

Recap. For a scheme  $X$  with  $\mathcal{K}_X =$  sheaf of  $\mathcal{O}_X$ -algebras associated to presheaf  $S(\mathcal{U})^{-1} \cdot \mathcal{O}_X(\mathcal{U})$ ,  $S(\mathcal{U}) := \{s \in \mathcal{O}_X(\mathcal{U}) \mid s: \mathcal{O}_U \rightarrow \mathcal{O}_U \text{ injective}\}$ ,  $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$  is the group of invertible  $\mathcal{O}_X$ -submodules of  $\mathcal{K}_X$ , Cartier divisors. If  $X$  is integral and normal, then the induced map of sheaves,  $(\nu_0)_0: \mathcal{K}_X^*/\mathcal{O}_X^* \rightarrow \bigoplus_{D \text{ prime}} \mathbb{Z} \cdot "D"$ , is injective, where the direct sum ranges over all irreducible closed subsets  $D \subseteq X$  whose prime ideal has height one, i.e., whose stalk  $\mathcal{O}_{X, \nu_0}$  is a DVR with valuation  $\nu_0$  (with convention that  $\nu_0^{-1}(\{1\})$  equals  $\mathfrak{m}_{X, \nu_0} - \mathfrak{m}_{X, \nu_0}^2$ ). The group of global sections of this sheaf is the group of Weil divisors, and the induced map of global sections is the natural map from the group of Cartier divisors to the group of Weil divisors.

Fact. A Cartier divisor is effective if and only if each  $\nu_0$  is  $\geq 0$ .

Proof. This is the commutative algebra result that an integrally closed integral domain equals the intersection of all localizations at height-one prime ideals.  $\square$

Algebra Fact. A Noetherian integral domain  $R$  is a UFD if & only if the Weil divisor class group is zero.

Corollary. For a Noetherian UFD  $R$ , the Weil divisor class group of  ~~$\mathbb{P}^n$~~   $\mathbb{P}_R^n = \text{Proj } R[t_0, \dots, t_n]$  ( $n \geq 0$ ) is  $\mathbb{Z}$ .

Proof. By Gauss,  $R[t_0, \dots, t_n]$  is a UFD. Thus, every homogeneous prime ideal  $P_D$  of height one is principal with a homogeneous generator,  $P_D = \langle F \rangle$ ,  $\deg(F) = e$ . The isomorphism of graded modules,  $R[t_0, \dots, t_n][e] \xrightarrow{F} P_D$ , shows that the class of  $D$  equals  $\frac{\deg(F)}{e}$ .

Proposition. For  $X$  integral, normal, for  $C \subset X$  closed with open complement  $U$ , there is a diagram of exact sequences

$$\begin{array}{ccccccc}
 0 \rightarrow \bigoplus_{C_i \subset C, \text{codim } 1} \mathbb{Z} \cdot [C_i] & \rightarrow & \text{Weil}(X) & \hookrightarrow & \text{Weil}(U) & \rightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \bigoplus_{C_i, \text{codim } 1} \mathbb{Z} \cdot [C_i] & \rightarrow & \text{Cl}(X) & \rightarrow & \text{Cl}(U) & \rightarrow & 0
 \end{array}$$

Proof. The splitting map  $\text{Weil}(U) \rightarrow \text{Weil}(X)$  is given by closure.  $\square$

Proposition. For  $X$  integral, normal,  $\text{Cl}(X) \rightarrow \text{Cl}(\mathbb{A}_X^n)$  is an isom.

Proof. The restriction of each Weil divisor on  $\mathbb{A}_X^n$  to the generic fiber  $\mathbb{A}_{\text{Frac}(X)}^n$  is principal. After modifying by this principal divisor, the prime divisors in the support are disjoint from the generic fiber, i.e., they are inverse images of prime Weil divisors in  $X$ .  $\square$

Definition. For a quasi-separated, finite type morphism  $\pi: X \rightarrow B$ , an invertible  $\mathcal{O}_X$ -module  $\mathcal{L}$  is  $\pi$ -relatively very ample if there exists a dense open immersion of  $B$ -schemes,  $U \hookrightarrow \mathbb{P}_B^r$

$$U: X \hookrightarrow \text{Proj}_B S$$

with  $\mathcal{L} \cong \mathcal{L}^*(\mathcal{O}(1))$ .

