<u>Recap</u>. For a scheme X with $\mathcal{K}_X$ = sheaf of $\mathcal{O}_X$-algebras associated to presheaf $S(\mathcal{U})^{-1}\cdot\mathcal{O}_X(\mathcal{U})$, $S(\mathcal{U}) := \{s\in\mathcal{O}_X(\mathcal{U}) \mid s:\mathcal{O}_{\mathcal{U}}\to\mathcal{O}_{\mathcal{U}} \text{ injective}\}$, $\Gamma(X,\mathcal{K}_X^*/\mathcal{O}_X^*)$ is the group of invertible $\mathcal{O}_X$-submodules of $\mathcal{K}_X$, <u>Cartier divisors</u>. If X is integral and normal, then the induced map of sheaves, $(v_D)_D : \mathcal{K}_X^*/\mathcal{O}_X^* \to \bigoplus_{D \text{ prime}} \iota_{D*}(\mathbb{Z})\cdot\text{"}\underline{D}\text{"}$, is injective, where the direct sum ranges over all irreducible closed subsets $D\subseteq X$ whose prime ideal has height one, i.e, whose stalk $\mathcal{O}_{X,\eta_D}$ is a DVR with valuation v_D (with convention that $v_D^{-1}(\{1\})$ equals $\mathfrak{m}_{X,\eta_D}\setminus\mathfrak{m}_{X,\eta_D}^2$). The group of global sections of this sheaf is the group of <u>Weil divisors</u>, and the induced map of global sections is the natural map from the group of Cartier divisors to the group of Weil divisors.

<u>Fact</u>. A Cartier divisor is effective if and only if each v_D is ≥ 0.
<u>Proof</u>. This is the commutative algebra result that an integrally closed integral domain equals the intersection of all localizations at height-one prime ideals. $\square$

<u>Algebra Fact</u>. A Noetherian integral domain R is a UFD if & only if the Weil divisor class group is zero.

<u>Corollary</u>. For a Noetherian UFD R, the Weil divisor class group of ~~Pick~~ $\mathbb{P}^n_R = \text{Proj } R[t_0,\dots,t_n]$ $(n\geq 0)$ is $\mathbb{Z}$.
<u>Proof</u>. By Gauss, $R[t_0,\dots,t_n]$ is a UFD. Thus, every homogeneous prime ideal P_D of height one is principal with a homogeneous generator, $P_D = \langle F\rangle$, $\deg(F)=e$. The isomorphism of graded modules, $R[t_0,\dots,t_n][e] \xrightarrow{F\cdot} P_D$, shows that the class of $\underline{D}$ equals $\mathcal{O}(e)$. $\square$

Proposition. For X integral, normal, for $C \subsetneq X$ closed with open complement $\mathcal{U}$, there is a diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{C_i \subseteq C,\, \mathrm{codim}\, 1} \mathbb{Z}\cdot\text{"}\underline{C_i}\text{"} \arrow[r] \arrow[d] & \mathrm{Weil}(X) \arrow[r] \arrow[d] & \mathrm{Weil}(\mathcal{U}) \arrow[r] \arrow[d] \arrow[l, bend right, dashed] & 0 \\ & \bigoplus_{C_i,\, \mathrm{cod}\, 1} \mathbb{Z}\cdot\text{"}[C_i]\text{"} \arrow[r] & \mathrm{Cl}(X) \arrow[r] & \mathrm{Cl}(\mathcal{U}) \arrow[r] & 0 \end{tikzcd}$$

Proof. The splitting map $\mathrm{Weil}(\mathcal{U}) \to \mathrm{Weil}(X)$ is given by closure. □

Proposition. For X integral, normal, $\mathrm{Cl}(X) \to \mathrm{Cl}(\mathbb{A}^n_X)$ is an isomor.

Proof. The restriction of each Weil divisor on $\mathbb{A}^n_X$ to the generic fiber $\mathbb{A}^n_{\mathrm{Frac}(X)}$ is principal. After modifying by this principal divisor, the prime divisors in the support are disjoint from the generic fiber, i.e., they are inverse images of prime Weil divisors in X. □

Definition. For a quasi-separated, finite type morphism $\pi : X \to B$, an invertible ~~sh~~ $\mathcal{O}_X$-module $\mathcal{L}$ is <u>π-relatively very ample</u> if there exists a dense open immersion of B-schemes, ~~$\iota: B \to P$~~

$$\iota : X \hookrightarrow \underline{\mathrm{Proj}}_B\, S$$

with $\mathcal{L} \cong \iota^* \mathcal{O}(1)$.

