MAT 552 PROBLEM SET 7

This problem set focuses on the tensor algebra and its important quotients, the symmetric algebra and the exterior algebra. In addition to their familiar properties, these each carry a structure of "graded Hopf algebra". (**Nota bene.** A graded Hopf algebra is not quite the same as a Hopf algebra whose structure morphisms respect the gradings). For a Lie algebra, the universal enveloping algebra is another quotient of the tensor algebra that has a structure of Hopf algebra. This Hopf algebra structure is the key ingredient in one proof of the Poincaré-Birkhoff-Witt Theorem.

Problems.

Problem 1. (Universal property of tensor algebra.) For every field k, for every k-vector space V, for every integer $n \geq 0$, inductively define the k-vector space $T^n_k(V)$ by the rule,

$$T^0_k(V) = k, \quad T^{n+1}_k(V) = V \otimes_k T^n_k(V).$$

For every integer n, denote by $\beta_{V,n}$ the universal k-bilinear operation,

$$\beta_{V,n} : V \times T^n_k(V) \to T^{n+1}_k(V).$$

Define $T^\bullet_k(V)$ to be the $\mathbb{Z}_{\geq 0}$-graded k-vector space,

$$T^\bullet_k(V) := \bigoplus_{n \geq 0} T^n_k(V), \quad q_{V,n} : T^n_k(V) \hookrightarrow T^\bullet_k(V).$$

Denote by β_V the unique k-bilinear operation

$$\beta_V : V \times T^\bullet_k(V) \to T^\bullet_k(V),$$

such that for every $n \geq 0$, the composition $\beta_V \circ (\mathrm{Id}_V \times q_{V,n})$ equals $q_{V,n+1} \circ \beta_{V,n}$.

(a) For $n = 0$, show that $\beta_{V,0}(v, 1)$ equals v for all $v \in V$, and this gives the (standard) identification of V with $T^1_k(V)$. Via this identification, prove that β_V extends to a unique k-bilinear pairing,

$$\beta_{T(V)} : T^\bullet_k(V) \times T^\bullet_k(V) \to T^\bullet_k(V),$$

whose restriction to $T^1_k(V) \times T^\bullet_k(V)$ equals β_V, such that $\beta_{T(V)}$ is associative, and such that $1 \in T^0_k(V)$ is a left-right multiplicative identity for this operation. Thus, with this unique k-bilinear pairing, $T^\bullet_k(V)$ is a unital, associative k-algebra, the **tensor k-algebra of V**. Also check that the given direct sum decomposition of $T^\bullet_k(V)$ makes $T^\bullet_k(V)$ into a $\mathbb{Z}_{\geq 0}$-graded k-algebra, i.e., $\beta_{T(V)}$ maps $T^m_k(V) \times T^n_k(V)$ to the summand $T^{m+n}_k(V)$ for every $m, n \in \mathbb{Z}_{\geq 0}$.

(b) For every associative k-algebra,

$$(A, b : A \times A \to A),$$

a k-bilinear operation,

$$\alpha_V : V \times A \to A,$$

is **right** A**-associative** if $\alpha_V(v, b(a, a'))$ equals $b(\alpha_V(v, a), a')$ for every $v \in V$ and for every $a, a' \in A$. In this case, prove that α_V extends to a unique k-bilinear pairing,

$$\alpha_{T(V)} : T_k^\bullet(V) \times A \to A,$$

whose restriction to $T_k^1(V) \times A$ equals α_V, such that this operation is both left $T_k^\bullet(V)$-associative and right A-associative, and such that $1 \in T_k^0(V)$ acts as the identity on A. This is the universal property of $T_k^\bullet(V)$ among associative k-algebras (that are not necessarily unital).

(c) If A is unital, then prove that every right A-associative k-bilinear operation α_V is equivalent to a k-linear transformation,

$$\widetilde{\alpha}_V : V \to A,$$

by the rule $\widetilde{\alpha}_V(v) = \alpha_V(v, 1)$. Moreover, prove that the induced k-linear transformation,

$$\widetilde{\alpha}_{T(V)} : T_k^\bullet(V) \to A, \quad t \mapsto \alpha_{T(V)}(t, 1),$$

is the unique homomorphism of unital, associative k-algebras whose restriction to $T_k^1(V) = V$ equals the k-linear transformation $\widetilde{\alpha}_V$. This is the universal property of $T_k^\bullet(V)$ among unital, associative k-algebras.

(d) Define $J_s \subset T_k^\bullet(V)$ to be the left-right ideal generated by all elements $q_{V,2}(v \otimes w - w \otimes v)$ for $v, w \in V$. The **symmetric** k**-algebra of** V, $\text{Sym}_k^\bullet(V)$, is defined to be the quotient of $T_k^\bullet(V)$ by the left-right ideal J_s. Prove that this is also a $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebra,

$$\text{Sym}_k^\bullet(V) = \bigoplus_{n \geq 0} \text{Sym}_k^n(V), \quad r_{V,n} : \text{Sym}_k^n(V) \hookrightarrow \text{Sym}_k^\bullet(V),$$

and the quotient map is a morphism of $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebras,

$$s_{V,n} : T_k^n(V) \twoheadrightarrow \text{Sym}_k^n(V), \quad n \in \mathbb{Z}_{\geq 0}.$$

Prove that is an isomorphism on the degree 0 and degree 1 summands,

$$s_{V,0} : k \xrightarrow{\cong} \text{Sym}_k^0(V), \quad s_{V,1} : V \xrightarrow{\cong} \text{Sym}_k^1(V).$$

(e) Prove that $\text{Sym}_k^\bullet(V)$ is a commutative k-algebra, i.e., for every pair of elements $u, v \in \text{Sym}_k^\bullet(V)$, the product $v \cdot u$ equals $u \cdot v$. For every associative k-algebra A that is commutative and for every right A-associative k-bilinear operation α_V, prove that the k-bilinear pairing $\alpha_{T(V)}$ factors uniquely as a composition of $s_V \times \text{Id}_A$ and a k-bilinear pairing,

$$\alpha_{\text{Sym}(V)} : \text{Sym}_k^\bullet(V) \times A \to A.$$

Prove that this pairing restricts on $\text{Sym}_k^1(V) \times A$ as α_V, prove that this pairing is left $\text{Sym}_k^\bullet(V)$-associative and right A-associative, and prove that $1 \in \text{Sym}_k^0(V)$ acts as the identity on A. This is the universal property of $\text{Sym}_k^\bullet(V)$ among associative, commutative k-algebras.

(f) If the associative, commutative k-algebra A above is also unital, formulate and prove the universal property of $\text{Sym}_k^\bullet(V)$ among associative, unital, commutative k-algebras.

(g) Define $J_a \subset T_k^\bullet(V)$ to be the unique left-right ideal generated by all elements $q_{V,2}(v \otimes v)$ for $v \in V$. The **exterior** k**-algebra of** V, $\bigwedge_k^\bullet(V)$, is defined to be the quotient of $T_k^\bullet(V)$ by the left-right ideal J_a. Prove that this is also a $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebra,

$$\bigwedge_k^\bullet(V) = \bigoplus_{n\geq 0} \bigwedge_k^n(V), \quad t_{V,n} : \bigwedge_k^n(V) \hookrightarrow \bigwedge_k^\bullet(V),$$

and the quotient map is a morphism of $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebras,

$$e_{V,n} : T_k^n(V) \twoheadrightarrow \bigwedge_k^n(V), \quad n \in \mathbb{Z}_{\geq 0}.$$

Prove that is an isomorphism on the degree 0 and degree 1 summands,

$$s_{V,0} : k \xrightarrow{\cong} \mathrm{Sym}_k^0(V), \quad s_{V,1} : V \xrightarrow{\cong} \mathrm{Sym}_k^1(V).$$

(h) Assume that the characteristic is different from 2. Prove that the $\mathbb{Z}_{\geq 0}$-graded, associative k-algebra $\bigwedge_k^\bullet(V)$ is **graded commutative**, i.e., for every pair of homogeneous elements $u \in \bigwedge_k^m(V)$ and $v \in \bigwedge_k^n(V)$, the product $v\wedge u$ equals $(-1)^{mn}u\wedge v$. For every $\mathbb{Z}_{\geq 0}$-graded associative k-algebra $A^\bullet$ that is graded commutative and for every right $A^\bullet$-associative k-bilinear operation that is homogeneous of degree $+1$,

$$(\alpha_{V,n} : V \times A^n \to A^{n+1})_{n\geq 0},$$

prove that the k-bilinear pairing $\alpha_{T(V)}$ factors uniquely as a composition of $e_V \times \mathrm{Id}_A$ and a k-bilinear pairing,

$$\alpha_{\bigwedge(V)} : \bigwedge_k^\bullet(V) \times A^\bullet \to A^\bullet.$$

Prove that this pairing restricts on $\bigwedge_k^1(V) \times A^\bullet$ as α_V, prove that this pairing is left $\bigwedge_k^\bullet(V)$-associative and right $A^\bullet$-associative, and prove that $q \in \bigwedge_k^0(V)$ acts as the identity on $A^\bullet$. This is the univeral property of $\bigwedge_k^\bullet(V)$ among $\mathbb{Z}_{\geq 0}$-graded associative, graded commutative k-algebras. Formulate and prove the analogous universal property when $A^\bullet$ is also unital. (**Challenge problem.** Formulate the correct analogue in characteristic 2.)

