normal, proj of dimn ≥ 2. (S_2: depth $A_p \geq \min(2, \mathrm{ht}\,p)$.)

then $H^1(X, \mathcal{F}(-q)) = 0$ $\forall$ locally free $\mathcal{F}$ & $q \gg 0$.

$$\underset{\substack{\| \\ H^1(\mathbb{P}^n, i_*\mathcal{F}(-q))}}{} = \mathrm{Ext}^{n-1}_{\mathcal{O}_{\mathbb{P}^n}}(i_*\mathcal{F}(-q), \omega_{\mathbb{P}^n}) = \Gamma(\mathbb{P}^n, \mathcal{E}xt^{n-1}_{\mathcal{O}_{\mathbb{P}^n}}(i_*\mathcal{F}(q), \omega_{\mathbb{P}^n}))$$

Cor 7.9 [LHT for π_0]. Let X be an integral, normal (connected) proj. variety of dimn ≥ 2 / $k = \bar{k}$.
Let D be an effective ample divisor. Then supp (D) is connected.

Pf: $0 \to \mathcal{O}_X(-qD) \to \mathcal{O}_X \to \mathcal{O}_{qD} \to 0$

For $q >> 0$, $H^1(X, \mathcal{O}_X(qD)) = 0$. Thus

$$H^0(X, \mathcal{O}_X) \twoheadrightarrow H^0(X, \mathcal{O}_{qD}) = H^0(qD, \mathcal{O}_{qD}). \quad \square$$

L.H.T in general.

Koszul, etc.

Hodge numbers. Serre duality $H^{p,q} = H^q(X, \Omega^p)$
Hodge theorem

$$\cong H^{n-q}(X, \Omega^{p\vee} \otimes \Omega^n)^\vee$$
$$= H^{n-q}(X, \Omega^{n-p})^\vee$$
$$= (H^{n-p, n-q})^\vee.$$

Residues for curves: Read Tate's paper.

Higher direct images.

Defn: Sheafify of $V \mapsto H^i(f^{-1}V, \mathcal{F}|_{f^{-1}V})$

If Y is aff. & $\mathcal{F}$ q-coh, then $= H^i(X, \mathcal{F})^\sim$
q-coh & finiteness.

1. Properties of differentials.

a) Base-change:
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$
, $\mathcal{O}_X \xrightarrow{d} \Omega_{X/Y} \rightsquigarrow$
$(g')^{-1}\mathcal{O}_X \xrightarrow{d} (g')^{-1}\Omega_{X/Y} \rightsquigarrow$
$(f')^{-1}\mathcal{O}_{Y'} \otimes_{\mathcal{O}_Y} (g')^{-1}\mathcal{O}_X \xrightarrow{d} (f')^{-1}\mathcal{O}_{Y'} \otimes_{\mathcal{O}_Y} (g')^{-1}\Omega_{X/Y}$

But $(f')^{-1}\mathcal{O}_{Y'} \otimes_{\mathcal{O}_Y} (g')^{-1}\mathcal{O}_X = \mathcal{O}_{X'}$. And $(f')^{-1}\mathcal{O}_{Y'} \otimes_{\mathcal{O}_Y} (g')^{-1}\Omega_{X/Y}$ $= (g')^*\Omega_{X/Y}$. So get $\Omega_{X'/Y'} \to (g')^*\Omega_{X/Y}$. This is an isomorphism.

b) $X \xrightarrow{f} Y$, $g_X : X \to Z$, $g_Y : Y \to Z$. There is a short exact sequence
$$f^*\Omega_{Y/Z} \xrightarrow{\alpha} \Omega_{X/Z} \xrightarrow{\beta} \Omega_{X/Y} \to 0$$

α: $f^{-1}d_{Y/Z} : f^{-1}\mathcal{O}_Y \to f^{-1}\Omega_{Y/Z}$ univ., but also have $f^{-1}\mathcal{O}_Y \xrightarrow{f^\#} \mathcal{O}_X \xrightarrow{d_{X/Z}} \Omega_{X/Z} \rightsquigarrow f^{-1}\Omega_{Y/Z} \to \Omega_{X/Z} \Leftrightarrow f^*\Omega_{Y/Z} \xrightarrow{\alpha} \Omega_{X/Z}$.