Fact. If B affine, then $\mathcal{L}$ is very ample $\iff \exists$ locally closed immersion of B-sch $X \hookrightarrow \mathbb{P}^n_B$ pulling back $\mathcal{O}(1)$ to $\mathcal{L}$. Also $f : X \to \mathbb{P}^n_B$ is a locally closed immersion of B-schemes $\iff$ true for each geometric fiber over B.

Criterion. For a field k, a morphism of finite type k-schemes $f : X \to \mathbb{P}^n_k$ is a locally closed immersion $\iff$ both

(i) $\mathcal{O}_X^{\oplus(n+1)} \to f^*\mathcal{O}(1)$ separates points, and

(ii) each $\kappa(p)^{\oplus(n+1)} \to f^*\mathcal{O}(1)_p / \mathfrak{m}_{X,p}^2 \cdot f^*\mathcal{O}(1)_p$ is surjective ("separates tangent vectors").

$(1) \Rightarrow (2)$, f. type & q-sepd, (2) &, q-cpct & q-sepd, $\Downarrow$ (3), (3), f. type & q-sepd $\Downarrow$ (1)

I. Let $X \to \operatorname{Spec} A$ be a finite type, q-sepd. morphism. Let $\mathcal{L}$ be an invertible sheaf on X.

<u>Theorem, Part I</u>. TFAE.

(1) $\exists n>0$ s.t. $\mathcal{L}^{\otimes n}$ is very ample.

(2) $\exists n>0$ and sections $s_0, \dots, s_r \in \Gamma(X, \mathcal{L}^{\otimes n})$ s.t. each $D(s_i)$ is quasi-affine.

(3) $\exists n>0$ and sections $s_0, \dots, s_r \in \Gamma(X, \mathcal{L}^{\otimes n})$ s.t. each $D(s_i)$ is affine.

<u>Proof</u>. <u>$(1) \Rightarrow (2)$</u>. Let

$$\begin{tikzcd} X \arrow[r, "i", "\text{open}"'] \arrow[rr, bend right, "f"'] \arrow[dr] & \overline{X} \arrow[r, "j", "\text{closed}"'] & \mathbb{P}^r_A \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$

be a comm. dig. w/ $f^*\mathcal{O}(1) \cong \mathcal{L}^{\otimes n}$.

Let $s_i = f^* x_i$. Then $D(s_i) = f^{-1}(D_+(x_i)) = i^{-1}(j^{-1}(D_+(x_i)))$. Because j is a closed imm. & $D_+(x_i)$ is affine, $j^{-1}(D_+(x_i))$ is affine. Because i is open, $i^{-1}(j^{-1}(D_+(x_i)))$ is quasi-affine.

<u>$(2) \Rightarrow (3)$</u>. For every $p \in X$, $\exists i = i_p$ and $f = f_p \in \Gamma(D(s_i), \mathcal{O})$ s.t. $p \in D(f)$ is an open affine nbhd. Because X is q-cpct, suffices to consider finitely many f_j. By basic prop, $\exists N$ s.t. each $s_i^N f_j$ extends to a section t_j of $\mathcal{L}^{\otimes(nN)}$. Form $s_i \cdot s_i^N f_j$ for safe measure. Then $D(s_i \cdot s_i^N f_j) = D(s_i) \cap D(s_i^N f_j) = D(f_j) \subset D(s_i)$ is affine.

<u>$(3) \Rightarrow (1)$</u>. Each $D(s_i) \to \operatorname{Spec} A$ is finite type.

So $\exists$ finitely many generators $\{=g_{i0}\}\ g_{i1},\dots,g_{ir_i}$ for $\Gamma(D(s_i),\mathcal{O})$ as an A-algebra. By basic prop., $\exists N$ s.t. each $t_{ij}=s_i^N g_{ij}$ lifts to a section of $\mathcal{L}^{\otimes(nN)}$. The sections t_{i0} generate $\mathcal{L}^{\otimes(nN)}$. Thus $\exists$ a unique morphism $f: X \to \mathbb{P}_A^{\Sigma r_i+n-1}$ s.t. $f^*\mathcal{O}(1)=\mathcal{L}^{\otimes(nN)}$, $f^* x_{ij}=t_{ij}$.
Thus $f^{-1}(D_+(x_{i0}))=D(t_{i0})=D(s_i)$. Also the pullback map $f^\#_{D_+(x_{i0})}:\Gamma(D_+(x_{i0}),\mathcal{O})\to\Gamma(D(s_i),\mathcal{O})$ factors

$$\begin{tikzcd}
 & A[u_{ij} \mid j=1,\dots,r_i] \arrow[dl] \arrow[dr, "g_i"] & \\
\Gamma(D_+(x_{i0}),\mathcal{O}) \arrow[rr, "f^\#"] & & \Gamma(D(s_i),\mathcal{O})
\end{tikzcd}$$