Problem 2. (Functoriality of tensor algebras and direct sum decompositions.) Prove that the tensor algebra, the symmetric algebra, and the exterior algebra are each covariant in V, and thus the graded components give k-linear representations of $\mathbf{GL}_k(V)$ and $\mathbf{SL}_k(V)$.

(a) For every short exact sequence of k-vector spaces,

$$0 \to U \xrightarrow{\iota} V \xrightarrow{\pi} W \to 0,$$

for every integer $n \geq 0$, prove that the induced morphisms

$$T_k^n(\iota) : T_k^n(U) \to T_k^n(V), \ \mathrm{Sym}_k^n(\iota) : \mathrm{Sym}_k^n(U) \to \mathrm{Sym}_k^n(V), \ \bigwedge_k^n(\iota) : \bigwedge_k^n(U) \to \bigwedge_k^n(V),$$

are injective, and prove that the induced morphisms

$$T_k^n(\pi) : T_k^n(V) \to T_k^n(W), \ \mathrm{Sym}_k^n(\pi) : \mathrm{Sym}_k^n(V) \to \mathrm{Sym}_k^n(W), \ \bigwedge_k^n(\pi) : \bigwedge_k^n(V) \to \bigwedge_k^n(W),$$

are surjective.

(a) Denote by I_π the left-right ideal that is the kernel of the morphism $T_k^\bullet(\pi)$ of $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebras. For every integer $r \geq 0$, denote by I_π^r the left-right ideal generated by r-fold products of elements of I_π, so $I_\pi^0 = T_k^\bullet(V)$, $I_\pi^1 = I_\pi$, etc. Prove that every two-sided ideal I_π^r is a homogeneous ideal, i.e.,

$$I_\pi^r = \bigoplus_{n \geq 0} I_{\pi,n}^r, \quad I_{\pi,n}^r := I_\pi^r \cap T_k^n(V).$$

Describe the components $I_{\pi,n}^r$ and prove that the multiplication map gives a k-isomorphism of the associated subquotients,

$$I_{\pi,n}^r / I_{\pi,n}^{r+1} \cong \bigoplus_{\Sigma \subset \{1,\dots,n\}, \#\Sigma = r} T_k^r(U) \otimes_k T_k^{n-r}(W).$$

In particular, this is zero if $r > n$, so this decreasing filtration is exhaustive on each graded component $T_k^n(V)$. Finally, prove that these k-isomorphisms assemble into an isomorphism of the $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebra

$$\mathrm{Gr}_\pi^\bullet T_k(V) := \bigoplus_{r \geq 0} I_{\pi,n}^r / I_{\pi,n}^{r+1},$$

with the free product of $T_k^\bullet(U)$ and $T_k^\bullet(W)$, i.e., the coproduct in the category of $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebras.

(b) For every integer $r \geq 0$, denote by $I_{s,\pi}^r$, resp. $I_{e,\pi}^r$, the image of I_π^r in $\mathrm{Sym}_k^\bullet(V)$, resp. in $\bigwedge_k^\bullet(V)$. Prove that the k-isomorphisms above give k-isomorphisms of associated subquotients,

$$I_{s,\pi,n}^r / I_{s,\pi,n}^{r+1} \cong \mathrm{Sym}_k^r(U) \otimes_k \mathrm{Sym}_k^{n-r}(W),$$

$$I_{e,\pi,n}^r / I_{e,\pi,n}^{r+1} \cong \bigwedge_k^r(U) \otimes_k \bigwedge_k^{n-r}(W).$$

Prove that these assemble into an isomorphism of the $\mathbb{Z}_{\geq 0}$-graded, associative, commutative, unital k-algebra

$$\mathrm{Gr}_\pi^\bullet \mathrm{Sym}_k(V) := \bigoplus_{r \geq 0} I_{s,\pi,n}^r / I_{s,\pi,n}^{r+1},$$

and the tensor product of $\mathbb{Z}_{\geq 0}$-graded, associative, commutative, unital k-algebras $\mathrm{Sym}_k(U) \otimes_k \mathrm{Sym}_k(W)$. Similarly, prove that these assemble into an isomorphism of the $\mathbb{Z}_{\geq 0}$-graded, associative, graded commutative, unital k-algebra

$$\mathrm{Gr}_\pi^\bullet \bigwedge_k(V) := \bigoplus_{r \geq 0} I_{e,\pi,n}^r / I_{e,\pi,n}^{r+1},$$

and the tensor product of $\mathbb{Z}_{\geq 0}$-graded, associative, graded commutative, unital k-algebras $\bigwedge_k(U) \otimes_k \bigwedge_k(W)$.

(c) When V equals k, prove that the surjection $T_k^\bullet(V) \to \mathrm{Sym}_k^\bullet(V)$ is an isomorphism, and compute that every graded piece is 1-dimensional. Similarly, prove that $\bigwedge_k^n(k)$ is zero for every $n \geq 2$. Combine this with the previous isomorphisms and induction on the dimension of V to prove that for every V of finite dimension m, for every $n \geq 0$, the graded component $T_k^n(V)$ has dimension m^n, the graded component $\mathrm{Sym}_k^n(V)$ has dimension $\binom{m+n-1}{n}$, and the graded component $\bigwedge_k^n(V)$ has dimension $\binom{m}{n}$.

(d) With V of finite dimension m as above, for every ordered k-basis $(x_1, \dots, x_m)$ of V, prove that one k-basis of $\text{Sym}^n_k(V)$ consists of the monomials $x_1^{n_1} \cdots x_i^{n_i} \cdots x_m^{n_m}$ for all $(n_1, \dots, n_m) \in \mathbb{Z}^m_{\geq 0}$ with $n_1 + \cdots + n_m = n$. Similarly, prove that one k-basis of $\bigwedge^n_k(V)$ for $n \leq m$ consists of the elements $x_{i_1} \wedge \cdots \wedge x_{i_n}$ for subsets $\{i_1, \dots, i_n\} \subset \{1, \dots, m\}$ of size n with the usual ordering $i_1 < \cdots < i_n$.

Problem 3. (Representations of the special linear group.) This problem continues the previous problem. Let V be a k-vector space of finite dimension m.

(a) For every nonzero element of $\text{Sym}^n_k(V)$, prove that there exists an ordered k-basis $(x_1, \dots, x_m)$ of V with respect to which the element has nonzero coefficient of x_1^n. For the maximal torus $T \subset \mathbf{SL}_k(V)$ corresponding to this basis, prove that the smallest T-invariant k-subspace of $\text{Sym}^n_k(V)$ that contains the element also contains the element x_1^n. Conclude that the smallest $\mathbf{SL}_k(V)$-invariant k-subspace also contains x^n for every $x \in V$. Using the multinomial expansion, conclude that also this k-subspace contains every monomial $x_1^{n_1} \cdots x_m^{n_m}$ whose multinomial coefficient in $(t_1 x_1 + \cdots + t_m x_m)^n$ is nonzero. Assuming that the characteristic of the field k is $> n$, e.g., as in the case of $k = \mathbb{R}$ and $k = \mathbb{C}$, show that the $\mathbf{SL}_k(V)$-invariant k-subspace equals all of $\text{Sym}^n_k(V)$. Conclude that $\text{Sym}^n_k(V)$ is an irreducible k-linear representation of $\mathbf{SL}_k(V)$ for every integer $n \geq 0$. (This fails if the characteristic of k is positive and less than n.)

