Missing step from last time. Let R be a Noetherian ring. Let $\langle a_1, \dots, a_n \rangle$ be a finitely generated ideal. For every $e \geq 1$ and $m = 1, \dots, n$, let $J_{m,e} := \langle a_1^e, \dots, a_m^e \rangle$.

Thm [Deligne] The natural R-module homomorphism $\varinjlim_e \mathrm{Hom}_R(J_{n,e}, M) \to \widetilde{M}(\mathcal{U})$ is an isomorphism, $\mathcal{U} = \mathrm{Spec}\, R \setminus Z(J)$.

Proof. Induction on n. Base case, $n = 1$, is our oft-used result, $\varinjlim_e M \cdot$"$\frac{1}{a^e}$" equals $\widetilde{M}(D(a))$. By way of induction, assume $n > 1$ and the result holds for $n-1$. Then $\mathcal{U} = V \cup D(a_n)$ where $V = \mathrm{Spec}\, R \setminus Z(J_{n-1}) = \bigcup_{1 \leq i \leq n-1} D(a_i)$. By the induction hypothesis, for every section of $\widetilde{M}(V)$, there exists $e \geq 1$ and $s : J_{n-1,e} \to M$ giving the section. By the base case, every section of $\widetilde{M}(D(a_n))$ comes from $t \in M$"$\frac{1}{a_n^d}$" for some $d \geq 1$. If the sections agree on $V \cap D(a_n)$, then $\forall k \geq k_0 \geq 1$, $t| : a_n^k \cdot (a_n^d R \cap J_{n-1,e}) \to M$ equals $s| : a_n^k \cdot (a_n^d R \cap J_{n-1,e}) \to M$. By the Artin-Rees Theorem (which Hartshorne calls "Krull's Intersection Theorem"), for $c \gg 0$, $a_n^c R \cap J_{n-1,e}$ is contained in such an ideal. Thus, s and t "glue" to define an R-module morphism on $a_n^c R + J_{n-1,e}$. So for all integers $b \geq \max(c, e)$, we have an R-module morphism $J_{n,b} \to M$ giving the section of $\widetilde{M}(\mathcal{U})$. ∎

Corollary. For every integer $r \geq 0$, $H^r(\mathcal{U}, \widetilde{M})$ equals $\varinjlim_e \mathrm{Ext}^r_R(J_{n,e}, M)$.

Cohomology of invertible sheaves on projective space. Let M be a free R-mod of rank $n > 1$. Let S be the $\mathbb{Z}_{\geq 0}$-graded R-algebra $S = \mathrm{Sym}^\bullet_R(M)$. Recall diagram

$$\begin{tikzcd} \mathbb{E}_R(M)^* \arrow[r, hook, "e"] \arrow[d, "\pi"] & \mathbb{E}_R(M) \\ \mathbb{P}_R(M) \arrow[u, bend left, "\sigma_M"] & \end{tikzcd}$$

where $q : M \otimes_R \mathcal{O}_{\mathbb{P}_R(M)} \to \mathcal{O}_{\mathbb{P}(M)}(1)$ is the universal invertible quotient whose pullback by π agrees with e^* of the universal morphism of coherent sheaves, $p : M \otimes_R \mathcal{O}_{\mathbb{E}(M)} \to \mathcal{O}_{\mathbb{E}(M)}$.

The $\mathbb{Z}$-graded $\mathcal{O}_{\mathbb{P}(M)}$-algebra $\pi_* \mathcal{O}_{\mathbb{E}(M)}$ equals $\bigoplus_{\ell \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}(M)}(\ell)$. Thus, $H^r(\mathbb{P}_R(M), \oplus \mathcal{O}(\ell))$ equals $\varinjlim \mathrm{Ext}^r_R(J_e, S)$, where $J = S_+$ is the irrelevant ideal. By facts about Koszul complexes, $\mathrm{Ext}^r = 0$ except for $r = 0$ and $r = n$.

