MAT 589 Introduction to Algebraic Geometry — **Jason Starr**
Due Thursday, April 13, 2017 — **Spring 2017**

Problem Set 10

Disclaimer For open-ended problems, part of the problem is to give a precise formulation. Especially for the problems in Part II, you should do as much of the problems as is useful to you. For each problem, it is important you understand how to verify all details. However, if you are pressed for time, you may write-up only the most important steps, instead of every detail.
Late homework policy. Late work will be accepted only with a medical note or for another approved reason.

Cooperation policy. You are strongly encouraged to work with others, but the final write-up must be entirely your own and based on your own understanding.

Part I. These problems are from the textbook. You are expected to read *all* the problems from the sections of the textbook covered that week. You are asked to write-up and turn-in only the problems assigned below.

Part II. These problems are not necessarily from the textbook. Often they will be exercises in commutative algebra, category theory, homological algebra or sheaf theory.

Part I(25 points)

(a)	(5 points)	p. 208,	Section III.2,	Read the proof of Thm. II.2.7	See note below
(b)	(10 points)	p. 216,	Section III.3,	Problem 3.1	See note below
(c)	(10 points)	p. 216,	Section III.3,	Problem 3.2	See note below

Note (a). Everybody gets full credit for this one. But please do actually look at this exercise. Although the result is somewhat special, the method is very close to the "devissage" method which we will use again later.

Note (b). The Noetherian hypothesis is used only to conclude the filtration on $\mathcal{F}$ terminates. Thus the result holds whenever the filtration terminates, e.g., if $\mathcal{N}^r = (0)$ for some integer r. In fact, the result holds even if the filtration does not terminate. Can you see how to prove the result in general?

For one approach, first use Exercise I.1.12 to deduce that for every inverse system $(\mathcal{F}^i)$ of Abelian sheaves, for every p, the following natural homomorphism is an isomorphism,

$$H^p(X, \varprojlim \mathcal{F}^i) \to \varprojlim H^p(X, \mathcal{F}^i).$$

Now form the inverse system of sheaves $\mathcal{F}^i = \mathcal{N}^i\mathcal{F}$. Use the hypothesis that X_{red} is affine and Thm. III.3.7 (or just Prop. II.5.6) to deduce that for every i the following homomorphism is surjective,

$$H^1(X, \mathcal{F}^{i+1}) \to H^1(X, \mathcal{F}^i).$$

Deduce that the following homomorphism is surjective,

$$\varprojlim H^1(X, \mathcal{F}^i) \to H^1(X, \mathcal{F}).$$

Finally, since $\mathcal{F}$ is quasi-coherent and $\mathcal{N}$ is nilpotent, deduce that $\varprojlim \mathcal{F}^i$ is the zero sheaf.

Note (c). You should definitely use Lemma III.2.10. Now mimic the reduction in Step 1 on p. 210, but instead of $\mathcal{F}_Y$ (which is probably not quasi-coherent because $\mathcal{F}|_Y := j^{-1}\mathcal{F}$ is not quasi-coherent!), use the following short exact sequence,

$$0 \longrightarrow \mathrm{Ker}\phi \longrightarrow \mathcal{F} \xrightarrow{\phi} j_*j^*\mathcal{F} \longrightarrow 0.$$

What can you say about $\mathrm{Ker}(\phi)$ and $\overline{U} = \overline{X - Y}$? Just for fun, see if you can remove the Noetherian hypotheses along the same lines as the note above. These two exercises will be completed by Exercise III.4.2 next time. One application will be a "general principles" solution to Problem 2, Part II of Problem Set 8.

Part II(25 points)

Problem 1(10 points) In the proof of Thm. III.3.7 in the non-Noetherian case (just that (i) is equivalent to (iii)), we used the following unproved assertion: For every quasi-compact scheme X, the only open subset U containing every closed point of X is all of X. As pointed out to me during lecture, this is equivalent to proving that every nonempty quasi-compact scheme Y contains a closed point (the complement Y of U is a quasi-compact scheme). Prove that every nonempty quasi-compact scheme contains a closed point.

Here is one approach (you must fill in the details if you follow this approach). Let $\mathfrak{U} = \{U_i\}$ be a finite cover of Y by open affines. Say that a point p of Y is $\mathfrak{U}$-locally closed if there exists an i such that p is a closed point of U_i. For every $\mathfrak{U}$-locally closed point p of Y define n_p to be the number of i such that U_i intersects $\overline{\{p\}}$. Prove that for every $\mathfrak{U}$-locally closed point p such that n_p is minimal, p is a closed point of X.

Problem 2(10 points) As Dorian correctly pointed out during lecture (and I failed to recognize!), one does not need the scheme X to be quasi-compact and quasi-separated in the proof of Thm. III.3.7, because one does not need X to be quasi-compact and quasi-separated in Exercise 2.17(b).

More generally, let X be a scheme and let $f_1, \dots, f_r$ be elements of $\Gamma(X, \mathcal{O}_X)$ such that the open subsets X_{f_i} give an affine covering of X (but do not necessarily assume that $f_1, \dots, f_r$ generate the unit ideal). Prove that X is quasi-compact and separated. (Warning: It is not necessarily the case that X is affine.)

Problem 3(5 points) Let $\mathcal{A}$ be an Abelian category. Let $\mathcal{A}'$ (or perhaps more correctly, $\mathcal{A}^{\Delta_1}$) denote the Abelian category whose objects are morphisms in A, $f : S \to T$ ("S" is for "source" and "T" is for "target"). A morphism from f' to f'' in $\mathcal{A}'$ is a commutative diagram,

$$\begin{array}{ccc} S' & \xrightarrow{f'} & T' \\ {\scriptstyle g_S}\downarrow & & \downarrow{\scriptstyle g_T} \\ S'' & \xrightarrow{f''} & T'' \end{array}$$

There is an obvious Abelian group structure on $\text{Hom}_{\mathcal{A}'}(f', f'')$ making $\mathcal{A}'$ into an Abelian category.

Denote by $Z : \mathcal{A}' \to \mathcal{A}$ the left-exact additive functor associating to f the module

$$Z(f) = \text{Ker}(f),$$

and associating to $(g_S, g_T) : f' \to f''$ the unique morphism $Z(g_S, g_T) : \text{Ker}(f') \to \text{Ker}(f'')$ compatible with g_S.

Describe the universal cohomological δ-functor $F = ((F^p)_{p \geq 0}, (\delta^p)_{p \geq 0})$ whose degree-0 term is Z. Say a little about why it is a δ-functor, why it is universal, etc.

Extra credit(5 points) Give an example of a nonempty scheme that has no closed point. For a challenge, try to find an example that is separated, irreducible and locally Noetherian (I don't know how to do this, or even whether it is possible).