MAT 552 PROBLEM SET 6

Problem 1. (**Duals of irreducibles are irreducible**.) For every representation of a Lie group, prove that the representation is indecomposable, resp. irreducible, resp. completely reducible, if and only if the dual representation also has this property. Show by example that a tensor product of two indecomposable representations, resp. irreducible representations, need not have this property. (Hint. Consider the identity and the trace for the tensor product of a representation with its dual.)

Problem 2. (**Natural decomposition into irreducibles**.) Let G be a Lie group. Let (U, σ) and (V, ρ) be $\mathbb{C}$-linear (left) G-representation. The following $\mathbb{C}$-bilinear map,

$$\mathrm{Hom}_{\mathbb{C}}(U, V) \times U \to V, \quad ((T : U \to V), u) \mapsto T(u),$$

induces a $\mathbb{C}$-linear map,

$$c_{U,V} : \mathrm{Hom}_{\mathbb{C}}(U, V) \otimes_{\mathbb{C}} U \to V.$$

This is natural in both U and V.

(a) For the induced $\mathbb{C}$-linear (left) G-representations on Hom and tensor product of representations, prove that $c_{U,V}$ is a morphism of $\mathbb{C}$-linear (left) G-representations. In particular, conclude that for the G-invariant subrepresentation,

$$\mathrm{Hom}_{\mathrm{Rep}^{\mathbb{C}}_G}((U, \sigma), (V, \rho)) \subseteq \mathrm{Hom}_{\mathbb{C}}(U, V),$$

the following restriction of $c_{U,V}$ is a morphism of $\mathbb{C}$-linear (left) G-representations,

$$H \otimes_{\mathbb{C}} U \to V, \quad H := \mathrm{Hom}_{\mathrm{Rep}^{\mathbb{C}}_G}((U, \sigma), (V, \rho)).$$

(b) Denote by $I = I(V, \rho)$ the finite set of isomorphism classes of irreducible $\mathbb{C}$-linear G-representations (V_i, ρ_i) that are isomorphic to a $\mathbb{C}$-linear G-subrepresentation of (V, ρ). Conclude the existence of a natural morphism of $\mathbb{C}$-linear G-representations,

$$a_{(V,\rho)} : \bigoplus_{i \in I} H_i \otimes_{\mathbb{C}} (V_i, \rho_i) \to (V, \rho), \quad H_i := \mathrm{Hom}_{\mathrm{Rep}^{\mathbb{C}}_G}((V_i, \rho_i), (V, \rho)).$$

(c) Complete the proof from lecture of the corollary of Schur's Lemma: (V, ρ) is completely reducible if and only if $a_{(V,\rho)}$ is an isomorphism.

(d) When (V, ρ) is completely reducible, use Schur's Lemma to prove that the isomorphism $a_{(V,\rho)}$ induces a natural decomposition of unital, associative, $\mathbb{C}$-algebras,

$$\widetilde{a}_{(V,\rho)} : \mathrm{Hom}_{\mathrm{Rep}^{\mathbb{C}}_G}((V, \rho), (V, \rho)) \xrightarrow{\cong} \prod_{i \in II} \mathrm{Hom}_{\mathbb{C}}(H_i, H_i).$$

In particular, conclude an isomorphism of the centers of these algebras,

$$Z(\widetilde{a}_{(V,\rho)}) : Z(\mathrm{Hom}_{\mathrm{Rep}^{\mathbb{C}}_G}((V, \rho), (V, \rho))) \xrightarrow{\cong} \prod_{i \in I} \mathbb{C} \cdot \mathrm{Id}_{H_i}.$$

(e) In particular, conclude that the cardinality of I equals the $\mathbb{C}$-vector space dimension of the center of this algebra. For each $i \in I$, the **multiplicity** of (V_i, ρ_i) in (V, ρ) is defined to be $m_i := \dim_{\mathbb{C}}(H_i)$. Conclude that the algebra has $\mathbb{C}$-vector space dimension equal to

$$\sum_{i \in I} m_i^2.$$

Problem 3. (**Schur's Lemma and Weyl's Trick for Lie algebras**.) Formulate the analogue of Schur's Lemma, the Weyl Unitarian Trick, etc., for real Lie algebras instead of real Lie groups. In particular, for a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ whose associated simply connected, real Lie group $G_{\mathbb{R}}$ is compact, conclude that all finite dimensional $\mathbb{C}$-linear, (left) $\mathfrak{g}$-modules are completely reducible. Also conclude that for the associated complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$, the same holds for finite dimensional, $\mathbb{C}$-linear, (left) $\mathfrak{g}_{\mathbb{C}}$-modules. Finally, conclude that the same holds for every simply connected, complex Lie group $G_{\mathbb{C}}$ whose Lie algebra is a complexification of $\mathfrak{g}_{\mathbb{R}}$.