Theorem, Part I. $H^0(\mathbb{P}_R(M), \mathcal{O}(\ell))$ equals $\mathrm{Sym}^{\ell}_R(M)$ for $\ell \geq 0$, and 0 otherwise. $H^r(\mathbb{P}_R(M), \mathcal{O}(\ell))$ is 0 for all $\ell \in \mathbb{Z}$ if $0 < r < \mathrm{rank}(M)$.

Finally, choosing a free R-basis, $M = R\cdot a_1 \oplus \cdots \oplus R\cdot a_n$, then the $\mathbb{Z}$-graded module $\mathrm{Ext}^n_S(J_e, S)$ equals $(S/J_e)\cdot \frac{``da_1\wedge\cdots\wedge da_n"}{a_1^e\cdots a_n^e}$. Thus, the graded R-module in degree $-n$ is free, generated by "$\frac{da_1\wedge\cdots\wedge da_n}{a_1\cdots a_n}$", and for every $\ell \leq -n$, the multiplication map

$$\mathrm{Sym}^{-\ell-n}_R(M) \times H^{n-1}(\mathbb{P}_R(M), \mathcal{O}(\ell)) \to H^{n-1}(\mathbb{P}_R(M), \mathcal{O}(-n)) \cong \Lambda^n_R(M)$$

is a perfect pairing, ~~i.e., $H^{n-1}(\mathbb{P}$~~

Theorem, Part II. $H^{n-1}(\mathbb{P}_R(M), \mathcal{O}(\ell))$ equals $\mathrm{Hom}_R(\mathrm{Sym}^{-\ell-n}_R(M), H^{n-1}(\mathbb{P}_R(M), \mathcal{O}(-n)))$ for all $\ell \leq -n$, and 0 otherwise.

New notation: $R \to A$, $M \to P$, $n \to \mathrm{rank}_A(P) = r$, $\ell \to d$

0. Recall last time: $\begin{cases} H^0(\mathbb{P}^{r-1}, \mathcal{O}(d)) \cong \mathrm{Sym}^d P \\ H^{r-1}(\mathbb{P}^{r-1}, \mathcal{O}(d)) \cong [\mathrm{Sym}^{-d-r} P, (\Lambda^r P)^\vee] \\ H^i(\mathbb{P}^{r-1}, \mathcal{O}(d)) = 0 \quad \text{otherwise} \end{cases}$

<u>Reformulation</u>. $\mathbb{P} \xrightarrow{\pi} \mathrm{Spec}\, A$, $\pi^* \tilde{P} \to \mathcal{O}(1) \rightsquigarrow$
$\phi : \pi^* \tilde{P} \otimes \mathcal{O}(-1) \longrightarrow \mathcal{O}$. Now form Koszul complex of ϕ,

$$0 \leftarrow \mathcal{O}_{\mathbb{P}} \xleftarrow{d_0} \pi^* \tilde{P} \otimes \mathcal{O}(-1) \xleftarrow{d_1} \pi^* \Lambda^2 \tilde{P} \otimes \mathcal{O}(-2) \leftarrow \cdots \leftarrow \boxed{\pi^* \Lambda^r \tilde{P} \otimes \mathcal{O}(-r)}$$

Ω $\qquad$ Ω^2

$0 \qquad 1 \qquad 2 \qquad r$

dualizing sheaf ω_π $= \Omega^{r-1}$

<u>Notation</u>:
$\Omega^k = \mathrm{Ker}(d_{k-1}) = \mathrm{Image}(d_k)$. Then $\Omega^{r-1} = \omega_\pi$.