$u_{ij} \mapsto x_{ij}/x_{i0}$, $\quad u_{ij} \mapsto g_{ij}$

Since g_i is surj, so is $f^\#_{D_+(x_{i0})}$. Thus $f: f^{-1}(D_+(x_{i0})) \to D_+(x_{i0})$ is a closed imm.
Therefore $f: X \to \mathcal{U} := \bigcup_i D_+(x_{i0}) \overset{\text{open}}{\subset} \mathbb{P}_A^{-}$ is a locally closed imm. $\square$

(3), q-cpt & q-sep $\Downarrow$ (4)

(4) & Noeth. $\Downarrow$ (3)

suffices to assume $X \to \operatorname{Spec} A$ is f. pres & q-sepd.

Theorem, Part II. TFAE ~~(X Noeth.)~~
(3) $\exists\, n>0$ and sections $s_0,\dots,s_r\in\Gamma(X,\mathcal{L}^{\otimes n})$ s.t. each $D(s_i)$ is affine
(4) $\mathcal{L}$ is ample.

Proof. (3) $\Rightarrow$ (4). Same argument as in Serre's thm.
(4) $\Rightarrow$ (3). For every $p\in X$, $\exists$ open affine $p\in\mathcal{U}\subset X$. Let $Z=X-\mathcal{U}$. There exists $N>0$ & $s\in\mathcal{I}_Z\otimes\mathcal{L}^{\otimes N}$ s.t. $s(p)\neq 0$. Considered as a section of $\mathcal{L}^{\otimes N}$, $p\in D(s)\subset\mathcal{U}$, affine $\square$

Definitions of relatively ample & relatively very ample. $f: X \to Y$. $\mathcal{L}$ on X.
~~f f.type & q-sepd~~ Y q-cpct, $\mathcal{L}$ f-relatively very ample $\Rightarrow \mathcal{L}$ f-relatively ample.
~~for~~ f f. presd & q-sepd, Y q-cpct, $\mathcal{L}$ f-rel. ample $\Rightarrow \exists N > 0$ s.t. $\mathcal{L}^{\otimes N}$ is f-relatively very ample.

Relative Proj. $\mathbb{P}\mathcal{J}$. Blowing up $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$.
$\nu^{-1}\mathcal{I} \cdot \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(1)$. "Universal property": Denote $g: Y \to X$
$(g^*\mathcal{I})_{pure} := \text{Image}(g^*\mathcal{I} \to \mathcal{H}om(\mathcal{H}om(g^*\mathcal{I}, \mathcal{O}_Y), \mathcal{O}_Y))$.
If $(g^*\mathcal{I})_{pure} \to \mathcal{O}_Y$ is injective, then g factors through ν if and only if $g^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ is invertible, in which case the factorization is unique.
<u>Local description</u>. $X = \operatorname{Spec} A$, $I = \langle f_0, \dots, f_r \rangle$. Form $\mathbb{P}^r_A$ and $V = \mathbb{V}(x_i f_j - x_j f_i)_{0 \leq i < j \leq r}$. If X is reduced, then $\tilde{X} =$ closure of $\mathbb{V} \cap \pi^{-1}(\operatorname{Spec} A - \mathbb{V}(I))$.
In general $\tilde{X} = \mathbb{V}(P(x_i) \mid \underset{P \text{ homog.}}{\wedge} P(f_i) = 0 \text{ in } A)$.

<u>Thm 7.17</u>. $f: X \to Y$ biratl & proj, Y q-proj, then f is isomorphic to $Bl_{\mathcal{I}} Y$ for some (non-unique) $\mathcal{I}$.

