18.725 PROBLEM SET 8

Due date: Wednesday, November 24 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1, 2, 3, and 4. There will be more problems posted soon, and you will be asked to do 1 more problem to a total of 5.

Required Problem 1: Recall from Definition 14.12 that a regular morphism of varieties $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is *projective* if for every open affine $U \subset Y$ there exists a projective variety Z, and a closed immersion $i : F^{-1}(U) \to U \times Z$ such that the restriction morphism $F : F^{-1}(U) \to U$ equals $\mathrm{pr}_U \circ i$. To be precise, this is the definition of *weakly projective.* A regular morphism of varieties $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is *strongly projective* if there exists a projective variety Z and a closed immersion $i : X \to Y \times Z$ such that $F = \mathrm{pr}_Y \circ i$.

Let X be a quasi-projective variety and denote by $j : X \hookrightarrow \mathbb{P}^n_k$ a locally closed immersion. Let $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a regular morphism of algebraic varieties. Prove the following are equivalent,

(i) F is weakly projective,
(ii) F is proper,
(iii) the graph morphism $F \times j : X \to Y \times \mathbb{P}^n_k$ has closed image, and
(iv) F is strongly projective.

Required Problem 2 In each of the following cases, X is an irreducible affine variety and $L/k(X)$ is a finite algebraic field extension. In each case compute the associated normalization $F : Y \to X$, i.e., write down the equations defining F in some affine space and the coordinates of the morphism F. In all cases, $\mathrm{char}(k) = 0$.

(a) $X = \mathbb{V}(y^2 - x^3) \subset \mathbb{A}^2_k$, $L = k(X)$.

(b) $X = \mathbb{V}(y^p - x^q) \subset \mathbb{A}^2_k$, p and q are relatively prime positive integers, $L = k(X)$.

(c) $X = \mathbb{A}^1_k$, $L = k(X)[t]/\langle t^2 + (1/x)t + 1\rangle$,

(d) $X = \mathbb{V}(y^2 - x^2(x - z)) \subset \mathbb{A}^3_k$, $L = k(X)$,

Required Problem 3 Let X be a variety. A *rank r subbundle of* $X \times \mathbb{A}^n_k$ is a pair (E, ϕ) of a rank r vector bundle E on X together with a morphism of Abelian cones on X, $\phi : E \to X \times \mathbb{A}^n_k$ such that for every point $p \in X$, the corresponding map $\phi_p : E_p \to \mathbb{A}^n_k$ is injective, where E_p denotes the fiber of E over p. An *equivalence of rank r subbundles*, $\psi : (E_1, \phi_1) \to (E_2, \phi_2)$ is a morphism of Abelian cones on X, $\psi : E_1 \to E_2$ such that $\phi_2 \circ \psi = \phi_1$. For every regular morphism $F : Y \to X$ and every rank r subbundle of $X \times \mathbb{A}^n_k$, (E, ϕ), the *pullback subbundle* is defined to be $(Y \times_X E, F^*\phi)$ where $F^*\phi : Y \times_X E \to Y \times \mathbb{A}^n_k$ is $\mathrm{pr}_Y \times (\mathrm{pr}_{\mathbb{A}^n_k} \circ \phi \circ \mathrm{pr}_E)$.

(i) Prove that $F^*\phi$ is injective on fibers.

(ii) Prove that (E_1, ϕ_1) and (E_2, ϕ_2) are equivalent rank r subbundles of $X \times \mathbb{A}^n_k$, then $(Y \times_X E_1, F^*\phi_1)$ and $(Y \times_X E_2, F^*\phi_2)$ are equivalent rank r subbundles of $Y \times \mathbb{A}^n_k$.

(iii) Let $G : Z \to Y$ be a regular morphism. For every rank r subbundle of $X \times \mathbb{A}^n_k$, (E, ϕ), prove that $(Z \times_X E, (F \circ G)^*\phi)$ is equivalent to $(Z \times_Y (Y \times_X E), G^*(F^*\phi))$.