Fact. If  $B$  affine, then  $\mathcal{L}$  is very ample  $\Leftrightarrow \exists$  locally closed immersion  $f: X \hookrightarrow \mathbb{P}_B^r$  pulling back  $\mathcal{O}(1)$  to  $\mathcal{L}$ . Also  $f: X \rightarrow \mathbb{P}_B^r$  is a locally closed immersion of  $B$ -schemes  $\Leftrightarrow$  true for each geometric fiber over  $B$ .

Criterion. For a field  $k$ , a morphism of finite type  $k$ -schemes  $f: X \rightarrow \mathbb{P}_k^r$  is a locally closed immersion  $\Leftrightarrow$  both

- (i)  $\mathcal{O}_X^{\oplus (r+1)} \rightarrow f^*(\mathcal{O}(1))$  separates points, and
- (ii) each  $x(p) \otimes_{\mathcal{O}_{X,p}} \rightarrow f^*(\mathcal{O}(1))_p / \mathfrak{m}_{X,p}^2 \cdot f^*(\mathcal{O}(1))_p$  is surjective ("separates tangent vectors").

I. Let  $X \rightarrow \text{Spec } A$  be a finite type,  $q$ -sepd. morphism. Let  $\mathcal{L}$  be an invertible sheaf on  $X$ .  
Theorem, Part I. TFAE.

(1)  $\Rightarrow$  (2),  
 f. type &  $q$ -sepd.

(2)  $\Rightarrow$  (3),  
 $q$ -cpt &  $q$ -sepd.

$\Downarrow$

(3),

(3), f. type &  $q$ -sepd.

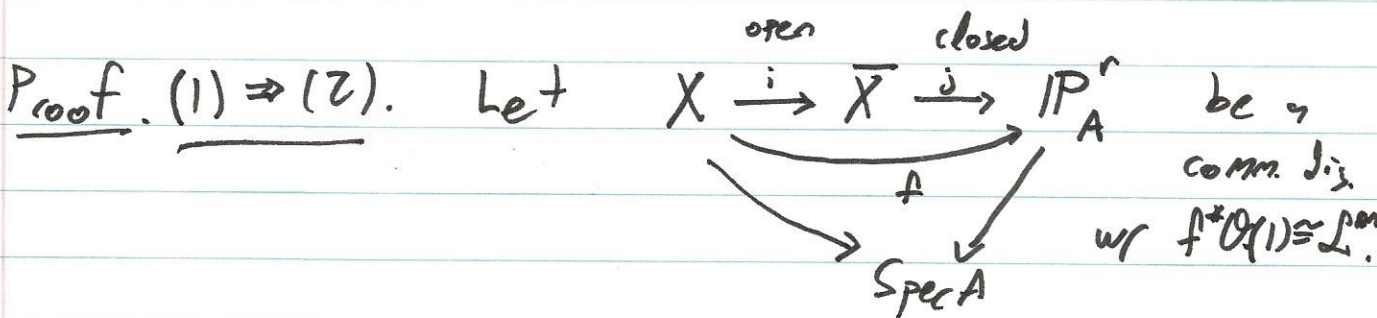
$\Downarrow$

(1)

(1)  $\exists n > 0$  s.t.  $\mathcal{L}^{\otimes n}$  is very ample.

(2)  $\exists n > 0$  and sections  $s_0, \dots, s_r \in \Gamma(X, \mathcal{L}^{\otimes n})$  s.t. each  $D(s_i)$  is quasi-affine.

(3)  $\exists n > 0$  and sections  $s_0, \dots, s_r \in \Gamma(X, \mathcal{L}^{\otimes n})$  s.t. each  $D(s_i)$  is affine.



