18.725 PROBLEM SET 3

Due date: Friday, October 1 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1, 2, 3, and 4, together with 2 others of your choice to a total of 6 problems. The last 5 problems on this problem set are taken from Problem Set 2 (the solutions to these problems were not given). You can use them for the non-required problems only if you did not use them for Problem Set 2.

Required Problem 1: Let V be a quasi-affine algebraic set and let $U \subset V$ be a closed subset. For every quasi-affine algebraic set T and every function $F : T \to U$, prove F is regular iff the induced function $F : T \to V$ is regular (this has already been used implicitly a few times in the course; do not simply quote the result from someplace it was used).

Required Problem 2: Let V be a quasi-affine algebraic set and let $U \subset V$ be an open subset. For every quasi-affine algebraic set T and every function $F : T \to U$, prove F is regular iff the induced function $F : T \to V$ is regular (this has already been used implicitly a few times in the course; do not simply quote the result from someplace it was used). **Not to be written up:** Conclude that for every subset $U \subset V$ that is a quasi-affine algebraic set and every $F : T \to U$, F is regular iff the induced function $F : T \to V$ is regular.

Required Problem 3: Let V be a quasi-affine algebraic set. By Problem 2 on Problem Set 2, there exists a product $(V \times V, \pi_1, \pi_2)$ for (V, V) in the category of quasi-affine algebraic sets. Define $\Delta_V : V \to V \times V$ to be the unique morphism such that $\pi_1 \circ \Delta_V = \pi_2 \circ \Delta_V = \mathrm{Id}_V$. Prove the image of Δ_V is a Zariski closed subset of $V \times V$. (**Hint:** First consider the case that $V = \mathbb{A}^n_k$.)

Required Problem 4: Consider the action of $\mathbb{G}_m$ on $X = \mathbb{A}^3_k$ by $m_X(\lambda, (a_1, a_2, a_3)) = (\lambda^{-1}a_1, a_2, \lambda a_3)$.

(a) Determine the associated grading of $k[X] = k[x_1, x_2, x_3]$, and in particular write a finite set of generators of the k-subalgebra $k[X]_0 \subset k[X]$.

(b) Find an affine algebraic set Y and a morphism $F : X \to Y$ such that $F^* : k[Y] \to k[X]$ is injective with image $k[X]_0$. Prove that $F(m_X(\lambda, p)) = F(p)$ for every $\lambda \in \mathbb{G}_m$ and every $p \in X$.

Problem 5: For the morphism F in Problem 4, write down all elements $q \in Y$ such that $F^{-1}(q)$ is not a single orbit of $\mathbb{G}_m$, and for each element q write the decomposition of $F^{-1}(q)$ as a union of $\mathbb{G}_m$-orbits.

Problem 6: Let $F : X \to Y$ be a regular morphism of quasi-affine algebraic sets. Let $(X \times Y, \pi_1, \pi_2)$ be a product of (X, Y) in the category of quasi-affine algebraic sets. Define $\Gamma_F : X \to X \times Y$, the *graph morphism of* F, to be the unique morphism such that $\pi_1 \circ \Gamma_F = \mathrm{Id}_X$ and $\pi_2 \circ \Gamma_F = F$. Prove the image of Γ_F is a Zariski closed subset of $X \times Y$. (**Hint**: Can you use Problem 3?)

Problem 7. A weighted projective space: Consider the action of $\mathbb{G}_m$ on $X = \mathbb{A}^3$ by $m_X(\lambda, (a_0, a_1, a_2)) = (a_0, \lambda a_1, \lambda^2 a_2)$. Define $V = X - \mathbb{V}(x_1, x_2)$, and define $F : V \to \mathbb{P}^3_k$ by $F(a_0, a_1, a_2) = [a_1^2, a_2, a_0 a_1^2, a_0 a_2]$.

(a) Prove that F is a well-defined function on V.

(b) Prove that every nonempty fiber of F is an orbit.

(c) Find the ideal of the Zariski closure of Image(F) and give an element in the Zariski closure of Image(F) that is not in Image(F).

Problem 8: This problem gives another example of an affine group variety. Let $n \geq 1$ be an integer and choose coordinates on $\mathbb{A}^{n^2}_k$ of the form $x_{i,j}$, $1 \leq i, j \leq n$. Define the determinant polynomial $\det \in k[x_{i,j} | 1 \leq i, j \leq n]$ in the usual way,

$$\det = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)},$$

where $\mathrm{sgn} : \mathfrak{S}_n \to \{+1, -1\}$ is the unique nontrivial group homomorphism. Define $\mathbf{GL}_n \subset \mathbb{A}^{n^2}_k$ to be $D(\det)$. Define $m : \mathbf{GL}_n \times \mathbf{GL}_n \to \mathbb{A}^{n^2}_k$ to be $m((a_{i,j}), (b_{i,j})) = (c_{i,j})$, where $c_{i,j} = \sum_{h=1}^{n} a_{i,h} b_{h,j}$. Define $e \in \mathbf{GL}_n$ to be the unique element such that $x_{i,j}(e) = 1$ iff $i = j$ and is 0 otherwise.

(a) Prove the image of m is contained in $\mathbf{GL}_n$.

(b) Prove there exists a regular morphism $i : \mathbf{GL}_n \to \mathbf{GL}_n$ such that for every $A \in \mathbf{GL}_n$, $m(A, i(A)) = e$.

(c) Prove the regular morphism $\det : \mathbf{GL}_n \to \mathbb{G}_m$ is a group homomorphism.

Problem 9: Assume $\mathrm{char}(k) \neq 2$. A *projective plane conic* is a proper closed subset $C \subset \mathbb{P}^2_k$ of the form $\mathbb{V}(a_{2,0,0}X_0^2 + a_{1,1,0}X_0X_1 + a_{1,0,1}X_0X_2 + a_{0,2,0}X_1^2 + a_{0,1,1}X_1X_2 + a_{0,0,2}X_2^2)$. Determine the analogue of Problem 6 from Problem Set 1 for projective plane conics, and solve the corresponding problem. How does your answer compare to the answer to Problem 6 from Problem Set 1?

Problem 10: Let $d \geq 1$ be an integer and assume that $\mathrm{char}(k)$ does not divide d. Define $\mu_d \subset \mathbb{A}^1_k$ to be $\mathbb{V}(x^d - 1)$.

(a) Prove this is a subgroup of $\mathbb{G}_m$.

(b) Let $n \geq 0$ be an integer, and restrict the standard action of $\mathbb{G}_m$ on $\mathbb{A}^n_k - \{0\}$ to an action of μ_d on $\mathbb{A}^n_k - \{0\}$. Prove the Veronese morphism from Problem 9 on Problem Set 2 is a quotient of this action in the sense that every nonempty fiber is an orbit under μ_n.

Difficult Problem 11: Problem 10 from Problem Set 2. (I've decided this is rather difficult after all.)

Difficult Problem 12: Problem 11 from Problem Set 2.

Very Difficult Problem 13: Problem 14 from Problem Set 2.

Problem 14: Problem 16 from Problem Set 2.

Difficult Problem 15: Problem 17 from Problem Set 2.