Together, (i)–(iii) prove the existence of a contravariant functor,

$$\underline{\mathrm{Grass}(r,n)} : k - \mathrm{Varieties} \to \mathrm{Sets},$$

where $\underline{\mathrm{Grass}(r,n)}(X)$ is the set of equivalence classes of rank r subbundles of $X \times \mathbb{A}^n_k$, and where $\underline{\mathrm{Grass}(r,n)}(F) : \underline{\mathrm{Grass}(r,n)}(X) \to \underline{\mathrm{Grass}(r,n)}(Y)$ is the set map that sends the equivalence class $[(E, \phi)]$ to the equivalence class $[(Y \times_X E, F^*\phi)]$. This functor is called the *Grassmann functor*.

Required Problem 4: This problem proves the existence of a universal object for the Grassmann functor, i.e., a k-variety $\mathrm{Grass}(r,n)$ together with a rank r subbundle of $\mathrm{Grass}(r,n) \times \mathbb{A}^n_k$, (E, ϕ), such that for every variety X and every rank r subbundle (E', ϕ'), there is a unique morphism $F : X \to \mathrm{Grass}(r,n)$ such that $F^*(E, \phi)$ is equivalent to (E', ϕ').

(i) For every r-tuple $\underline{i} = (i_1, \ldots, i_r)$ of integers satisfying $1 \leq i_1 < \cdots < i_r \leq n$, define $U_{\underline{i}} \subset \mathrm{Hom}(\mathbb{A}^r_k, \mathbb{A}^n_k)$ to be the closed subvariety of $n \times r$ matrices such that for every $k, l = 1, \ldots, r$,

$$A_{i_k,l} = \begin{cases} 1, & k = l, \\ 0, & k \neq l \end{cases}$$

Denote by $\phi_{\underline{i}} : U_{\underline{i}} \times \mathbb{A}^r_k \to U_{\underline{i}} \times \mathbb{A}^n_k$ the morphism given by the matrix A. Prove that $(U_{\underline{i}} \times \mathbb{A}^r_k, \phi_{\underline{i}})$ is a rank r subbundle.

(ii) Let $\underline{i}$ be an r-tuple as above. Denote by $\chi_{\underline{i}} : \mathbb{A}^n_k \to \mathbb{A}^r_k$ the projection of $\mathbb{A}^n_k$ onto the coordinates x_{i_k}, $k = 1, \ldots, r$. Let X be a variety and let (E, ϕ) be a rank r subbundle of $X \times \mathbb{A}^n_k$ such that composition of ϕ with $\mathrm{Id}_X \times \chi_{\underline{i}} : X \times \mathbb{A}^n_k \to X \times \mathbb{A}^r_k$ is an isomorphism. Prove there exists a unique morphism $F : X \to U_{\underline{i}}$ such that $F^*(U_{\underline{i}} \times \mathbb{A}^r_k, \phi_{\underline{i}})$ is equivalent to (E, ϕ).

(iii) For every pair of r-tuples $(\underline{i}, \underline{j})$, define $U_{\underline{i},\underline{j}} \subset U_{\underline{i}}$ to be the open set where the $r \times r$ submatrix $(A_{j_k,l})$ is invertible, i.e., the distinguished open affine of the determinant of this $r \times r$ matrix. Restricting $(U_{\underline{i}}, \phi_{\underline{i}})$ to $U_{\underline{i},\underline{j}}$, prove the composition of $\phi_{\underline{i}}$ with $\mathrm{Id} \times \chi_{\underline{j}}$ is an isomorphism. Deduce existence of a morphism $u_{\underline{i},\underline{j}} : U_{\underline{i},\underline{j}} \to U_{\underline{j},\underline{i}}$.

(iv) Prove the image of $u_{\underline{i},\underline{j}}$ is contained in $U_{\underline{j},\underline{i}}$ and that $u_{\underline{i},\underline{j}}$ and $u_{\underline{j},\underline{i}}$ are inverse isomorphisms.

(v) Prove the collection $((U_{\underline{i}}), (U_{\underline{i},\underline{j}}), (u_{\underline{i},\underline{j}}))$ satisfies the gluing lemma for varieties. Denote the associated variety by $\iota_{\underline{i}} : U_{\underline{i}} \hookrightarrow \mathrm{Grass}(r,n)$.