β. $d_{X/Y} : \mathcal{O}_X \to \Omega_{X/Y}$ is $f^{-1}\mathcal{O}_Y$-linear, thus also $g_X^{-1}\mathcal{O}_Z$ $=$ ~~...~~ $f^{-1}g_Y^{-1}\mathcal{O}_Z$ -linear.

c) If $f : X \to Y$ is an open immersion, then $\alpha : f^*\Omega_{Y/Z} \to \Omega_{X/Z}$ is an isomorphism. In particular, $\Omega_{X/Y} = 0$. A morphism $f : X \to Y$ is <u>unramified</u> if it is locally finitely presented and $\Omega_{X/Y} = 0$.

d) If $f : X \to Y$ is a locally closed immersion, then f is unramified and there is a s.e.s.
$$f^*\mathcal{I} \xrightarrow{\delta} f^*\Omega_{Y/Z} \xrightarrow{\alpha} \Omega_{X/Z} \to 0.$$

$d_{Y/Z}|_{\mathcal{I}} : \mathcal{I} \to \Omega_{Y/Z} \rightsquigarrow f^{-1}d_{Y/Z}|_{\mathcal{I}} : f^{-1}\mathcal{I} \to f^{-1}\Omega_{Y/Z} \to f^*\Omega_{Y/Z}$.
This is $f^{-1}\mathcal{O}_Y$-linear b/c $d(f_1 f_2) = f_1\,df_2 + f_2\,df_1 \equiv 0 \mod f^{-1}\mathcal{I}\cdot f^{-1}\Omega_{Y/Z}$ for $f_1, f_2 \in f^{-1}\mathcal{I}(\mathcal{U})$.

e) Let $\mathcal{E}$ be a locally free sheaf on X.
Let $S = \mathrm{Sym}^\bullet \mathcal{E} = \bigoplus_{d \geq 0} \mathrm{Sym}^d \mathcal{E}$. Let $\mathbb{A} = \underline{\mathrm{Spec}}\, S \xrightarrow{\pi} X$,
$\mathbb{P} = \underline{\mathrm{Proj}}\, S \xrightarrow{\pi} X$. Let $\varphi : \pi^* \mathcal{E}[-1] \to \mathcal{O}_{\mathbb{A}}$, $\tilde{\varphi} : \pi^* \mathcal{E}(-1) \to \mathcal{O}_{\mathbb{P}}$
be the universal homomorphisms.

There is a unique graded $\mathcal{O}_X$-module homomorphism
$$d_{S/\mathcal{O}_X} : S \longrightarrow S \otimes_{\mathcal{O}_X} \mathcal{E}[-1]$$
such that
(i) $d_{S/\mathcal{O}_X}^0 = 0$
(ii) $d_{S/\mathcal{O}_X}^1 : S_1 \to S_0 \otimes_{\mathcal{O}_X} \mathcal{E}$, i.e. $\mathcal{E} \to \mathcal{E}$, is the identity.
(iii) $d_{S/\mathcal{O}_X}$ is an $\mathcal{O}_X$-derivation.

This gives an isomorphism $\psi : \Omega_{\mathbb{A}/X} \to \pi^* \mathcal{E}[-1]$.
Restricting to $\mathcal{U}$, this gives $\Omega_{\mathbb{A}/X} \cong \rho^*(\pi^* \mathcal{E}(-1))$.
And φ gives $\Omega_{\mathbb{A}/X} \xrightarrow{\varphi} \mathcal{O}_{\mathcal{U}}$. The claim is that
$$0 \to \rho^* \Omega_{\mathbb{P}/X} \to \Omega_{\mathbb{A}/X} \xrightarrow{\varphi} \mathcal{O}_{\mathcal{U}} \to 0$$
is exact. Check locally over $D_+(x)$'s.
This gives the Euler sequence on $\mathbb{P}$

$$\boxed{0 \to \Omega_{\mathbb{P}/X} \to \pi^* \mathcal{E}(-1) \xrightarrow{\tilde{\varphi}} \mathcal{O}_{\mathbb{P}} \to 0}$$

This is an element in Yoneda-Ext^1: $h \in \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}, \Omega_{\mathbb{P}/X})$
$= H^1(\mathbb{P}, \Omega_{\mathbb{P}/X})$.
The other syzygies of $K^\bullet(\tilde{\varphi})$ are of the form
$$0 \to \Omega^q_{\mathbb{P}/X} \to \pi^* \Lambda^q \mathcal{E}(-q) \to \cdots \to \pi^* \mathcal{E}(-1) \to \mathcal{O}_{\mathbb{P}} \to 0$$
$\rightsquigarrow$ elt. $h_q \in H^r(\mathbb{P}, \Omega^r_{\mathbb{P}/X})$

<u>Fact</u>: $h_q = h^q$.