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) \xrightarrow{\delta} H^1(\mathbb{P}, \Omega^1) \xrightarrow{\delta} H^2(\mathbb{P}, \Omega^2) \to \cdots \to H^{r-1}(\mathbb{P}, \Omega^{r-1}) = H^{r-1}(\mathbb{P}, \omega_\pi).$$

So this defines (1) $A \to H^{r-1}(\mathbb{P}, \omega_\pi)$.
(2) There is a pairing $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \otimes_A H^{r-1}(\mathbb{P}, \omega_\pi \otimes \mathcal{O}_{\mathbb{P}}(-d)) \to H$

<u>Thm reformulated</u>. (i) $A \to H^{r-1}(\mathbb{P}, \omega_\pi)$ is an isomorphism. Its inverse is called the <u>trace map</u> $\mathrm{Tr} : H^{r-1}(\mathbb{P}, \omega_\pi) \to A$.
(ii) The induced map $\mathrm{Sym}^d P \to H^0(\mathbb{P}, \mathcal{O}(d))$ is an isom $\forall d$.

(iii) $\forall d$, the pairing below is perfect

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \otimes_A H^{r-1}(\mathbb{P}, \omega_\pi \otimes \mathcal{O}_{\mathbb{P}}(-d)) \longrightarrow H^{r-1}(\mathbb{P}, \omega_\pi) \xrightarrow{\mathrm{Tr}} A.$$

i.e., $H^{r-1}(\mathbb{P}, \omega_\pi \otimes \mathcal{O}_{\mathbb{P}}(-d)) \longrightarrow \mathrm{Hom}_A(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)), A)$

is an isom. of A-modules.

$\rightsquigarrow$ $\Lambda^r P \otimes H^{r-1}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-r-d)) \longrightarrow \mathrm{Hom}_A(\mathrm{Sym}^d P, A)$ is an iso.

(iv) $H^i(\mathbb{P}, \rightsquigarrow) = 0$ for $i \neq 0, r-1$.

Assum A Noeth

<u>Consequence</u>. Let $\mathcal{F}$ be a ~~locally f. presented~~ coh't. shef on $\mathbb{P}$. Then

(i) $\forall i \geq r$, $H^i(\mathbb{P}, \mathcal{F}) = 0$.

(ii) $\forall i$, $H^i(\mathbb{P}, \mathcal{F})$ is a f. gen'd. A-module

(iii) $\exists n_0$ depending only on $\mathcal{F}$ s.t. $\forall n \geq n_0$, $H^{i>0}(\mathbb{P}, \mathcal{F}(n)) = 0$.

(iv) $\bigoplus_{n \geq 0} H^0(\mathbb{P}, \mathcal{F}(n))$ is a f. gen'd. S-module.

<u>Pf</u>: (i) Čech covering.

(ii) & (iii) Downward induction on i with $i = r$ being the base case. Fix $\mathcal{O}_{\mathbb{P}}(-d)^{\oplus N} \twoheadrightarrow \mathcal{F}$ $\rightsquigarrow$

$$0 \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}}(-d)^{\oplus N} \to \mathcal{F} \to 0.$$

$$H^i(\mathcal{O}_{\mathbb{P}}(n-d))^{\oplus N} \to H^i(\mathcal{F}(n)) \to H^{i+1}(\mathcal{E}(n))$$

f.gen'd by cor, $=0$ for $n \geq d-r$ & $i>0$. f.gen'd, resp 0 by ind. hyp

(iv) Suffices to prove $\bigoplus_{n\geq n_0} H^0(\mathbb{P}, \mathcal{F}(n))$ is f.g.

$0 \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}}(-d)^{\oplus N} \to \mathcal{F} \to 0.$

For $n \geq n_0$, $H^1(\mathcal{E}(n)) = 0$ so

$\bigoplus_{n\geq n_0} H^0(\mathbb{P}, \mathcal{F}(n))$ is a qt. of $\bigoplus_{n\geq n_0} H^0(\mathbb{P}, \mathcal{O}(n-d))^{\oplus N}$

$= M^{\oplus N}$ where $M = \{(S[d])_{\geq n_0}\}$ which is an ~~f.g.~~ S-submodule of $S[d]$ ... $\square$.