(vi) Prove there exists a unique rank r subbundle of $\mathrm{Grass}(r,n) \times \mathbb{A}^n_k$, (E, ϕ), such that for every $\underline{i}$, $(\iota_{\underline{i}})^*(E, \phi)$ is equivalent to $(U_{\underline{i}} \times \mathbb{A}^r_k, \phi_{\underline{i}})$.

(vii) Use (ii) to prove that $\mathrm{Grass}(r,n)$ and (E, ϕ) have the universal property.

Problem 5: In this problem, do at least 2 of the parts (but you don't have to do all the parts). Recall for every integer $r \geq 0$, every vector space V and every vector space W, an *alternating, r-multilinear map* is a map $T : V^r \to W$ such that,

(i) for every $i = 1, \ldots, r$, and for every $(r-1)$-tuple $\underline{\mathbf{v}} = (\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \mathbf{v}_r) \in V^{r-1}$, the map $T_{\underline{\mathbf{v}}} : V \to W$, $\mathbf{v} \mapsto T(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_r)$, is a k-linear map, and

(ii) for every $1 \leq i < j \leq r$, for every r-tuple $\underline{\mathbf{v}} = (\mathbf{v}_1, \ldots, \mathbf{v}_r) \in V^r$, $T(\underline{\mathbf{v}}) = \mathbf{0}$ if $\mathbf{v}_i = \mathbf{v}_j$.

A pair $(\bigwedge^r(V), \tau)$ of a k-vector space $\bigwedge^r(V)$ and an alternating, r-multilinear map $\tau : V^r \to \bigwedge^r(V)$ is an r^{th} *exterior power of* V if for every alterating, r-multilinear map $T : V^r \to W$, there exists a unique k-linear map $L : \bigwedge^r(V) \to W$ such that $T = L \circ \tau$. If the r^{th} exterior power of V exists (which it does!), it is unique up to unique isomorphism.

Let V be a finite-dimensional k-vector space and let $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an ordered basis for V. Define $\bigwedge^r(V)$ to be the free k-vector space with finite basis denoted $\mathcal{B}^{(r)} = (\mathbf{v}_{\underline{i}} | \underline{i} \in \Sigma_{n,r})$ where $\Sigma_{n,r}$ is the finite set,

$$\Sigma_{n,r} = \{\underline{i} = (i_1, \ldots, i_r) | 1 \leq i_1 < \cdots < i_r \leq n\}.$$

Define $\tau : V^r \to \bigwedge^r(V)$ to be the unique alternating, k-linear map such that for every $\underline{i} \in \Sigma_{n,r}$, $\tau(\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_r}) = \mathbf{v}_{\underline{i}}$.

(i) Prove that $(\bigwedge^r(V), \tau)$ is an r^{th} exterior power of V.

(ii) Let $L : V_1 \to V_2$ be a k-linear map of vector spaces, let $(\bigwedge^r(V_1), \tau_1)$ be an r^{th} exterior power of V_1 and let $(\bigwedge^r(V_1), \tau_2)$ be an r^{th} exterior power of V_2. Prove there exists a unique k-linear map $\bigwedge^r(L) : \bigwedge^r(V_1) \to \bigwedge^r(V_2)$ such that $\bigwedge^r(L) \circ \tau_1 = \tau_2 \circ (L^r)$.

(iii) Let $\bigwedge^r$ be a rule that assigns to every k-vector space V an r^{th} exterior power $(\bigwedge^r(V), \tau)$. Prove there exists an associated covariant functor $\bigwedge^r : k -$ Vector spaces $\to k -$ Vector spaces which associates to every vector space V the vector space $\bigwedge^r(V)$ and which associates to every k-linear map $L : V_1 \to V_2$ the k-linear map $\bigwedge^r(L)$, i.e., check this rule respects identity morphisms and composition of k-linear maps. **Remark:** The only issue in defining such a functor is that the r^{th} exterior power is not unique – it is only unique up to unique isomorphism. This is not a serious issue (there is a canonical choice which is a quotient vector space of the free vector space with basis V^r).