(b) Similarly, for every integer n with $n \leq m$, for every nonzero element of $\bigwedge^n_k(V)$, prove that there exists an ordered k-basis $(x_1, \dots, x_m)$ of V with respect to which the element has nonzero coefficient of $x_1 \wedge \cdots \wedge x_n$. For the corresponding maximal torus T, prove that the smallest T-invariant k-subspace of $\bigwedge^n_k(V)$ that contains this element also contains the element $x_1 \wedge \cdots \wedge x_n$. Conclude that the smallest $\mathbf{SL}_k(V)$-invariant k-subspace contains $x_{i_1} \wedge \cdots \wedge x_{i_n}$ for every subset $\{i_1, \dots, i_n\} \subset \{1, \dots, m\}$ of size n with the usual ordering $i_1 < \cdots < i_n$. Conclude that $\bigwedge^n_k(V)$ is an irreducible k-linear representation of $\mathbf{SL}_k(V)$ for every integer $0 \leq n \leq m$. (Note that this holds with no hypothesis on the characteristic of k.)

(c) For the natural action of $\mathfrak{S}_n$ on $V^n = V \times \cdots \times V$ by permuting factors, prove that there exists a unique k-linear action of $\mathfrak{S}_n$ on $T^n_k(V)$ such that the following set map is $\mathfrak{S}_n$-equivariant,

$$V^n = V \times \cdots \times V \to V \otimes_k \cdots \otimes_k V = T^n_k(V), \quad (v_1, \dots, v_n) \mapsto v_1 \otimes \cdots \otimes v_n.$$

(d) Let $k[\mathfrak{S}_n] \to A$ be a morphism of associative, unital k-algebras. Show that the induced left $k[\mathfrak{S}_n]$-module,

$$\mathbf{S}_A(V) := A \otimes_{k[\mathfrak{S}_n]} T^n_k(V),$$

is functorial in V, hence defines a k-linear (left) representation of $\mathbf{SL}_k(V)$. When k has characteristic 0 or positive characteristic $p > n$, for the simple algebra factors $k[\mathfrak{S}_n] \to A_\lambda = \text{Hom}_k(V_\lambda, V_\lambda)$ associated to an irreducible k-linear (left) representation V_λ of $\mathfrak{S}_n$, the corresponding functor $\mathbf{S}_\lambda$ is a **Schur functor**. The two most familiar examples are the symmetric power $\text{Sym}^n_k(V)$ corresponding to the trivial 1-dimensional representation of $\mathfrak{S}_n$ and the exterior power $\bigwedge^n_k(V)$ corresponding to the sign representation, i.e., the 1-dimensional representation whose associated group homomorphism to $\{-1, +1\} \subset k^\times$ is the sign homomorphism.

(e) In every characteristic, prove that there are unique k-linear actions of $\mathfrak{S}_n$ on $\mathrm{Sym}^n_k(V)$ and $\bigwedge^n_k(V)$ such that $s_{V,n}$ and $e_{V,n}$ are morphisms of k-linear representations of $\mathfrak{S}_n$.

(f) Assuming that k has characteristic 0 or positive characteristic $p > n$, prove that there exist morphisms of k-linear representations of $\mathfrak{S}_n$,

$$s^*_{V,n} : \mathrm{Sym}^n_k(V) \to T^n_k(V), \quad v_1 \cdots v_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)},$$

$$e^*_{V,n} : \bigwedge_k^n(V) \to T^n_k(V), \quad v_1 \wedge \cdots \wedge v_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)},$$

such that $s_{V,n} \circ s^*_{V,n}$ and $e_{V,n} \circ e^*_{V,n}$ equal the identity maps. For this reason, often the symmetric algebra and exterior algebra are considered as $\mathbb{Z}_{\geq 0}$-graded k-vector subspaces of the tensor algebra.

Nota bene. The maps s^*_V and e^*_V do *not* respect the product structures on the respective algebras.

Problem 4. (Hopf algebra structure on the tensor algebra.) Please read Problem 4 from Problem Set 6 about the Hopf algebra structure on the group k-algebra of a finite group Γ. This problem explains the construction of the Hopf algebra structure on the tensor algebra, $T^\bullet_k(V)$, which then induces a Hopf algebra structure on the symmetric algebra. When the k-vector space is a Lie k-algebra, this also induces a Hopf algebra structure on the universal enveloping algebra. The exterior algebra has a structure of "graded Hopf algebra". It is typically not a Hopf algebra ("graded Hopf algebras" are *not* the same as Hopf algebras with a grading), and the quotient map from the tensor algebra to the exterior algebra typically does not respect the comultiplication structure.

(a) For associative, unital k-algebras A and B, prove that there is a unique structure of associative, unital k-algebra on the tensor product $A \otimes_k B$ such that both of the following k-linear maps are morphisms of associative, unital k-algebras,

$$\alpha : A \to A \otimes_k B, \quad a \mapsto a \otimes 1,$$

$$\beta : B \to A \otimes_k B, \quad b \mapsto 1 \otimes b,$$

and the images **strictly commute**, i.e., $\alpha(a) \cdot \beta(b)$ equals $\beta(b) \cdot \alpha(a)$ for every $a \in A$ and for every $b \in B$. Prove that the triple $(A \otimes_k B, \alpha, \beta)$ is universal (i.e., initial) among all triples (R, α', β') of an associative, unital k-algebra R, and a pair of morphisms of associative, unital k-algebras,

$$\alpha' : A \to R, \quad \beta' : B \to R,$$

such that $\alpha'(a) \cdot \beta'(b)$ equals $\beta'(b) \cdot \alpha'(a)$ for every $a \in A$ and for every $b \in B$. Finally, if A and B are $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebras, prove that there exists a unique $\mathbb{Z}_{\geq 0}$-grading of $A \otimes_k B$ such that both α and β are morphisms of $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebras.

(b) Prove that there exists a unique morphism of associative, unital k-algebras,

$$\tau_{A,B} : A \otimes_k B \to B \otimes_k A,$$

permuting α and β. Check that $\tau_{A,B} \circ \tau_{B,A}$ equals the identity. In particular, when B equals A, this defines an automorphism of order 2 of associative, unital k-algebras,

$$\tau_A : A \otimes_k A \to A \otimes_k A.$$

If A and B are $\mathbb{Z}_{\geq 0}$-graded, check that also $\tau_{A,B}$ respects the induced $\mathbb{Z}_{\geq 0}$-gradings.

(c) Now consider the case when B equals A equals the $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebra $T_k^\bullet(V)$. Define $\Delta_{V,1}$ to be the k-linear transformation from V to the first graded piece of $T_k^\bullet(V) \otimes_k T_k^\bullet(V)$ given by

$$\Delta_{V,1} : V \to (V \otimes_k k) \oplus (k \otimes_k V), \quad v \mapsto v \otimes 1 + 1 \otimes v.$$

Denote the morphism of $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebras restricting as $\Delta_{V,1}$ on $T_k^1(V) = V$ by

$$\Delta_V : T_k^\bullet(V) \to T_k^\bullet(V) \otimes_k T_k^\bullet(V).$$

Prove that the following two morphisms of $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebras are equal,

$$T_k^\bullet(V) \xrightarrow{\Delta_V} T_k^\bullet(V) \otimes_k T_k^\bullet(V) \xrightarrow{\Delta_V \otimes \mathrm{Id}_{T(V)}} T_k^\bullet(V) \otimes_k T_k^\bullet(V) \otimes_k T_k^\bullet(V),$$

$$T_k^\bullet(V) \xrightarrow{\Delta_V} T_k^\bullet(V) \otimes_k T_k^\bullet(V) \xrightarrow{\mathrm{Id}_{T(V)} \otimes \Delta_V} T_k^\bullet(V) \otimes_k T_k^\bullet(V) \otimes_k T_k^\bullet(V).$$

Thus, the map Δ_V is **coassociative**.