Most important: $\Omega^r_{\mathbb{P}/X} = \omega_{\mathbb{P}/X} = \pi^* \Lambda^{top} \mathcal{E}(-r)$.
But $\Omega^r_{\mathbb{P}/X} = \Lambda^r \Omega^1_{\mathbb{P}/X}$. Thus the dualizing sheaf is the canonical sheaf.

~~2. a) Products.~~ $\cancel{Ext^q_{\mathcal{O}_X}(A,B) \otimes_{\mathcal{O}(X)} Ext^{q'}_{\mathcal{O}_X}(A',B') \to Ext^{q+q'}_{\mathcal{O}_X}(A\otimes A', B\otimes B')}$

2. a) Internal products.

$$Ext^j_{\mathcal{O}_X}(B,C) \underset{\mathcal{O}(X)}{\otimes} Ext^i_{\mathcal{O}_X}(A,B) \longrightarrow Ext^{i+j}_{\mathcal{O}_X}(A,C)$$

Associativity & ~~skew~~ graded commutativity (if $A=B=C$).

b) Let $\iota: X \to Y$ be a regular embedding, i.e., a closed immersion s.t. $\mathcal{I}$ is everywhere locally generated by a regular sequence.
Fact: This is automatic if Y is regular & X is a local complete intersection.

Proposition. If $\iota: X \to Y$ is a regular embedding of codim c, then

$$Ext^q_{\mathcal{O}_Y}(\iota_* \mathcal{O}_X, \mathcal{O}_Y) = \begin{cases} \iota_*(\Lambda^c \iota^* \mathcal{I})^\vee, & q = c \\ 0, & q \neq c \end{cases}.$$

Pf:
Locally clear b/c of Koszul complexes. How to reduce to local case?
(i) $Ext^c_{\mathcal{O}_Y}(\iota_* \mathcal{O}_X, \mathcal{O}_Y) \xrightarrow{\sim} Ext^c_{\mathcal{O}_Y}(\iota_* \mathcal{O}_X, \iota_* \mathcal{O}_X)$.
(ii) Internal product: $\Lambda^c Ext^1_{\mathcal{O}_Y}(\iota_* \mathcal{O}_X, \iota_* \mathcal{O}_X) \xrightarrow{\sim} Ext^c_{\mathcal{O}_Y}(\iota_* \mathcal{O}_X, \iota_* \mathcal{O}_X)$
(iii) $0 \to \mathcal{I} \to \mathcal{O}_Y \to \iota_* \mathcal{O}_X \to 0$ gives

$$\iota_*(\iota^* \mathcal{I})^\vee = Hom_{\mathcal{O}_Y}(\mathcal{I}, \iota_* \mathcal{O}_X) \xrightarrow{\delta} Ext^1_{\mathcal{O}_Y}(\iota_* \mathcal{O}_X, \iota_* \mathcal{O}_X)$$

is an isom. $\square$.

Consequence: Let X be a reduced, local complete intersection scheme that is projective / $k = \bar{k}$. There is an isomorphism $\omega_X^\circ \cong \det(\Omega_{X/k})$.

Pf: $\iota : X \to \mathbb{P}^n_k$ of codim c. Then, by constr,

$\iota_* \omega_X^\circ \cong \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n_k}}(\iota_* \mathcal{O}_X, \omega_{\mathbb{P}^n}) = \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n_k}}(\iota_* \mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n}) \otimes \omega_{\mathbb{P}^n}$

$= \iota_*(\iota^* \omega_{\mathbb{P}^n} \otimes \Lambda^c \iota^* \mathcal{I}^\vee)$.