(iv) In the same manner as Problem 8 from Problem Set 5, extend the notion of exterior power to vector bundles.

Problem 6: Let $n, r \geq 0$ be integers. Define $N = \binom{n}{r}$. Let X be a variety.

(i) Using Problem 5(i) and (iv), give an isomorphism of the r^{th} exterior power of $X \times \mathbb{A}^n_k$ with $X \times \mathbb{A}^N_k$.

(ii) Applying (i) and Problem 5(iii) to the Grassmannian $\text{Grass}(r, n)$, define a tautological rank 1 subbundle of $\text{Grass}(r, n) \times \mathbb{A}^N_k$, $\bigwedge^r(\phi) : \bigwedge^r(S) \to \text{Gras}(r, n) \times \mathbb{A}^N_k$. Combined with Problem 7 from Problem Set 5, deduce existence of a regular morphism $F : \text{Grass}(r, n) \to \mathbb{P}^{N-1}$. This is the *Plücker embedding.*

(iii) For every $\underline{i} \in \Sigma_{n,r}$, denote by $x_{\underline{i}}$ the corresponding coordinate on $\mathbb{A}^N_k$. Prove that $F^{-1}(D_+(x_{\underline{i}}))$ equals $\iota(U_{\underline{i}})$. Conclude that F is an affine morphism.

Problem 7: This problem continues the previous problem, proving the Plücker embedding is a closed immersion.

(i) Assume $n \geq r$. Let $\underline{i} = (1, \dots, r)$. The variety $U_{\underline{i}}$ is the closed subvariety of affine space $\mathbb{A}_k^{nr}$ of $n \times r$ matrices such that the first $r \times r$ rows form the identity matrix. Identify $U_{\underline{i}}$ with the affine space $\mathbb{A}_k^{(n-r)r}$ of $(n-r) \times r$ matrices A via the rule,

$$A \leftrightarrow \left(\frac{I_{r \times r}}{A} \right).$$

Denote the entries of A by $(a_{i,j} | 1 \leq i \leq n-r, 1 \leq j \leq r)$. These are coordinates on the affine space $U_{\underline{i}}$. For every $1 \leq i \leq n-r$ and $1 \leq j \leq r$, denote by $\underline{k} \in \Sigma_{n,r}$ the r-tuple,

$$\underline{k} = (1, \dots, j-1, j+1, \dots, r, r+i).$$

On the affine space $D_+(x_{\underline{i}})$, the rational function $x_{\underline{k}}/x_{\underline{i}}$ is a coordinate. Prove that $F^{\#}(x_{\underline{k}}/x_{\underline{i}}) = a_{i,j}$.

(ii) Deduce that $F^{\#} : k[D_+(x_{\underline{i}})] \to k[U_{\underline{i}}]$ is surjective. Therefore $F : U_{\underline{i}} \to D_+(x_{\underline{i}})$ is a closed immersion. Argue this is true for every $\underline{i} \in \Sigma_{n,r}$, therefore $F : \text{Grass}(r, n) \to \mathbb{P}_k^{N-1}$ is a closed immersion.

Problem 8: Here is a way to find generators for the homogeneous ideal of the projective variety $F(\text{Grass}(r, n)) \subset \mathbb{P}_k^{N-1}$. Denote by V the vector space $\mathbb{A}_k^n$ so that $\mathbb{A}_k^N$ equals $\bigwedge^r(V)$. Let $\tau_r : V^r \to \bigwedge^r(V)$ be the universal alternating r-linear map. Denote by $(\bigwedge^{r+1}(V), \tau_{r+1})$ an $(r+1)^{\text{st}}$ exterior power of V.