Problem 4. (**Hopf algebra structure on the group algebra**.) Let Γ be a finite group considered as a (compact, totally disconnected) Lie group of dimension 0. As usual, denote the identity element by e. Denote the unital, associative group $\mathbb{C}$-algebra by $(\mathbb{C}[\Gamma], \mathbf{b}_e, *)$.

The **trace** (sometimes called the **counit**) is defined to be

$$\mathrm{Tr}_{\Gamma} : \mathbb{C}[\Gamma] \to \mathbb{C}, \quad \sum_{g \in \Gamma} z_g \mathbf{b}_g \mapsto \sum_{g \in \Gamma} z_g.$$

The **comultiplication** is defined to be

$$\Delta_{\Gamma} : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma], \quad \sum_{g \in \Gamma} z_g \mathbf{b}_g \mapsto \sum_{g \in \Gamma} z_g (\mathbf{b}_g \otimes \mathbf{b}_g).$$

The **antipode** is defined to be

$$S_{\Gamma} : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma], \quad \sum_{g \in \Gamma} z_g \mathbf{b}_g \mapsto \sum_{g \in \Gamma} z_g \mathbf{b}_{g^{-1}}.$$

Check that these operations (together with the usual unital, associated $\mathbb{C}$-algebra operations) make $\mathbb{C}[\Gamma]$ into a *Hopf* $\mathbb{C}$*-algebra*. Precisely, check all of the following.

(a) The comultiplication is coassociative, i.e., the following two compositions are equal,

$$\mathbb{C}[\Gamma] \xrightarrow{\Delta_{\Gamma}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \xrightarrow{\Delta_{\Gamma} \otimes \mathrm{Id}} (\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]) \otimes_{\mathbb{C}} \mathbb{C}[\Gamma],$$

$$\mathbb{C}[\Gamma] \xrightarrow{\Delta_{\Gamma}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \xrightarrow{\mathrm{Id} \otimes \Delta_{\Gamma}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} (\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]).$$

(b) The counit is a left-right coidentity, i.e., the following two compositions both equal the identity map,

$$\mathbb{C}[\Gamma] \xrightarrow{\Delta_{\Gamma}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \xrightarrow{\mathrm{Tr}_{\Gamma} \otimes \mathrm{Id}} \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] = \mathbb{C}[\Gamma],$$

$$\mathbb{C}[\Gamma] \xrightarrow{\Delta_{\Gamma}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \xrightarrow{\mathrm{Id} \otimes \mathrm{Tr}_{\Gamma}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}[\Gamma].$$

(c) The unital, associative $\mathbb{C}$-algebra structure and the counital, coassociative $\mathbb{C}$-coalgebra structure satisfy the axioms of a **bialgebra**, i.e., each of the following diagram commute.

$$
\begin{tikzcd}
\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] \arrow[r, "\Delta_\Gamma\circ(-*-)"] \arrow[d, "\Delta\otimes\Delta_\Gamma\otimes\Delta_\Gamma"'] & \mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] \\
\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] \arrow[r, "\mathrm{pr}_1\otimes\mathrm{pr}_3\otimes\mathrm{pr}_2\otimes\mathrm{pr}_4"'] & \mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] \arrow[u, "(-*-)\otimes(-*-)"']
\end{tikzcd}.
$$

$$
\begin{tikzcd}
\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] \arrow[r, "(-*-)"] \arrow[d, "\mathrm{Tr}_\Gamma\otimes\mathrm{Tr}_\Gamma"'] & \mathbb{C}[\Gamma] \arrow[d, "\mathrm{Tr}_\Gamma"] \\
\mathbb{C}\otimes_{\mathbb{C}}\mathbb{C} \arrow[r, "\cong"'] & \mathbb{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\cong"] \arrow[d, "\mathbf{b}_e"'] & \mathbb{C}\otimes_{\mathbb{C}}\mathbb{C} \arrow[d, "\mathbf{b}_e\otimes\mathbf{b}_e"] \\
\mathbb{C}[\Gamma] \arrow[r, "\Delta_\Gamma"'] & \mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma]
\end{tikzcd}.
$$