(d) For the associative, unital k-algebra that equals k itself, for the zero homomorphism from V to k, denote by

$$\epsilon_V : T_k^\bullet(V) \to k,$$

the unique morphism of associative, unital k-algebras whose restriction to $T_k^1(V) = V$ equals the zero homomorphism. Prove that the following compositions both equal the identity map,

$$T_k^\bullet(V) \xrightarrow{\Delta_V} T_k^\bullet(V) \otimes_k T_k^\bullet(V) \xrightarrow{\epsilon_V \otimes \mathrm{Id}_{T(V)}} k \otimes_k T_k^\bullet(V) = T_k^\bullet(V),$$

$$T_k^\bullet(V) \xrightarrow{\Delta_V} T_k^\bullet(V) \otimes_k T_k^\bullet(V) \xrightarrow{\mathrm{Id}_{T(V)} \otimes \epsilon_V} T_k^\bullet(V) \otimes_k k = T_k^\bullet(V).$$

Thus, ϵ_V is a left-right **counit** for the comultiplication Δ_V.

(e) Denoting the algebra multiplication on $T_k^\bullet(V)$ by ∇_V and denoting the k-algebra map by $\eta_V : k \to T_k^\bullet(V)$, check that $(T_k^\bullet(V), \nabla_V, \eta_V, \Delta_V, \epsilon_V)$ forms a **bialgebra**. Explicitly, this means that each of the following diagrams commute,

$$\begin{array}{ccc} T_k^\bullet(V) \otimes_k T_k^\bullet(V) & \xrightarrow{\Delta_V \circ \nabla_V} & T_k^\bullet(V) \otimes_k T_k^\bullet(V) \\ \Delta_V \otimes \Delta_V \downarrow & & \uparrow \nabla_V \otimes \nabla_V \\ T_k^\bullet(V) \otimes_k T_k^\bullet(V) \otimes_k T_k^\bullet(V) \otimes_k T_k^\bullet(V) & \xrightarrow[\mathrm{Id}_{T(V)} \otimes \tau_V \otimes \mathrm{Id}_{T(V)}]{} & T_k^\bullet(V) \otimes_k T_k^\bullet(V) \otimes_k T_k^\bullet(V) \otimes_k T_k^\bullet(V) \end{array}.$$

$$\begin{array}{ccc} T_k^\bullet(V) \otimes_k T_k^\bullet(V) & \xrightarrow{\nabla_V} & T_k^\bullet(V) \\ \epsilon_V \otimes \epsilon_V \downarrow & & \downarrow \epsilon_V \\ k \otimes_k k & \xrightarrow[\cong]{} & k \end{array}.$$

$$\begin{array}{ccc} k & \xrightarrow{\cong} & k \otimes_k k \\ {\scriptstyle \eta_V}\downarrow & & \downarrow{\scriptstyle \eta_V \otimes \eta_V} \\ T_k^\bullet(V) & \xrightarrow[\Delta_V]{} & T_k^\bullet(V) \otimes_k T_k^\bullet(V) \end{array} .$$

$$\begin{array}{ccc} k & \xrightarrow{\mathrm{Id}} & k \\ {\scriptstyle \eta_V}\downarrow & & \uparrow{\scriptstyle \epsilon_V} \\ T_k^\bullet(V) & \xrightarrow[\mathrm{Id}_{T(V)}]{} & T_k^\bullet(V) \end{array} .$$

More briefly, it means that $(T_k^\bullet(V), \nabla_V, \eta_V)$ is an associative, unital k-algebra, and both of the following maps are morphisms of associative, unital k-algebras,

$$\epsilon_V : T_k^\bullet(V) \to k, \quad \Delta_V : T_k^\bullet(V) \to T_k^\bullet(V) \otimes_k T_k^\bullet(V),$$

where $T_k^\bullet(V) \otimes_k T_k^\bullet(V)$ is the algebra structure where the two factors *strictly commute* with each other. Equivalently, it means that $(T_k^\bullet(V), \Delta_V, \epsilon_V)$ is a coassociative, counital k-coalgebra and the morphisms η_V and ∇_V are morphisms of coassociative, counital k-algebras.

(f) For every associative, unital k-algebra $(A, \nabla_A : A \times A \to A)$, define A^{op} to be the k-vector space A with the following k-bilinear operation,

$$\nabla_A^{\mathrm{op}} : A \times A \to A, \quad \nabla_A^{\mathrm{op}}(a_1, a_2) := \nabla_A(a_2, a_1).$$

Prove that this is also an associative, unital k-algebra with the same left-right identity as in (A, ∇_A). Check that the following k-bilinear pairing,

$$\alpha_A : (A \otimes_k A^{\mathrm{op}}) \times A \to A, \quad (a \otimes c, b) \mapsto a \cdot b \cdot c = \nabla_A(a, \nabla_A(b, c)) = \nabla_A(\nabla_A(a, b), c).$$

defines a structure of *left* $A \otimes_k A^{\mathrm{op}}$-module on A. Prove that this is equivalent to a structure of *right* $A^{\mathrm{op}} \otimes_k A$-module on A (another name for such a structure is a $A - A$-bimodule). Check that the multiplication is a map of left $A \otimes_k A^{\mathrm{op}}$-modules,

$$\nabla_A : A \otimes A^{\mathrm{op}} \to A, \quad a \otimes c \mapsto a \cdot c = \nabla_A(a, c),$$

and check that the multiplication is also a map of right $A^{\mathrm{op}} \otimes_k A$-modules,

$$\nabla_A : A^{\mathrm{op}} \otimes A \to A, \quad c \otimes a \mapsto c \cdot a.$$

(g) Denote by S_V the unique morphism of $\mathbb{Z}_{\geq 0}$-associative, unital k-algebras,

$$S_V : T_k^\bullet(V) \to T_k^\bullet(V)^{\mathrm{op}},$$

whose restriction to $T_k^1(V) = V$ maps to the summand $T_k^1(V)$ and equals $-\mathrm{Id}_V$ (the negative of the identity map). This is the **antipode map**. Prove that both of the following compositions equal $\eta_V \circ \epsilon_V$ as k-linear maps from the $\mathbb{Z}_{\geq 0}$-graded k-vector space $T_k^\bullet(V)$,

$$T_k^\bullet(V) \xrightarrow{\Delta_V} T_k^\bullet(V) \otimes_k T_k^\bullet(V) \xrightarrow{S_V \otimes \mathrm{Id}_{T(V)}} T_k^\bullet(V) \otimes_k T_k^\bullet(V) \xrightarrow{\nabla_V} T_k^\bullet(V),$$

$$T_k^\bullet(V) \xrightarrow{\Delta_V} T_k^\bullet(V) \otimes_k T_k^\bullet(V) \xrightarrow{\mathrm{Id}_{T(V)} \otimes S_V} T_k^\bullet(V) \otimes_k T_k^\bullet(V) \xrightarrow{\nabla_V} T_k^\bullet(V).$$

Together with the previous operations, the antipode S_V makes $T_k^\bullet(V)$ into a **Hopf k-algebra**. **Hint.** For the upper commutative square, for $A = T_k^\bullet(V)$, first check that the right A-module structure on A induced from the right $A^{\mathrm{op}} \otimes A$-module structure via $(S_V \otimes \mathrm{Id}_{T(V)}) \circ \Delta_V$ restricts to a right action on A by $V = T_k^1(V)$

as an action by k-derivations, $(a, v) \mapsto v \otimes a - a \otimes v$, which annihilate the span of 1. Next, for the natural right module structure of $A^{\text{op}} \otimes_k A$ on itself, check that ∇_V sends the generator 1 to the element $1 \in A$ that is annihilated by each of these derivations. Conclude that the upper composition annihilates the left-right ideal of $T_k^\bullet(V)$ generated by $V = T_k^1(V)$.

(h) For a Hopf k-algebra $(A, \nabla, \eta, \Delta, \epsilon, S)$, a left-right ideal $I \subset A$ (for the multiplication ∇ on A) is a **Hopf ideal** if all of the following hold. The kernel of ϵ contains I. The image of I under the antipode map is contained in I. The image of I under Δ is contained in the k-subspace,

$$(I \otimes_k A) + (A \otimes_k I) \subseteq A \otimes_k A.$$

For the quotient algebra homomorphism $A \to A/I$, check that there is a unique structure of Hopf k-algebra on A/I making this quotient homomorphism a morphism of Hopf k-algebras if and only if the ideal is a Hopf ideal.