(i) Prove there is a unique 2-multilinear map $L : \bigwedge^r(V) \times V \to \bigwedge^{r+1}(V)$ such that $\tau_{r+1} = L \circ ((\tau_r \circ \text{pr}_{1,\dots,r}) \times \text{pr}_{r+1})$. Using adjointness, deduce existence of a map $\widetilde{L} : \bigwedge^r(V) \to \text{Hom}_k(V^\vee, \bigwedge^{r+1}(V))$ such that for every $\mathbf{w} \in \bigwedge^r(V)$, for every $x \in V^\vee$, and every basis $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ for V,

$$(\widetilde{L}(\mathbf{w}))(x) = \sum_{i=1}^{n} \langle x, \mathbf{v}_i \rangle L(\mathbf{w}, \mathbf{v}_i).$$

(ii) Let $\mathbf{w}$ be an element of $\bigwedge^r(V) - \{\mathbf{0}\}$. Prove the image $[\mathbf{w}] \in \mathbb{P}(\bigwedge^r(V)) = \mathbb{P}_k^{N-1}$ is in $F(\text{Grass}(r, n))$ iff $\widetilde{L}(\mathbf{w})$ has rank at most r, i.e., iff the $(r+1) \times (r+1)$ minors of the matrix are all zero.

(iii) Let $n = 4$ and $r = 2$. Let $(\mathbf{v}_1, \dots, \mathbf{v}_4)$ be an ordered basis for V and let $(\mathbf{v}_{1,2}, \mathbf{v}_{1,3}, \mathbf{v}_{1,4}, \mathbf{v}_{2,3}, \mathbf{v}_{2,4}, \mathbf{v}_{3,4})$ be an ordered basis for $\bigwedge^2(V)$. Denote by $(x_{1,2}, \dots, x_{2,4}$ the dual ordered basis for $(\bigwedge^2(V))^\vee$. Let $(x_1, \dots, x_4)$ be the dual ordered basis for $V^\vee$ and let $(\mathbf{v}_{1,2,3}, \mathbf{v}_{1,2,4}, \mathbf{v}_{1,3,4}, \mathbf{v}_{2,3,4})$ be an ordered basis for $\bigwedge^3(V)$. With respect to these ordered bases, write down the linear transformation $\widetilde{L}$ as a 4×4 matrix whose entries are linear polynomials in $x_{1,2}, \dots, x_{3,4}$.

(iv) After performing elementary row and column operations, reduce this matrix to a skew-symmetric matrix. The rank of a skew-symmetric matrix is always even, therefore the 3×3 minors vanish iff the determinant vanishes. Prove there exists a quadratic polynomial in $x_{1,2}, \dots, x_{3,4}$ such that the determinant of the skew-symmetric matrix is the square of this polynomial. The polynomial is called the *Pfaffian*, and generates the homogeneous ideal of $F(\text{Grass}(2, 4)) \subset \mathbb{P}_k^5$.

Problem 9: Serre's criterion says that an irreducible variety X is normal if,

(i) the singular locus of X has codimension at least 2, and

(ii) for every pair of open subset $U \subset V \subset X$, if $V - U \subset V$ has codimension at least 2, the restriction map is an isomorphism, $\rho_U^V : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

Here is an example of a non-normal variety that satisfies the first condition, but not the second. Let $A \subset k[x, y]$ be the set of polynomials $f(x, y)$ such that $f(1, 0) = f(0, 1)$.

(i) Prove that A is a finitely generated k-subalgebra of $k[x, y]$.

(ii) Let X be an affine variety with $k[X] \cong A$, and let $F : \mathbb{A}_k^2 \to X$ be the unique morphism such that $F^\#$ induces the inclusion $A \subset k[x, y]$. Prove that F is a birational, finite morphism that is not an isomorphism. Therefore X is not normal.

(iii) Let $U = \mathbb{A}_k^2 - \{(1, 0), (0, 1)\}$. Prove that $F(U) \subset X$ is an open set and $F : U \to F(U)$ is an isomorphism. In particular $F(U)$ is smooth, and $X - F(U)$ is finite because the inverse image $\mathbb{A}_k^2 - U$ is finite. So the singular locus of X has codimension 2.

(iv) Prove that the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(F(U))$ is not an isomorphism.

Problem 10: In a commutative algebra textbook, read the proof that an integrally closed, Noetherian local ring of dimension 1 is a DVR, and thus is regular. Sketch a proof that every normal 1-dimensional variety is smooth.