Problem Set 2

Disclaimer For open-ended problems, part of the problem is to give a precise formulation. Especially for the problems in Part II, you should do as much of the problems as is useful to you. For each problem, it is important you understand how to verify all details. However, if you are pressed for time, you may write-up only the most important steps, instead of every detail.
Late homework policy. Late work will be accepted only with a medical note or for another approved reason.

Cooperation policy. You are strongly encouraged to work with others, but the final write-up must be entirely your own and based on your own understanding.

Part I. These problems are from the textbook. You are expected to read *all* the problems from the sections of the textbook covered that week. You are asked to write-up and turn-in only the problems assigned below.

Part II. These problems are not necessarily from the textbook. Often they will be exercises in commutative algebra, category theory, homological algebra or sheaf theory.

Part I(25 points)

(a)	(5 points)	p. 66,	Section II.1,	Problem 1.2
(b)	(5 points)	p. 66,	Section II.1,	Problem 1.6
(c)	(5 points)	p. 67,	Section II.1,	Problem 1.13
(d)	(5 points)	p. 67,	Section II.1,	Problem 1.16
(e)	(5 points)	p. 69,	Section II.1,	Problem 1.22

Part II(25 points)

Problem 1(10 points) Solve Problem 1.18, p. 68, §II.1 of the textbook. Also prove the natural maps are homomorphisms of sheaves of Abelian groups when $\mathcal{F}$ is a sheaf of Abelian groups. Also state and prove the compatibility of the natural homomorphisms with respect to composition, i.e., given a pair of continuous maps $f : X \to Y$ and $g : Y \to Z$, what compatibilities do the natural homomorphisms satisfy? Finally, state and prove the compatibility of the natural homomorphisms with respect to taking stalks, i.e., for a point p of X, find the relation between the stalks of the adjoint maps,

$$(f^{-1}\mathcal{G})_p \to \mathcal{F}_p,$$

and,

$$\mathcal{G}_{f(p)} \to (f_*\mathcal{F})_{f(p)}.$$

Problem 2(10 points) Let $\mathcal{C}$ be a category with arbitrary products. Formulate a definition of a sheaf on X of objects in $\mathcal{C}$. You may want to use the categorical definitions of monics and epis, or rather the idea behind these definitions. A morphism $f : a \to b$ is *monic* if for every object c of $\mathcal{C}$, the following map is injective,

$$\text{Hom}_{\mathcal{C}}(c, f) : \text{Hom}_{\mathcal{C}}(c, a) \to \text{Hom}_{\mathcal{C}}(c, b).$$

The morphism is *epi* if for every object c of $\mathcal{C}$, the following map is injective,

$$\text{Hom}_{\mathcal{C}}(f, c) : \text{Hom}_{\mathcal{C}}(b, c) \to \text{Hom}_{\mathcal{C}}(a, c).$$

Not to be turned in: Does your definition agree with the earlier definition for the categories of sets, groups, rings, etc.?

Problem 3(5 points) Assume $\mathcal{C}$ also has colimits over (filtering) directed systems. Define stalks and formulate an analogue of Proposition 1.1 from the textbook. **Not to be turned in.** How much of the usual proof goes through?

Extra credit(5 points) Give an example of an Abelian category $\mathcal{C}$ having arbitrary products and colimits over (filtering) directed systems, but such that the analogue of Proposition 1.1 fails.