(i) For the tensor algebra considered as a $\mathbb{Z}_{\geq 0}$-graded k-vector space, check that the Hopf k-algebra structures are homogeneous operations of degree 0. Also check that Δ_V is cocommutative, i.e., the composition $\tau \circ \Delta_V$ equals Δ_V. Prove that the quotient of a Hopf k-algebra A by a Hopf ideal is commutative, resp. cocommutative, if A is commutative, resp. cocommutative. For a Hopf k-algebra with a $\mathbb{Z}_{\geq 0}$-grading such that all structures are homogeneous operations of degree 0, for a Hopf ideal that is a homogeneous ideal, prove that the quotient algebra has a $\mathbb{Z}_{\geq 0}$-grading making the quotient map homogeneous of degree 0 and such that all Hopf algebra structures are homogeneous of degree 0. Finally, if A is **connected**, i.e., the graded piece A_0 equals k, check that also the quotient is connected.

(j) For every pair $\ell, m \in \mathbb{Z}_{\geq 0}$, denote by $\Delta_V^{\ell,m}$ the graded component of Δ_V,

$$\Delta_V^{\ell,m} : T_k^{\ell+m}(V) \to T_k^\ell(V) \otimes_k T_k^m(V).$$

Denote by $P_{\ell,m}$ the set with $\binom{\ell+m}{m}$ elements that consist of ordered partitions (A, B) of $\{1, \dots, \ell+m\}$ into subsets A and B of respective cardinalities ℓ and m, say

$$A = (1 \leq a_1 < \cdots < a_\ell \leq \ell + m), \quad B = (1 \leq b_1 < \cdots < b_m \leq \ell + m).$$

Prove the formula,

$$\Delta_k^{\ell,m}(v_1 \otimes \cdots \otimes v_{\ell+m}) = \sum_{(A,B) \in P_{\ell,m}} (v_{a_1} \otimes \cdots \otimes v_{a_\ell}) \otimes (v_{b_1} \otimes \cdots \otimes v_{b_m}).$$

If the characteristic k equals 0 or $p > \ell + m$, check the following identity,

$$\Delta_k^{\ell,m}(s_{V,\ell+m}^*(v_1 \cdots v_{\ell+m})) = \sum_{(A,B) \in P_{\ell,m}} s_{V,\ell}^*(v_A) \otimes s_{V,m}^*(v_B),$$

$$v_A := v_{a_1} \cdots v_{a_\ell}, \quad v_B := v_{b_1} \cdots v_{b_m}.$$

Thus, the image of $\Delta_k^{\ell,m} \circ s_{V,\ell+m}^*$ is contained in the image of

$$s_{V,\ell}^* \otimes s_{V,m}^* : \mathrm{Sym}_k^\ell(V) \otimes_k \mathrm{Sym}_k^m(V).$$

Problem 5. (Hopf algebra structure on the symmetric algebra.) Recall that J_s is defined to be the left-right kernel ideal of the morphism of $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebras,

$$s : T_k^\bullet(V) \to \mathrm{Sym}_k^\bullet(V).$$

(a) Check that the graded left-right ideal $J_s \subset T_k^\bullet(V)$ is a Hopf ideal that is homogeneous. Deduce that there exists a unique structure of Hopf k-algebra on $\mathrm{Sym}_k^\bullet(V)$ for which s_V is a morphism of Hopf k-algebras, and every Hopf structure on $\mathrm{Sym}_k^\bullet(V)$ is homogeneous of degree 0. Also, deduce that this Hopf k-algebra structure is cocommutative and connected. The symmetric algebra is also commutative.

(b) For this Hopf k-algebra structure, check that the antipode map equals the unique morphism of $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras from $\mathrm{Sym}_k^\bullet(V)$ to itself (the same as the opposite algebra since the algebra is commutative) that restricts on $\mathrm{Sym}_k^1(V) = V$ as $-\mathrm{Id}_V$.

(c) Via the isomorphism of $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebras from the previous problems,

$$\mathrm{Sym}_k^\bullet(V) \otimes_k \mathrm{Sym}_k^\bullet(V) \cong \mathrm{Sym}_k^\bullet(V \oplus V),$$

check that Δ naturally corresponds to the unique morphism of $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras

$$\mathrm{Sym}_k^\bullet(V) \to \mathrm{Sym}_k^\bullet(V \oplus V),$$

functorially associated to the diagonal embedding of V in $V \oplus V$. If you know algebraic geometry, check that the Hopf k-algebra structure on $\mathrm{Sym}_k^\bullet(V)$ for a finite dimensional k-vector space V equals the Hopf k-algebra structure on the coordinate k-algebra of the dual k-vector space $V^\vee$ considered as a commutative group k-variety with group operation equal to the usual vector addition. (**Challenge problem**. If we instead identify a 3-dimensional vector space $V^\vee$ as the group of upper triangular, unipotent 3×3 matrices, how does the Hopf k-algebra structure on the commutative k-algebra $\mathrm{Sym}_k^\bullet(V)$ "deform"?)

(d) Check that $\nabla \circ \Delta$ is the unique morphism of $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras from $\mathrm{Sym}_k^\bullet(V)$ to itself that restricts on $\mathrm{Sym}_k^1(V) = V$ as $2\mathrm{Id}_V$. If k has characteristic 0 or positive characteristic $p > \ell + m$, check that $\nabla \circ \Delta^{\ell,m}$ equals the map that restricts on $\mathrm{Sym}_k^1(V) = V$ as $\binom{\ell+m}{m}$.

(e) Assuming that k has characteristic 0 or positive characteristic $p > n$, check that the following composite

$$\Delta_V^{\ell,m} \circ s_{V,n}^* : \mathrm{Sym}_k^n(V) \to T_k^n(V) \to T_k^\ell(V) \otimes_k T_k^m(V),$$

equals the bigraded component of the composite for the Hopf k-algebra structure on $\mathrm{Sym}_k^\bullet(V)$,

$$(s_{V,\ell}^* \otimes s_{V,m}^*) \circ \Delta^{\ell,m} : \mathrm{Sym}_k^n(V) \to \mathrm{Sym}_k^\ell(V) \otimes_k \mathrm{Sym}_k^m(V) \to T_k^\ell(V) \otimes_k T_k^m(V).$$

Problem 6. (Graded Hopf algebra structure on the exterior algebra.) Let $A = A_\bullet$ and $B = B_\bullet$ be $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras. As in **Problem 4(a)**, form the tensor product $A \otimes_k B$ with the k-linear maps $\alpha : A \to A \otimes_k B$ and $\beta : B \to A \otimes_k B$.

(a) Prove that there is a unique structure of associative, unital k-algebra on $A \otimes_k B$ such that α and β are morphisms of associative, unital k-algebras, and the images **graded commute**, i.e., for every $\ell, m \in \mathbb{Z}_{\geq 0}$, for every $a \in A_\ell$, and for every $b \in B_m$,

$$\beta_m(b) \cdot \alpha_\ell(a) = (-1)^{\ell \cdot m} \alpha_\ell(a) \cdot \beta_m(b).$$

Prove that the triple $(A \otimes_k B, \alpha, \beta)$ is universal (i.e., initial) among all triples (R, α', β') of an associative, unital k-algebra R, and a pair of morphisms of associative, unital k-algebras

$$\alpha' : A \to R, \quad \beta' : B \to R,$$

such that $\beta'_m(b) \cdot \alpha'_\ell(a) = (-1)^{\ell \cdot m} \alpha'_\ell(a) \cdot \beta'_m(b)$ for every $\ell, m \in \mathbb{Z}_{\geq 0}$ and for every $(a, b) \in A_\ell \times B_m$. If either A or B has nonzero graded components only in even degrees, check that this algebra structure is the same as the algebra structure from the previous exercise.