But also have

$$0 \to \iota^* \mathcal{I} \to \iota^* \Omega_{\mathbb{P}^n/k} \to \Omega_{X/k} \to 0.$$

Thus $\det(\Omega_{X/k}) \cong \iota^* \det(\Omega_{\mathbb{P}^n/k}) \otimes \det(\iota^* \mathcal{I})^\vee$

$= \iota^* \omega_{\mathbb{P}^n} \otimes \Lambda^c \iota^* \mathcal{I}^\vee$. $\square$.

Hodge numbers. Hodge thm. Serre duality & Poincaré duality.

Theorem on formal functions. X a proper ~~projective~~ scheme over $\operatorname{Spec} A$, A a Noetherian local ring.

$X_n :=$ closed subscheme of X w/ ideal sheaf $\mathfrak{m}_A^{n+1} \cdot \mathcal{O}_X$, i.e., $X_n = \operatorname{Spec} A/\mathfrak{m}^{n+1} \times_{\operatorname{Spec} A} X$.

Given a coh. sheaf $\mathcal{F}$ on X, have $\mathcal{F} \twoheadrightarrow \mathcal{F}_n$ compatibly $\rightsquigarrow H^i(X, \mathcal{F}) \to \varprojlim_n H^i(X_n, \mathcal{F}_n)$.

$\rightsquigarrow H^i(X, \mathcal{F})^\wedge \to \varprojlim_n H^i(X_n, \mathcal{F}_n)$.

Theorem on formal functions. Let X be a proper A-scheme, where A is a Noetherian local scheme. For every coherent sheaf $\mathcal{F}$ & every integer i, the following is an isomorphism

$$H^i(X, \mathcal{F})^\wedge \longrightarrow \varprojlim_n H^i(X_n, \mathcal{F}_n).$$

Corollary 11.2. If $d = \dim X_0$, then $H^i(X, \mathcal{F}) = 0$ for $i > d$.

Cor. 11.3. If $H^0(X, \mathcal{O}_X) = A$, then X_0 is connected.

Pf: If $X_0 = X_0' \sqcup X_0''$, same is true for every n

$$\rightsquigarrow \varprojlim H^0(X_n, \mathcal{O}_{X_n}) = \varprojlim H^0(X_n', \mathcal{O}_{X_n'}) \oplus \varprojlim H^0(X_n'', \mathcal{O}_{X_n''})$$

$$\Psi$$

$$(1, 0), (0, 1)$$

These are elts $e', e'' \in H^0(X, \mathcal{O}_X)^\wedge = \hat{A}$ that are not units, b/c $e' \cdot e'' = 0$, but whose sum is a unit $e' + e'' = 1$. $\times\!\!\times$. $\square$.

ZMT: Original form. Y normal, $X \xrightarrow{f} Y$ birat'l & quasi-finite. Then $f(X) \subset Y$ is open and $f: X \to f(X)$ is an isom., i.e., $f: X \to Y$ is an open immersion.

Groth. form: Let $f: X \to Y$ be any finite type, sep'd., quasi-finite morphism. Then exists an open imm $X \overset{i}{\hookrightarrow} \overline{X}$ and a finite morphism $\bar{f}: \overline{X} \to Y$ s.t. $f = \bar{f} \circ i$.

Connd thm. Let $f: X \to Y$ be a birational proper morphism. If Y is normal, all fibers of f are connected.

Stein factorization. $f: X \to Y$. Form $f_* \mathcal{O}_X$. Have a factorization $X \xrightarrow{g} \underline{\mathrm{Spec}}\, f_* \mathcal{O}_X \xrightarrow{h} Y$
$$\underline{\mathrm{Spec}}\, f_* \mathcal{O}_X = Z$$
where h is finite and $g_* \mathcal{O}_X = \mathcal{O}_Z$.

Cor. The fibers of g are connected.

Apps of ZMT

Consequences

1. Theorem on formal functions. Let A be a local Noetherian ring. Let X be a proper A-scheme and let $\mathcal{F}$ be a coherent sheaf. For every p, the following map is an isomorphism

$$\hat{A} \otimes_A H^p(X, \mathcal{F}) \longrightarrow \varprojlim_n H^p(X_n, \mathcal{F}_n).$$

Corollary 11.2. $H^p(X, \mathcal{F}) = 0$ for $p > \dim(X_0)$.