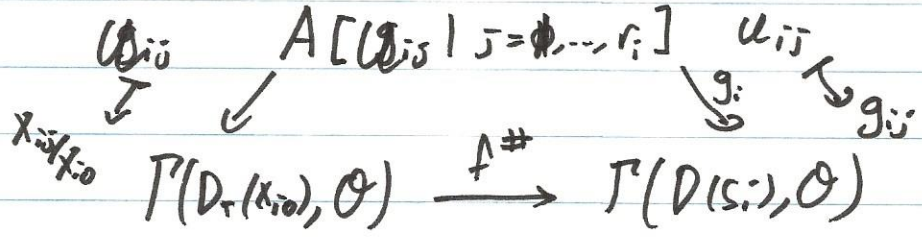
Let  $s_i = f^* x_i$ . Then  $D(s_i) = f^{-1}(D_+(x_i)) = i^{-1}(j^{-1}(D_+(x_i)))$ .  
 Because  $j$  is a closed im. &  $D_+(x_i)$  is affine,  $j^{-1}(D_+(x_i))$  is affine. Because  $i$  is open,  $i^{-1}(j^{-1}(D_+(x_i)))$  is quasi-affine.

(2)  $\Rightarrow$  (3). For every  $p \in X$ ,  $\exists i = \overset{i_p}{\text{any}}$  and  $f \in f_p \in \Gamma(D(s_i), \mathcal{O})$  s.t.  $p \in D(f)$  is an open affine nbhd. Because  $X$  is  $q$ -cpt, suffices to consider finitely many  $f_s$ . By basic prop,  $\exists N$  s.t. each  $s_i^N f_s$  extends to a section  $t_s$  of  $\mathcal{L}^{\otimes (nN)}$ . Form  $s_i \cdot s_i^N f_s$  for site measure. Then  $D(s_i \cdot s_i^N f_s) = D(s_i) \cap D(s_i^N f_s) = D(t_s) \subset D(s_i)$  is affine.

(3)  $\Rightarrow$  (1). Each  $D(s_i) \rightarrow \text{Spec } A$  is finite type.

So  $\exists$  finitely many generators  $\{g_{i0}, g_{i1}, \dots, g_{ir_i}\}$  for  $\Gamma(D_+(s_i), \mathcal{O})$  as an  $A$ -algebra. By basic prop.,  $\exists N$  s.t. each  $t_{ij} = s_i^N g_{ij}$  lifts to a section of  $\mathcal{L}^{\otimes(N)}$ . The sections  $t_{i0}$  generate  $\mathcal{L}^{\otimes(N)}$ . Thus  $\exists$  a unique morphism  $f: X \rightarrow \mathbb{P}_A^{r_1 + \dots + r_n - 1}$  s.t.  $f^* \mathcal{O}(1) = \mathcal{L}^{\otimes(N)}$ ,  $f^* x_{ij} = t_{ij}$ .

Thus  $f^{-1}(D_+(x_{i0})) = D(t_{i0}) = D(s_i)$ . Also the pullback map  $f_{D_+(x_{i0})}^\# : \Gamma(D_+(x_{i0}), \mathcal{O}) \rightarrow \Gamma(D(s_i), \mathcal{O})$  has factors



Since  $g_i$  is surj, so is  $f_{D_+(x_{i0})}^\#$ . Thus  $f: f^{-1}(D_+(x_{i0})) \rightarrow D_+(x_{i0})$  is a closed imm. Therefore  $f: X \rightarrow U := \bigcup D_+(x_{i0}) \subset \mathbb{P}_A$  is a locally closed imm.  $\square$

Theorem, Part II. TFAE (X Noeth.)

- (3)  $\exists n > 0$  and sections  $s_0, \dots, s_r \in \Gamma(X, \mathcal{L}^{\otimes n})$  s.t. each  $D(s_i)$  is affine
- (4)  $\mathcal{L}$  is ample.