(b) For k-vector spaces V and W, for the natural inclusions

$$\alpha'_1 : V \to V \oplus W, \quad \beta'_1 : W \to V \oplus W,$$

by the universal property of the exterior algebra, these extend uniquely to morphisms of $\mathbb{Z}_{\geq 0}$-graded, associative, unital, graded commutative k-algebras,

$$\alpha' : \bigwedge\nolimits_k^{\bullet}(V) \to \bigwedge\nolimits_k^{\bullet}(V \oplus W), \quad \beta' : \bigwedge\nolimits_k^{\bullet}(W) \to \bigwedge\nolimits_k^{\bullet}(V \oplus W),$$

whose restrictions to the degree 1 graded summands equal α'_1 and β'_1 respectively. By the universal property in (a), there is a unique morphism of associative, unital k-algebras,

$$e^{\bullet}_{V,W} : \bigwedge\nolimits_k^{\bullet}(V) \otimes_k \bigwedge\nolimits_k^{\bullet}(W) \to \bigwedge\nolimits_k^{\bullet}(V \oplus W),$$

such $e^{\bullet}_{V,W} \circ \alpha$ equals α' and such that $e^{\bullet}_{V,w} \circ \beta$ equals β'. Check that $e^{\bullet}_{V,W}$ is an isomorphism of $\mathbb{Z}_{\geq 0}$-graded associative, unital, graded commutative k-algebras. Check that this is compatible with the isomorphism from **Problem 2(b)**.

(c) This exercise is for those that know about the Künneth formula in algebraic topology. Let X and Y be topological spaces. Denote the product topological space by

$$(X \times Y, \chi : X \times Y \to X, \upsilon : X \times Y \to Y).$$

Since cohomology $H^*(-; k)$ is a contravariant functor from the category of topological spaces to the category of $\mathbb{Z}$-graded, associative, unital, graded commutative cohomology algebras, there are induced pullback maps,

$$H^*(\chi; k) : H^*(X; k) \to H^*(X \times Y; k), \quad H^*(\upsilon; k) : H^*(Y, k) \to H^*(X \times Y; k).$$

By the universal property in (a), there is an induced morphism of $\mathbb{Z}_{\geq 0}$-graded, associative, unital, graded commutative algebras, for the product from (a),

$$H^*(X; k) \otimes_k H^*(Y; k) \to H^*(X \times Y; k).$$

The Künneth Theorem states that this morphism of $\mathbb{Z}$-graded, associative, unital, graded commutative algebras is an isomorphism (but this is only valid with the product from (a), not for the strictly commuting product).

(d) For $\mathbb{Z}$-graded associative, unital k-algebras A and B, check that there is a unique $\mathbb{Z}_{\geq 0}$-grading on $A \otimes_k B$ making both α and β into morphisms of $\mathbb{Z}_{\geq 0}$-graded, associative, unital k-algebras. For the graded commuting product on $A \otimes_k B$ and $B \otimes_k A$ from (a), prove that there exists a unique morphism of associative, unital k-algebras

$$\tau'_{A,B} : A \otimes_k B \to B \otimes_k A,$$

permuting α and β. Note that this does **not** usually equal the morphism from **Problem 4(b)**. The two are related as follows,

$$\tau'_{A,B}(a_\ell \otimes b_m) = (-1)^{\ell\cdot m}\tau_{A,B}(a_\ell \otimes b_m) = (-1)^{\ell\cdot m} b_m \otimes a_\ell,$$

and thus they are equal if either A or B has nonzero graded components only in even degrees. Check that $\tau'_{A,B}$ is an isomorphism of $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras, and $\tau'_{B,A} \circ \tau'_{A,B}$ equals the identity.

(e) If A and B are each graded commutative, check that also $A \otimes_k B$ is graded commutative.

(f) Now consider the case of (b) when W equals V. For the natural diagonal inclusion,

$$\Delta'_1 : V \to V \oplus V, \quad \Delta'_1 = \alpha'_1 + \beta'_1,$$

by the universal property of the exterior algebra, this extends uniquely to a morphism of $\mathbb{Z}_{\geq 0}$-graded, associative, unital, graded commutative k-algebras,

$$\Delta'_V : \bigwedge\nolimits_k^{\bullet}(V) \to \bigwedge\nolimits_k^{\bullet}(V \oplus V)$$

whose restriction to the degree 1 graded summands equal Δ'_1. Thus, there exists a unique morphism of $\mathbb{Z}_{\geq 0}$-graded, associative, unital, graded commutative k-algebras,

$$\Delta_{e,V} : \bigwedge\nolimits_k^{\bullet}(V) \to \bigwedge\nolimits_k^{\bullet}(V) \otimes_k \bigwedge\nolimits_k^{\bullet}(V),$$

such that $e_{V,V} \circ \Delta_{e,V}$ equals Δ'_V. Check that Δ is a coassociative comultiplication that is graded cocommutative. Since k, concentrated in degree 0, is a $\mathbb{Z}_{\geq 0}$-graded associative, unital, graded commutative k-algebra, prove that the morphism $\epsilon_V : T_k^{\bullet}(V) \to k$ factors uniquely through a morphism of $\mathbb{Z}_{\geq 0}$-graded associative, unital, graded commutative k-algebras,

$$\epsilon_{e,V} : \bigwedge\nolimits_k^{\bullet}(V) \to k.$$

Altogether, $(\bigwedge_k^{\bullet}(V), \nabla_{e,V}, \eta_{e,V}, \Delta_{e,V}, \epsilon_{e,V})$ is a $\mathbb{Z}_{\geq 0}$-graded k-bialgebra. Moreover, it is graded commutative and graded cocommutative. Once more, because the graded commuting product is different from the strictly commuting product, a $\mathbb{Z}_{\geq 0}$-graded k-bialgebra is not the same as a k-bialgebra with a $\mathbb{Z}_{\geq 0}$-grading such that all bialgebra operations are homogeneous of degree 0; these notions are the same, however, if the only nonzero components for the grading occur in even degree.

(g) Since $\bigwedge_k^{\bullet}(V)$ is graded commutative, the product for the opposite algebra in the category of $\mathbb{Z}_{\geq 0}$-graded k-algebras is the same as the usual product. Define the antipode map $S_{e,V}$ to be the identity map. Check the analogue of the identities from **Problem 4(g)** for S. Together with the previous operations, the antipode $S_{e,V}$ makes $\bigwedge_k^{\bullet}(V)$ into a $\mathbb{Z}_{\geq 0}$-**graded Hopf** k-**algebra**.

(h) By way of caution, check that the projection from $T_k^{\bullet}(V)$ to $\bigwedge_k^{\bullet}(V)$, which is a morphism of $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras, preserves the comultiplication Δ only if $\dim(V) \leq 1$.

Problem 7. (Filtered algebras.) A $\mathbb{Z}_{\geq 0}$**-filtered** k**-algebra** is a pair $(A, (F_nA)_{n\geq 0})$ of an associative, unital k-algebra A and an increasing, exhaustive filtration of A by k-subspaces,

$$\{0\} = F_{-1}A \subseteq F_0A \subseteq \cdots \subseteq F_nA \subseteq F_{n+1}A \subseteq \cdots \subseteq A, \quad \cup_{n=0}^{\infty} F_nA = A,$$

such that for every $m, n \in \mathbb{Z}_{\geq 0}$, the image of $F_mA \times F_nA$ under the multiplication map is contained in $F_{m+n}A$. For filtered k-algebras $(A, F_\bullet A)$ and $(B, E_\bullet B)$, a **morphism of filtered** k**-algebras** is a morphism of associative, unital k-algebras,

$$f : A \to B,$$

such that for every $n \in \mathbb{Z}_{\geq 0}$, the f-image of F_nA is contained in E_nB.

(a) Check that the identity Id_A is a self-morphism for every $\mathbb{Z}_{\geq 0}$-filtered k-algebra $(A, F_\bullet A)$. Also, check that every composition of morphisms of $\mathbb{Z}_{\geq 0}$-filtered k-algebras is a morphism of $\mathbb{Z}_{\geq 0}$-filtered k-algebras. Conclude that these operations define a category of $\mathbb{Z}_{\geq 0}$-filtered k-algebras.

(b) For every $\mathbb{Z}_{\geq 0}$-filtered k-algebra $(A, F_\bullet A)$, for every integer $n \geq 0$, consider the following quotient k-vector space,

$$\mathrm{Gr}_n^F A := F_nA/F_{n-1}A, \quad \rho_{(A,FA),n} : F_nA \twoheadrightarrow \mathrm{Gr}_n^F A.$$

For every pair $m, n \in \mathbb{Z}_{\geq 0}$, prove that there exists a unique k-bilinear operation,

$$\mu_{m,n} : \mathrm{Gr}_m^F A \times \mathrm{Gr}_n^F A \to \mathrm{Gr}_{m+n}^F A,$$

such that $\mu_{m,n}(\rho_m(a), \rho_n(b))$ equals $\rho_{m+n}(a \cdot b)$ for every $a \in F_mA$ and for every $b \in F_nA$. Check that these maps assemble to a multiplication rule on the $\mathbb{Z}_{\geq 0}$-graded k-vector space,

$$\mathrm{Gr}_\bullet^F A := \bigoplus_{n\geq 0} \mathrm{Gr}_n^F A.$$

Check that this multiplication rule makes $\mathrm{Gr}_\bullet^F A$ into a $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebra.