Corollary 11.3. ~~If $\operatorname{Spec} H^0(X, \mathcal{O}_X)$ is connected, then X_0 is connected.~~ If $H^0(X, \mathcal{O}_X) \hat{=} A$, then X_0 is connected.

(or if $\operatorname{Spec} H^0(X, \mathcal{O}_X)/m_A H^0(X, \mathcal{O}_X)$ is conn.)

Proof. If X_0 is not connected, $\exists$ nonempty, open & closed subsets $X_0', X_0'' \subset X_0$ s.t. $X_0 = X_0' \sqcup X_0''$. So there are elts. $e_0', e_0'' \in H^0(X_0, \mathcal{O}_{X_0})$ s.t. $e_0' \cdot e_0'' = 0$, $e_0' + e_0'' = 1$, $e_0' e_0' = e_0'$, $e_0'' e_0'' = e_0''$ ... Because $X_n = X_0$ as a topological space, the same holds for X_n, i.e., $\exists$ $e_n', e_n'' \in H^0(X_n, \mathcal{O}_{X_n})$ ....
The sequences (e_n'), (e_n'') are elts. in $\varprojlim H^0(X_n, \mathcal{O}_{X_n})$.

Thus, $\exists e', e'' \in \hat{A} \otimes_A H^0(X_0, \mathcal{O}_{X_0})$ s.t. $e' \cdot e'' = 0$, $e' + e'' = 1$, ...
However, $\hat{A} \otimes_A H^0(X_0, \mathcal{O}_{X_0})$ is a local ring, $\times$. $\square$.

Stein factorization. By the finiteness theorem, $B := H^0(X, \mathcal{O}_X)$ is a finite A-module, i.e., $\operatorname{Spec} B \to \operatorname{Spec} A$ is finite.
There is a factorization $X \to \operatorname{Spec} B \to \operatorname{Spec} A$.

More generally, given a proper morphism $f: X \to Y$, there is a factorization

$$X \xrightarrow{g} \overset{Z}{\overset{\shortparallel}{\underline{\mathrm{Spec}}\, f_* \mathcal{O}_X}} \xrightarrow{h} Y$$

with h finite and $g_* \mathcal{O}_X = \mathcal{O}_Z$.

Cor. The fibers of g are connected.

Proof. Localization and Cor 11.3. $\square$.

Connectedness theorem. Let $f: X \to Y$ be a proper, birational morphism. If Y is normal, all fibers of f are conn'd. of reduced schemes.

Proof. ~~Consider $(f_* \mathcal{O}_X)^{red}$. This is an ~~ ~~integral~~

The $\mathcal{O}_Y$-algebra $f_* \mathcal{O}_X$ is finite and contained in $\mathcal{K}_Y$. Since Y is normal, $\mathcal{O}_Y$ is integrally closed in $\mathcal{K}_Y \Rightarrow f_* \mathcal{O}_X = \mathcal{O}_Y$. Thus $h = Id$ & previous cor $\Rightarrow f$ has connected fibers. $\square$.

State the 5 forms of Zariski's main theorem

The consequence about functions.

Proof Assuming finiteness theorem.

<u>Issue 1</u>, $\mathcal{F} \to \varprojlim_n H^p(X_n, \mathcal{F}_n)$ is a δ-functor.

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

$$0 \to \mathcal{K}_n \to \mathcal{F} \otimes \mathcal{O}_{X_n} \to \mathcal{F}'' \otimes \mathcal{O}_{X_n} \to 0$$

$$0 \to \mathcal{G}_n \to \mathcal{F}' \otimes \mathcal{O}_{X_n} \to \mathcal{K}_n \to 0$$

Then have a l.e.s.

$$\cdots \to \varprojlim_n H^p(X_n, \mathcal{K}_n) \to \varprojlim_n H^p(X_n, \mathcal{F}_n) \to \varprojlim_n H^p(X_n, \mathcal{F}_n'') \xrightarrow{\delta} \varprojlim_n H^{p+1}(X_n, \mathcal{K}_n) \to$$

and

$$\varprojlim_n H^p(X_n, \mathcal{G}_n) \to \varprojlim_n H^p(X_n, \mathcal{F}_n') \to \varprojlim_n H^p(X_n, \mathcal{K}_n) \to \varprojlim_n H^{p+1}(X_n, \mathcal{G}_n)$$

<u>Claim</u>. $\varprojlim H^i(X_n, \mathcal{G}_n) = 0$.