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathbf{b}_e"'] & \mathbb{C} \\
\mathbb{C}[\Gamma] \arrow[r, "\mathrm{Id}"'] & \mathbb{C}[\Gamma] \arrow[u, "\mathrm{Tr}_\Gamma"']
\end{tikzcd}
$$

(d) The antipode S satisfies the axioms of a **Hopf algebra**, i.e., the following diagram commutes.

$$
\begin{tikzcd}
\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] \arrow[r, "S_\Gamma\otimes\mathrm{Id}"] & \mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] \arrow[d, "-*-"] \\
\mathbb{C}[\Gamma] \arrow[u, "\Delta_\Gamma"] \arrow[r, "\mathrm{Tr}_\Gamma(-)\mathbf{b}_e"] \arrow[d, "\Delta_\Gamma"'] & \mathbb{C}[\Gamma] \\
\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] \arrow[r, "\mathrm{Id}\otimes S_\Gamma"] & \mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] \arrow[u, "-*-"']
\end{tikzcd}.
$$

(e) For every pair (U,σ) and (V,ρ) of left modules over a Hopf $\mathbb{C}$-algebra R, for every element $r\in R$ with

$$\Delta_R(t) = \sum_\alpha s_\alpha \otimes r_\alpha,$$

there is an associated left R-module structure on $U\otimes_{\mathbb{C}} V$ defined by

$$(\sigma\otimes\rho)(t)\cdot(u\otimes v) := \sum_\alpha (\sigma(s_\alpha)\cdot u)\otimes(\rho(r_\alpha)\cdot v)\,.$$

Check that for the comultiplication Δ_Γ defined above, this equals the structure of Γ-representation on $U\otimes_{\mathbb{C}} V$ as defined in lecture. Also, check that the trivial representation (i.e., the left-right identity for the tensor product operation on $\mathbb{C}$-linear left Γ-representations) is the unique representation such that the associated trace on $\mathbb{C}[\Gamma]$ equals Tr_Γ.

(f) Similarly, for every left module (V, ρ) over R, define a left R-module on the dual $\mathbb{C}$-vector space $V^\vee$ of $\mathbb{C}$-linear functional χ on V by

$$(t \cdot \chi)(v) := \chi(S(t) \cdot v).$$

Check that for the antipode S_Γ defined above, this equals the structure of Γ-representation on $V^\vee$ as defined in lecture. Thus, the "extra structures" on $\mathbb{C}$-linear Γ-representations are explained by the Hopf algebra structure on $\mathbb{C}[\Gamma]$. Conversely, these extra structures uniquely determine the Hopf algebra structures Δ_Γ, Tr_Γ, and S_Γ on the group $\mathbb{C}$-algebra $\mathbb{C}[\Gamma]$.

(g) Finally, check that the comultiplication is cocommutative, i.e., Δ equals its postcomposition with the involution

$$\mathrm{pr}_2 \otimes \mathrm{pr}_1 : \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma], \quad a_1 \otimes a_2 \mapsto a_2 \otimes a_1.$$

For every Hopf $\mathbb{C}$-algebra that is cocommutative and that is finite dimensional as a $\mathbb{C}$-vector space, there is an associated group Γ consisting of all elements b such that $\Delta(b)$ equals $b \otimes b$. Moreover, the Hopf $\mathbb{C}$-algebra is canonically isomorphic, as a Hopf $\mathbb{C}$-algebra, to the group $\mathbb{C}$-algebra of this group.

In particular, we can recover the finite group Γ from the structure of the group $\mathbb{C}$-algebra $\mathbb{C}[\Gamma]$ as a Hopf algebra, and we can recover this from the group $\mathbb{C}$-algebra as a unital, associative $\mathbb{C}$-algebra together with the extra structures on tensor product and duals of $\mathbb{C}$-linear, (left) Γ-representations.