(c) For every morphism of $\mathbb{Z}_{\geq 0}$-filtered k-algebras,

$$f : (A, F_\bullet A) \to (B, E_\bullet A),$$

for every $n \in \mathbb{Z}_{\geq 0}$, prove that there exists a unique k-linear map,

$$\mathrm{Gr}_n^{F,E} f : \mathrm{Gr}_n^F A \to \mathrm{Gr}_n^E B,$$

such that $\mathrm{Gr}_n^{F,E} f(\rho_{A,F,n}(a))$ equals $\rho_{B,E,n}(f(a))$ for every $a \in F_nA$. Prove that these k-linear maps define a morphism of $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras,

$$\mathrm{Gr}_\bullet^{F,E} f : \mathrm{Gr}_\bullet^F A \to \mathrm{Gr}_\bullet^E B.$$

Also prove that $\mathrm{Gr}_\bullet^{F,E} f$ is surjective if and only if every $f(F_nA) + E_{n-1}B$ equals E_nB; this holds if every $f(F_nA)$ equals E_nB.

(d) Check that the operation $f \mapsto \mathrm{Gr}_\bullet^{F,E} f$ sends the identity self-morphism of $(A, F \bullet A)$ to the identity self-morphism of $\mathrm{Gr}_\bullet^F A$. Check that the operation preserves compositions. Altogether, the rule associating $\mathrm{Gr}_\bullet^F A$ to every $(A, F_\bullet A)$ and associating $\mathrm{Gr}_\bullet^{F,E} f$ to every f is a covariant functor from the category of $\mathbb{Z}_{\geq 0}$-filtered associative, unital k-algebras to the category of $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras.

(e) For every $\mathbb{Z}_{\geq 0}$-filtered k-algebra, $(A, F_\bullet A)$, and for every surjection of associative, unital k-algebras, $p : A \to C$, check that the induced filtration on A/I,

$$p_* F_n C := p(F_n A),$$

is a structure of $\mathbb{Z}_{\geq 0}$-filtered k-algebra on C such that p is a morphism of $\mathbb{Z}_{\geq 0}$-filtered k-algebras. Moreover, for every $\mathbb{Z}_{\geq 0}$-filtered k-algebra $(B, E_\bullet B)$, for every morphism of associative, unital k-algebras, $g : C \to B$, such that $f = g \circ p$ is a morphism of $\mathbb{Z}_{\geq 0}$-filtered k-algebras, $f : (A, F_\bullet A) \to (B, E_\bullet B)$, prove that also g is a morphism of $\mathbb{Z}_{\geq 0}$-filtered k-algebras, $g : (C, p_* F_\bullet C) \to (B, E_\bullet B)$. Also, check that the induced morphism of $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras is surjective,

$$\mathrm{Gr}^F_\bullet p : \mathrm{Gr}^F_\bullet A \twoheadrightarrow \mathrm{Gr}^{p_* F}_\bullet C.$$

Finally, check that $\mathrm{Gr}^{p_* F}_\bullet C$ is commutative if and only if I is **nearly commuting**, i.e., for every $\ell, m \in \mathbb{Z}_{\geq 0}$, for every $a \in F_\ell A$, for every $b \in F_m A$, the commutator $[a, b]_A := a \cdot b - b \cdot a$ in $F_{\ell+m} A$ is contained in the k-subspace $F_{\ell+m} A \cap I + F_{\ell+m-1} A$.

(f) Conversely, for every $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebra $A_\bullet$, for every integer n, consider the following k-subspace of $A_\bullet$,

$$F_n A := A_{\leq n} = \bigoplus_{m=0}^{n} A_m \subset A_\bullet.$$

Prove that this sequence of k-subspaces makes $(A_\bullet, F_\bullet A)$ into a $\mathbb{Z}_{\geq 0}$-filtered k-algebra.

(g) For every morphism of $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras,

$$f_\bullet : A_\bullet \to B_\bullet, \quad f_n : A_n \to B_n,$$

check that $f_\bullet$ is also a morphism of the associated $\mathbb{Z}_{\geq 0}$-filtered k-algebras. Check that these rules define a covariant functor from the category of $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras to the category of $\mathbb{Z}_{\geq 0}$-filtered k-algebras. Check that the composition of this functor with the previous functor is naturally equivalent to the identity functor on the category of $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras.

Problem 8. (Hopf algebra structure on the universal enveloping algebra.) Let $\mathfrak{g}$ be a Lie k-algebra, i.e., a k-vector space together with a skew-symmetric, k-bilinear operation,

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (X, Z) \mapsto \mathrm{ad}_X(Z),$$

that satisfies the Jacobi identity, i.e., for every $X, Y \in \mathfrak{g}$, the following k-linear self-maps of $\mathfrak{g}$ are equal,

$$\mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_Y \circ \mathrm{ad}_X = \mathrm{ad}_{[X,Y]}.$$

As usual, denote by $I_\mathfrak{g}$ the left-right ideal in $T^\bullet_k(\mathfrak{g})$ generated by all elements $q_{\mathfrak{g},2}(X \otimes Y - Y \otimes X) - q_{\mathfrak{g},1}([X, Y])$ for $X, Y \in \mathfrak{g}$. The **universal enveloping k-algebra of** $\mathfrak{g}$, $U(\mathfrak{g})$, is defined to be the quotient of $T^\bullet_k(V)$ by the left-right ideal $I_\mathfrak{g}$,

$$p_\mathfrak{g} : T^\bullet_k(\mathfrak{g}) \to U(\mathfrak{g}).$$

For the $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebra $T^\bullet_k(\mathfrak{g})$, denote by $F_\bullet T_k(\mathfrak{g})$ the associated structure of $\mathbb{Z}_{\geq 0}$-filtered k-algebra,

$$F_n T^\bullet_k(\mathfrak{g}) := \bigoplus_{\ell=0}^{n} T^\ell_k(\mathfrak{g}), \quad F_{-1} T^\bullet_k(\mathfrak{g}) := \{0\}.$$

Denote by $F_\bullet U(\mathfrak{g})$ the associated structure of $\mathbb{Z}_{\geq 0}$-filtered k-algebra on the quotient associative, unital k-algebra,

$$F_n U(\mathfrak{g}) := p_{\mathfrak{g}}(F_n T_k^\bullet(\mathfrak{g})).$$

(a) For the morphism of $\mathbb{Z}_{\geq 0}$-filtered associative, unital k-algebras,

$$p_{\mathfrak{g}} : (T_k^\bullet(\mathfrak{g}), F_\bullet T_k(\mathfrak{g})) \to (U(\mathfrak{g}), F_\bullet U(\mathfrak{g})),$$

there exists an associated morphism of $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebras,

$$\mathrm{Gr}^F_\bullet p_{\mathfrak{g}} : T_k^\bullet(\mathfrak{g}) \to \mathrm{Gr}^F_\bullet U(\mathfrak{g}).$$

Since each map $p_{\mathfrak{g}} : F_n T_k(\mathfrak{g}) \to F_n U(\mathfrak{g})$ is surjective, conclude that $\mathrm{Gr}_\bullet p_{\mathfrak{g}}$ is surjective. Since the $\mathbb{Z}_{\geq 0}$-graded associative, unital k-algebra $T_k^\bullet(\mathfrak{g})$ is generated in degree 1, conclude that $\mathrm{Gr}^F_\bullet U(\mathfrak{g})$ is also generated by the images of the degree 1 elements $T_k^1(\mathfrak{g}) = \mathfrak{g}$.