This is proved using the same idea as in the <u>Artin-Rees lemma</u>: Let A be a Noetherian ring, M a finite A-module & $M' \subset M$ an A-submodule. Let $\mathfrak{a}$ be an ideal of A. There exists $c > 0$ s.t. $\forall n > c$, $(\mathfrak{a}^n M) \cap M' = \mathfrak{a}^{n-c}((\mathfrak{a}^c M) \cap M')$.

<u>ML</u>. Given inverse systems $0 \to (E_n') \to (E_n) \to (E_n'') \to 0$, $\varprojlim E_n \to \varprojlim E_n''$ is surj. if (E_n') satisfies

(ML): $\forall m$, $\text{Image}(E_n' \to E_m)$ stabilizes for $n \gg 0$.

In our case, locally on X we have

$$0 \to \underset{\Gamma(\mathcal{U}, \mathcal{F}')}{\overset{\shortparallel}{M'}} \longrightarrow \underset{\Gamma(\mathcal{U}, \mathcal{F})}{\overset{\shortparallel}{M}} \longrightarrow \underset{\Gamma(\mathcal{U}, \mathcal{F}'')}{\overset{\shortparallel}{M''}} \to 0 \quad , B = \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$$

$\mathfrak{a} = m_A B$. ...

$$0 \to K_n \to M/\mathfrak{a}^n M \to M''/\mathfrak{a}^n M'' \to 0$$

$$0 \to G_n \to M'/\mathfrak{a}^n M' \longrightarrow K_n \to 0$$

$$G_n = \frac{\mathfrak{a}^n M \cap M'}{\mathfrak{a}^n M'}, \quad \text{Image}(G_n \to G_m) = \frac{\mathfrak{a}^n M \cap M' + \mathfrak{a}^m M'}{\mathfrak{a}^m M'}$$

But $\mathfrak{a}^n M \cap M' = \mathfrak{a}^{n-c}(\mathfrak{a}^c M \cap M') \subset \mathfrak{a}^m M'$ for $n \geq m + c$. So $\text{Image}(G_n \to G_m)$ stabilizes to (0).

Thus $\text{Image}(H^p(X_n, \mathcal{G}_n) \to H^p(X_n, \mathcal{G}_m))$ stabilizes to (0)

$\Longrightarrow \varprojlim H^p(X_n, \mathcal{G}_n) = 0 \quad \forall p$.

~~Essential~~. Hartshorne. Reduce to $X = \mathbb{P}^N_A$.

Prove that $H^p(X, \mathcal{F})^\wedge \longrightarrow \varprojlim_n H^p(X_n, \mathcal{F}_n)$ is an iso. $\forall\ p \geq p_0$ by downward induction on p.

Clear for $p_0 = N+1$ by Čech complex.

Thus, assume ~~the~~ result $p_0 \leq N$ & the result is known for p_0+1.

$$0 \to \mathcal{F}' \to \bigoplus_{i=1}^{r} \mathcal{O}(a_i) \to \mathcal{F}'' \to 0.$$

$$"\mathcal{F}$$

$$\begin{array}{ccccccccc}
H^{p_0}(X,\mathcal{F}')^\wedge & \to & H^{p_0}(X,\mathcal{F})^\wedge & \to & H^{p_0}(X,\mathcal{F}'')^\wedge & \to & H^{p_0+1}(X,\mathcal{F}')^\wedge & \to & H^{p_0+1}(X,\mathcal{F})^\wedge \\
\downarrow & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr \\
\varprojlim_n H^{p_0}(\mathcal{F}'_n) & \to & \varprojlim_n H^{p_0}(\mathcal{F}_n) & \to & \varprojlim_n H^{p_0}(\mathcal{F}''_n) & \to & \varprojlim_n H^{p_0+1}(\mathcal{F}'_n) & \to & \cdots
\end{array}$$

Surj., then inj.. $\square$.