Proof. (3)  $\Rightarrow$  (4). Same argument as in Serre's thm.  
 (4)  $\Rightarrow$  (3). For every point  $p \in X$ ,  $\exists$  open affine  $p \in U \subset X$ . Let  $Z = X - U$ . There exists  $N > 0$  &  $s \in \Gamma_Z \otimes \mathcal{L}^{\otimes N}$  s.t.  $s(p) = 0$ . (considered as a section of  $\mathcal{L}^{\otimes N}$ ,  $p \in D(s) \subset U$ , affine)

(3), (4)  $\Leftrightarrow$  (5)  $\Leftrightarrow$  (4) & Noeth.  $\Leftrightarrow$  (3)  
 Suffices to assume  $X \rightarrow \mathbb{P}_A^1$  is f. prod & q-sepd.

Definitions of relatively ample & relatively very ample.  $f: X \rightarrow Y$ .  $\mathcal{L}$  on  $X$ .

~~$f$  flat &  $g$ -sepd,  $Y$   $g$ -cpt,  $\mathcal{L}$   $f$ -relatively very ample~~  
 $\Rightarrow \mathcal{L}$   $f$ -relatively ample.

for  $f$  flat, presd &  $g$ -sepd,  $Y$   $g$ -cpt,  $\mathcal{L}$   $f$ -rel. ample  
 $\Rightarrow \exists N > 0$  s.t.  $\mathcal{L}^{\otimes N}$  is relatively very ample.

Relative Proj.  $\mathbb{P}^n$ . Blowing up  $A = \bigoplus_{n \geq 0} I^n$ .

$\nu^{-1} I \cdot \mathcal{O}_X = \mathcal{O}_X(1)$ . "Universal property": Denote  $g: Y \rightarrow X$   
 $(g^* I)_{\text{pure}} := \text{Image}(g^* I \rightarrow \text{Hom}(\text{Hom}(g^* I, \mathcal{O}_Y), \mathcal{O}_Y))$ .

If  $(g^* I)_{\text{pure}} \rightarrow \mathcal{O}_Y$  is injective, then  $g$  factors through  $\nu$  if and only if  $g^* I \cdot \mathcal{O}_Y$  is invertible, in which case the factorization is unique.

Local description.  $X = \text{Spec } A$ ,  $I = \langle f_0, \dots, f_r \rangle$ . Form  $\mathbb{P}^r_A$  and  $V = V(x_i f_j - x_j f_i)_{0 \leq i, j \leq r}$ . If  $X$  is reduced, then  $\tilde{X} = \text{closure of } V \cap \pi^{-1}(\text{Spec } A - V(I))$ .  
 In general  $\tilde{X} = V(\underbrace{P(x_i)}_{\text{homog}} \mid P(f_i) = 0 \text{ in } A)$ .

Thm 7.17.  $f: X \rightarrow Y$  biratl & proj,  $Y$   $g$ -proj, then  $f$  is isomorphic to  $\text{Bl}_I Y$  for some (non-unique)  $I$ .

II. Differentials.

Definition. Let  $f: X \rightarrow Y$  be a morphism. Let  $\Delta_{X/Y}: X \rightarrow X \times_Y X$  be the diagonal. It is a locally closed imm. Let  $\mathcal{I}$  denote the ideal sheaf. Then  $\Omega_{X/Y} := \Delta^* \mathcal{I} \cong \mathcal{I}/\mathcal{I}^2$ .  
 Denote by  $d: \mathcal{O}_X \rightarrow \Omega_{X/Y}$  the m.p.  $a \mapsto p_1^* a - p_2^* a$ .

Then  $d$  is an  $f^{-1}\mathcal{O}_Y$ -derivation. Moreover, it is the universal  $f^{-1}\mathcal{O}_Y$ -derivation.

Local.  $B \subset A$ .  $\Omega_{B/A} = I/I^2$ ,  $I := \text{Ker}(B \otimes_A B \rightarrow B)$

$d: B \rightarrow \Omega_{B/A}$ ,  $b \mapsto b \otimes 1 - 1 \otimes b$ .