(b) For a $\mathbb{Z}_{\geq 0}$-filtered k-algebra $(A, F_\bullet A)$ such that the k-subalgebra $F_0 A$ is commutative and such that the k-subspace $F_1 A$ generates A as an $F_0 A$-algebra, check that a left-right ideal $I \subset A$ is nearly commuting if and only if for every $X, Y \in F_1 A$, the commutator $[X, Y]$ in $F_2 A$ is contained in $F_2 A \cap I + F_1 A$. Use this to prove that the ideal $I_{\mathfrak{g}}$ in $T_k^\bullet(\mathfrak{g})$ is nearly commuting for the natural $\mathbb{Z}_{\geq 0}$-filtration on $T_k^\bullet(\mathfrak{g})$. Thus, there is a unique surjective morphism of $\mathbb{Z}_{\geq 0}$-graded associative, unital, commutative k-algebras,

$$p_{s,\mathfrak{g},\bullet} : \mathrm{Sym}_k^\bullet(\mathfrak{g}) \twoheadrightarrow \mathrm{Gr}^F_\bullet U(\mathfrak{g}),$$

that factors $\mathrm{Gr}^F_\bullet p_{\mathfrak{g}}$.

(c) Now assume that k has characteristic 0 or positive characteristic $p > n$. Identify $\mathrm{Sym}_k^n(\mathfrak{g})$ with the image of $s^*_{\mathfrak{g},n}$ in $T_k^n(\mathfrak{g})$. Conclude that the kernel of $p_{s,\mathfrak{g},n}$ equals the inverse image under $s^*_{\mathfrak{g},n}$ of $I_{\mathfrak{g}} \cap T_k^n(\mathfrak{g})$. Therefore $p_{s,\mathfrak{g},n}$ is injective if and only if the following map is injective,

$$p_{\mathfrak{g}} \circ s^*_{\mathfrak{g},\leq n} : F_n \mathrm{Sym}_k^\bullet(\mathfrak{g}) \to F_n T_k^\bullet(\mathfrak{g}) \to F_n U(\mathfrak{g}).$$

In particular, in characteristic 0, conclude that

$$p_{s,\mathfrak{g}} : \mathrm{Sym}_k^\bullet(\mathfrak{g}) \to \mathrm{Gr}^F_\bullet U(\mathfrak{g})$$

is an isomorphism (i.e., injective since we already know it is surjective) if and only if the following map is injective,

$$p_{\mathfrak{g}} \circ s^*_{\mathfrak{g}} : \mathrm{Sym}_k^\bullet(\mathfrak{g}) \to T_k^\bullet(\mathfrak{g}) \to U(\mathfrak{g}).$$

Injectivity of this second map is the weak formulation of the Poincaré-Birhoff-Witt Theorem.

(d) Assuming injectivity of the map $p_{\mathfrak{g}} \circ s^*_{\mathfrak{g}}$ so that also $p_{s,\mathfrak{g}}$ is an isomorphism, use induction on n to prove that every map $p_{\mathfrak{g}} \circ s^*_{\mathfrak{g},\leq n}$ is also surjective. Conclude that the map $p_{\mathfrak{g}} \circ s^*_{\mathfrak{g}}$ is an isomorphism. Isomorphism of this map is the strong formulation of the Poincaré-Birhoff-Witt Theorem.

(e) For the Hopf k-algebra structure defined on $T_k^\bullet(\mathfrak{g})$, check that the left-right ideal $I_{\mathfrak{g}}$ is a Hopf ideal. Conclude that there is a unique Hopf k-algebra structure on $U(\mathfrak{g})$ such that $p_{\mathfrak{g}}$ is a morphism of Hopf k-algebras.

(f) For a k-vector space V, a left-right ideal $I \subset T_k^\bullet(V)$ is of **PBW type** if I is a Hopf ideal, i.e., $T_k^\bullet(V)/I$ has a structure of Hopf k-algebras such that the surjection $p_I : T^\bullet_k(V) \to T^\bullet_k(V)$ is a morphism of Hopf k-algebras, if I is nearly commuting, i.e., the associated graded k-algebra $\text{Gr}^{p_*F}_\bullet(T^\bullet_k(V)/I)$ is commutative, and if $F_1 T^\bullet_k(V) \cap I$ is the zero subspace. Assuming Ado's Theorem, prove that $F_1 T^\bullet_k(\mathfrak{g}) \to U(\mathfrak{g})$ is injective, so that $I_{\mathfrak{g}}$ is of PBW type.

(g) Let I be a left-right ideal of PBW type. Let $n > 1$ be an integer. Assume that k has characteristic 0 or positive characteristic $> n$. By way of induction, assume that for every integer $1 \leq m \leq n-1$, also $p_I \circ F_m s_V^*$ is injective. Thus, for every $m = 1, \dots, n-1$, the following composite is injective,

$$F^{n-m}\text{Sym}_k(V) \otimes_k F^m \text{Sym}_k(V) \xrightarrow{F_{n-m}s_V^* \otimes F_m s_V^*} F_{n-m}T_k(V) \otimes_k F_m T_k(V) \xrightarrow{p_I \otimes p_I} F_{n-m}T_k(V)/I \otimes_k F_m T_k(V)/I.$$

Use surjectivity of the induced morphism $p_{s,I} : \text{Sym}^\bullet_k(V) \to \text{Gr}^{p_*F}_\bullet(T_k(V)/I)$ and induction on m to prove that for every $m = 1, \dots, n-1$, also the composite

$$p_I \circ F_{\leq m} s_V^* : F_m \text{Sym}^\bullet_k(V) \to F_m T_k(V) \to F_m T_k(V)/I$$

is surjective and the following map is a bijection,

$$p_{s,I,m} : \text{Sym}^m_k(V) \to \text{Gr}^{p_*F}_m(T_k(V)/I).$$

(h) With the same hypotheses as above, by way of contradiction, assume that there exists nonzero $a \in F_n \text{Sym}^\bullet_k(V)$ that is in the kernel of $p_I \circ F_{\leq n} s_V^*$. Use the induction hypothesis to conclude that the component a_n in $\text{Sym}^n_k(V)$ is nonzero and maps to zero in $\text{Gr}^{p_*F}_n T_k(V)/I$. Since p preserves the Hopf algebra structures, conclude that also $\Delta(a)$ is in the kernel of $p_I \otimes p_I$. By hypothesis, the components $a \otimes 1$ and $1 \otimes a$ map to $0 \otimes 1 = 0$ and $1 \otimes 0 = 0$. Thus, conclude that also $\Delta(a) - (a \otimes 1 + 1 \otimes a)$ is in the kernel of $p_I \otimes p_I$.

(i) Conclude that for every $1 \leq m \leq n-1$, also the $(n-m, m)$-component of $\Delta(s^*_{V,n} a_n)$ maps to zero in $\text{Gr}^{p_*F}_{n-m} T_k(V)/I \otimes_k \text{Gr}^{p_*F}_m T_k(V)/I$. Using the induction hypothesis, conclude that every $(n-m, m)$-component of $\Delta^{n-m,m}(s^*_{V,n} a_n)$ equals 0. Finally, use **Problem 5(d)** to conclude that a_n equals 0, contrary to hypothesis. By way of induction, conclude that also $p_I \circ F_{\leq n} s_V^*$ is injective. Combined with (g) above, assuming that k has characteristic 0 and assuming Ado's Theorem, conclude the Poincaré-Birkhoff-Witt Theorem for Lie algebras, the induced map

$$p_I \circ s^*_{\mathfrak{g}} : \text{Sym}^\bullet_k(\mathfrak{g}) \to U(\mathfrak{g})$$

is a bijection of $\mathbb{Z}_{\geq 0}$-filtered k-vector spaces.

(j) Since the adjoint $\mathfrak{g}$-action on $T^n_k(\mathfrak{g})$ acts through $\text{Hom}_k(V, V)$, use **Problem 3** to conclude that $s^*_{\mathfrak{g}} \text{Sym}^\bullet_k(\mathfrak{g})$ is a $\mathfrak{g}$-subrepresentation. Conclude that $p_I \circ s^*_{\mathfrak{g}}$ is a morphism of $\mathfrak{g}$-representations. Thus, by Poincaré-Birkhoff-Witt, this is an isomorphism of $\mathfrak{g}$-representations. In particular, the k-subspace of invariants for the adjoint $\mathfrak{g}$-action on $U(\mathfrak{g})$ is the isomorphic image of the the invariants for the adjoint $\mathfrak{g}$-action on $\text{Sym}^\bullet_k(\mathfrak{g})$.

Nota bene. This proof of Poincaré-Birkhoff-Witt is adapted from an answer by David Speyer to a MathOverflow question.