<u>Prop. 5.3</u>. A Noeth. X a proper A-schd. $\mathcal{L}$ an inv. shef on X. TFAE

(i) $\mathcal{L}$ is ample

(ii) $\forall$ coht. $\mathcal{F}$ $\exists\, n_0 = n_0(\mathcal{F})$ s.t. $\forall n \geq n_0$,

$H^i(X, \mathcal{F}\otimes\mathcal{L}^n) = 0$.

<u>Pf</u>: (i) $\Rightarrow$ (ii). $\mathcal{L}^{\otimes d}$ is v. ample $\mathcal{O}(1)$.

Now for $\mathcal{G} = \mathcal{F}\oplus(\mathcal{F}\otimes\mathcal{L})\oplus\cdots\oplus(\mathcal{F}\otimes\mathcal{L}^{d-1})$, $\exists$ m_0 s.t. $\forall m \geq m_0$, $H^{i>0}(\mathcal{G}(m)) = 0$

i.e. $\bigoplus_{t=0}^{d-1} H^{i>0}(\mathcal{F}(md+t)) = 0 \Rightarrow \forall n \geq m_0 d$,

$H^{i>0}(\mathcal{F}(n)) = 0$.

(ii) $\Rightarrow$ (i) $0 \to \mathcal{I}_p \cdot \mathcal{F} \to \mathcal{F} \to \mathcal{F}\otimes k(p) \to 0$. at p

Definitions of $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, -)$ & $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, -)$.

<u>Observation</u>. $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = H^i(X, \mathcal{G})$.

$\mathrm{Ext}^i_{\mathcal{O}_X}(j_! \mathcal{O}_U, \mathcal{G}) = H^i(U, \mathcal{G}|_U)$.

<u>Lemma</u>. $j^{-1} \mathcal{I}$ is injective if $\mathcal{I}$ is inj.

<u>Conseq</u>. $j^{-1} \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathcal{E}xt^i_{\mathcal{O}_U}(j^{-1}\mathcal{F}, j^{-1}\mathcal{G})$.

⌈<u>Reminder</u> typically $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is not q-coh; but it is if $\mathcal{F}$ is loc. f. presd.⌋

$\mathcal{E}xt^{i>0}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = 0$.

<u>Prop. 6.4</u>. Given $\mathcal{E}: 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, have a $\overset{\text{contra}}{\delta}$- functor $\mathcal{E}xt^i_{\mathcal{O}_X}(-, \mathcal{G})$, $\mathrm{Ext}^i_{\mathcal{O}_X}(-, \mathcal{G})$.

Pf: $\mathcal{H}om(-, \mathcal{I})$ is exact for $\mathcal{I}$ injective.

...

$\square$.

Compatibility, bifunctor a diagram.

<u>Prop. 6.5</u>. Compute $\mathcal{E}xt^i_{\mathcal{O}_X}(-, \mathcal{G})$ using locally free sheaves

Lem: inj. $\otimes$ locally free is inj.

Prop. 6.7 $\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{E}^\vee \otimes \mathcal{G})$.

Prop. 6.8 X ~~Noeth,~~ $\mathcal{F}$ coh't, $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x = \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$.

Pf: Use free resolutions.

Prop. 6.9. X proj. over Noeth. A. $\mathcal{F}, \mathcal{G}$ coh't, $\exists\, n_0$ s.t. $\forall n >> n_0$

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n)) \xrightarrow{\sim} \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))).$$

Pf: True for $\mathcal{F} = \mathcal{O}_X$, true for $\mathcal{F}$ = locally free by Prop 6.7.

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

$$0 \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}(n)) \to \mathrm{Hom}(\mathcal{E}, \mathcal{G}(n)) \to \mathrm{Hom}(\mathcal{K}, \mathcal{G}(n)) \to$$
$$\to \mathrm{Ext}^1(\mathcal{F}, \mathcal{G}(n)) \to 0 \quad (n >> 0)$$

$$\mathrm{Ext}^i(\mathcal{K}, \mathcal{G}(n)) \xrightarrow{\sim} \mathrm{Ext}^{i+1}(\mathcal{F}, \mathcal{G}(n)) \quad \text{etc.} \; \square$$