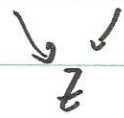
$\partial: B \rightarrow M$ ,  $\partial(b \otimes 1 - 1 \otimes b) := \partial b$ .  $\phi: B \otimes_A B \rightarrow M$

$\phi(b, \otimes b_2) = b, \partial(b_2) \dots$

Fundamental exact sequences. (1)  $X \xrightarrow{f} Y \xrightarrow{g} Z$ ,

$f^*\Omega_{Y/Z} \rightarrow \Omega_{X/Z} \rightarrow \Omega_{X/Y} \rightarrow 0$

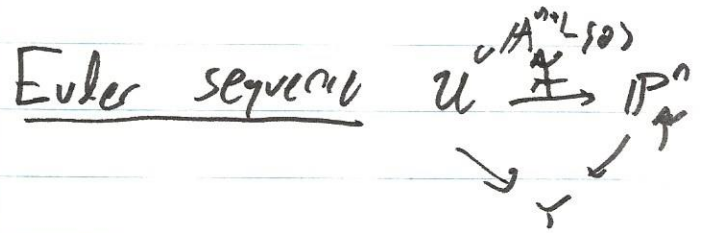
(2)  $X \xrightarrow{i} Y$ ,  $\mathcal{I} = \text{Ideal sheaf of } i(X)$



$\mathcal{I}/\mathcal{I}^2 \rightarrow i^*\Omega_{Y/Z} \rightarrow \Omega_{X/Z}$ .

Example.  $A_Y^n \rightarrow Y$ .  $\Omega_{A_Y^n/Y} \cong \mathcal{O}_{A_Y^n} \{dx_1, \dots, dx_n\}$ .

$X \hookrightarrow A_Y^n$ ,  $X = V(f_1, \dots, f_r)$ .  $\Omega_{X/Y} \cong \mathcal{O}_X \{dx_1, \dots, dx_n\} / \langle df_1, \dots, df_r \rangle$ .



$0 \rightarrow \pi^*\Omega_{P^n/Y} \rightarrow \Omega_{U/Y} \rightarrow \Omega_{U/P^n} \rightarrow 0$

$\mathcal{O}_U \{dx_1, \dots, dx_n\} \rightarrow \mathcal{O}_U$   
 $dx_i \mapsto x_i$

$\rightsquigarrow 0 \rightarrow \Omega_{P^n/Y} \rightarrow \mathcal{O}_{P^n}(-1)^{\oplus n+1} \rightarrow \mathcal{O}_{P^n} \rightarrow 0$

unramified also

Def. The morphism  $f: X \rightarrow Y$  is smooth if

- (1)  $f$  is flat of rel. dim  $d$ .
- (2)  $f$  is locally finitely presented
- (3)  $\Omega_{X/Y}$  is locally free of  $\text{rank} = d$ .

Jacobian Criterion

Ex. 1.11.  $Y = \text{Spec } k$ ,  $k = \bar{k}$ . Then  $X$  is smooth iff  $X$  is locally f. type &  $\forall$  closed point  $x$  of  $X$ ,  $\mathcal{O}_{x,x}$  is a regular local ring.

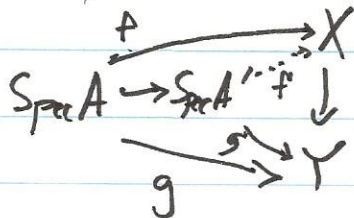
Non-example.  $\text{char}(k) = p$ ,  $\text{Spec } k \xrightarrow{F} \text{Spec } k$ ,  
 $F^\#: k \rightarrow k$   
 $a \mapsto a^p$ . If  $k$  is not a finite field, then  
 $k = \mathbb{F}_p(t)$ ,  $K = k[u]/u^p - t$ .  $\text{Spec } K \rightarrow \text{Spec } k$   
 is not smooth.  $K \otimes_k K \cong k[u_1, u_2]/\langle u_1^p - t, u_2^p - t \rangle$

$u_1 - u_2$  generates  $\Omega_{K/k}$ ,  $\Omega_{K/k} \cong K \langle du \rangle$

Bertini's theorem.

formally smooth:  $f'$  exists

formally unramified:  $f'$  is unique if it exists



formally étale:  $f'$  exists and is unique.