II. Differentials.
<u>Definition</u>. Let $f: X \to Y$ be a morphism. Let $\Delta_{X/Y}: X \to X \times_Y X$ be the diagonal. It is a locally closed imm. Let $\mathcal{I}$ denote the ideal sheaf.
Then $\Omega_{X/Y} := \Delta^* \mathcal{I} \cong \mathcal{I}/\mathcal{I}^2$.
Denote by $d: \mathcal{O}_X \to \Omega_{X/Y}$ the map $a \mapsto pr_1^\# a - pr_2^\# a$.

Then ∂ is an $f^{-1}\mathcal{O}_Y$-derivation. Moreover, it is the universal $f^{-1}\mathcal{O}_Y$-derivation.

<u>Local</u>. $B \xleftarrow{\varphi} A$. $\Omega_{B/A} = I/I^2$, $I := \mathrm{Ker}(B \otimes_A B \xrightarrow{m} B)$

$d : B \to \Omega_{B/A}$, $b \mapsto b \otimes 1 - 1 \otimes b$.

$\partial : B \to M$, ~~$\phi(b\otimes 1 - 1 \otimes b) := \partial b$.~~ $\phi : B \otimes_A B \to M$

$\phi(b_1 \otimes b_2) = b_1 \partial(b_2)$ ....

<u>Fundamental exact sequences</u>. (1) $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

(2)
$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd}$$
, $\mathcal{J}$ = Ideal sheaf of $i(X)$

$$\mathcal{J}/\mathcal{J}^2 \to i^*\Omega_{Y/Z} \to \Omega_{X/Z}.$$

<u>Example</u>. $\mathbb{A}^n_Y \to Y$. $\Omega_{\mathbb{A}^n_Y/Y} \cong \mathcal{O}_{\mathbb{A}^n_Y}\{dx_1, \dots, dx_n\}$.

$X \hookrightarrow \mathbb{A}^n_Y$, $X = \mathbb{V}(f_1, \dots, f_r)$. $\Omega_{X/Y} \cong \dfrac{\mathcal{O}_X\{dx_1, \dots, dx_n\}}{\langle df_1, \dots, df_r \rangle}$.

<u>Euler sequence</u>
$$\begin{tikzcd} \mathcal{U} = \mathbb{A}^{n+1} \setminus \{0\} \arrow[r, "\pi"] \arrow[dr] & \mathbb{P}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \pi^*\Omega_{\mathbb{P}^n_Y/Y} \arrow[r] & \Omega_{\mathcal{U}/Y} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \Omega_{\mathcal{U}/\mathbb{P}^n} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\ & & \mathcal{O}_{\mathcal{U}}\{dx_0, \dots, dx_n\} \arrow[r] & \mathcal{O}_{\mathcal{U}} & \end{tikzcd}$$

$$dx_i \mapsto x_i$$

$$\rightsquigarrow\ 0 \to \Omega_{\mathbb{P}^n_Y/Y} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

unramified also

Def. The morphism $f: X \to Y$ is <u>smooth</u> if
(1) f is flat of rel. dim d.
(2) f is locally finitely presented
(3) $\Omega_{X/Y}$ is locally free of rank $= d$.

<u>Jacobian Criterion</u>

~~Example~~. $Y = \operatorname{Spec} k$, $k = \bar{k}$. Then X is smooth iff ~~$\forall$~~ X is locally f. type & $\forall$ closed point x of X, $\mathcal{O}_{X,x}$ is a regular local ring.

<u>Non-example</u>. ~~$\operatorname{char}(k) = p$, $\operatorname{Spec} k \xrightarrow{F} \operatorname{Spec} k$, $F^{\#}: k \to k$, $a \mapsto a^p$. If k is not a finite field, then~~

$k = \overline{\mathbb{F}_p}(t)$, $K = k[u]/u^p - t$. $\operatorname{Spec} K \to \operatorname{Spec} k$ is <u>not</u> smooth. $K \otimes_k K \cong k[u_1, u_2]/\langle u_1^p - t, u_2^p - t\rangle$
$\downarrow$
$k[u]$

$u_1 - u_2$ generates $\Omega_{K/k}$, $\Omega_{K/k} \cong K\langle du\rangle$

Bertini's theorem.
~~formally~~ smooth: f' exists
~~formally~~ unramified: f' is unique if it exists

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r] \arrow[rr, bend left, "f"] \arrow[rrd, bend right, "g"'] & \operatorname{Spec} A' \arrow[r, dashed, "f'"] \arrow[rd, "g"] & X \arrow[d] \\
& & Y
\end{tikzcd}$$

formally étale: f' exists and is unique.