Problem 5. (**A universal property of the group algebra as a representation.**) As in the previous problem, let $\mathbb{C}[\Gamma]$ be the group $\mathbb{C}$-algebra of a finite group Γ. Give $\mathbb{C}[\Gamma]$ its natural structure of $\mathbb{C}$-linear (left) G-representation, i.e., $(g, \mathbf{b}_h) \mapsto \mathbf{b}_{gh}$.

(a) Prove the following claim from lecture. For every $\mathbb{C}$-linear Γ-representation (V, ρ), the following $\mathbb{C}$-linear map is an isomorphism,

$$\mathrm{Hom}_{\mathrm{Rep}_\Gamma^{\mathbb{C}}}(\mathbb{C}[\Gamma], (V, \rho)) \to V, \quad (T : \mathbb{C}[\Gamma] \to V) \mapsto T(\mathbf{b}_e).$$

Also, show that this isomorphism is natural in (V, ρ). Stated in terms of category theory, there is a **fiber functor**,

$$F : \mathrm{Rep}_\Gamma^{\mathbb{C}} \to \mathbb{C} - \mathrm{Vect}, \quad (V, \rho) \mapsto V, \quad \mathrm{Hom}_{\mathrm{Rep}_\Gamma^{\mathbb{C}}}((U, \sigma), (V, \rho)) \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(U, V).$$

This is a covariant functor, and it is represented by $\mathbb{C}[\Gamma]$.

(b) Invert the isomorphism above to get a $\mathbb{C}$-linear map,

$$V \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Rep}_\Gamma^{\mathbb{C}}}(\mathbb{C}[\Gamma], (V, \rho)) \subseteq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[\Gamma], V).$$

Use adjointness of Hom and tensor product to obtain an associated $\mathbb{C}$-linear map,

$$\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} V \to V.$$

Prove that this $\mathbb{C}$-linear map is a morphism of $\mathbb{C}$-linear G-representations for the following structures of $\mathbb{C}$-linear G-representation,

$$\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} (V, \mathrm{triv}) \to (V, \rho).$$

(c) Apply this in the special case that (V, ρ) equals $\mathbb{C}[\Gamma]$ itself, and deduce that the $\mathbb{C}$-linear map,

$$\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma],$$

is the usual $\mathbb{C}$-algebra multiplication on the group $\mathbb{C}$-algebra.

Thus, the natural isomorphism above is enough to deduce the algebra structure. Altogether, this means that we can recover the group $\mathbb{C}$-algebra as a unital, associative $\mathbb{C}$-algebra from the data of the category of finite dimensional, $\mathbb{C}$-linear (left) Γ-representations as a category whose Hom sets are binaturally given structures of finite dimensional $\mathbb{C}$-vector spaces with $\mathbb{C}$-bilinear composition operations and the fiber functor.

The previous problem sketched how the extra operations of tensor product of representations, the trivial (1-dimensional) representation, and Hom objects define the additional Hopf algebra structures. Altogether, we can (explicitly) recover the finite group Γ from the category of $\mathbb{C}$-linear (left) Γ-representations with these extra operations (a structure of "rigid, symmetric, monoidal category") and the fiber functor (a structure of "Tannakian category").

(d) By Maschke's Theorem, every finite dimensional $\mathbb{C}$-linear Γ-representation is completely reducible. Combine this with Schur's Lemma and (a) above to conclude that every finite dimensional, irreducible, $\mathbb{C}$-linear Γ-representation (V_i, ρ_i) is isomorphic to a $\mathbb{C}$-linear subrepresentation of $\mathbb{C}[\Gamma]$.

(e) Use the morphism of $\mathbb{C}$-linear Γ-representations from (c) to conclude that right multiplication of $\mathbb{C}[\Gamma]$ on itself (the "right regular representation") gives an isomorphism of unital, associative $\mathbb{C}$-algebras,

$$\mathbb{C}[\Gamma]^{\mathrm{opp}} \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Rep}_\Gamma^{\mathbb{C}}}(\mathbb{C}[\Gamma], \mathbb{C}[\Gamma]).$$

Here, for every unital, associative $\mathbb{C}$-algebra A, the **opposite algebra** A^{opp} is the same $\mathbb{C}$-vector space but with multiplication defined by $a \bullet b := ba$.

(f) Combine the isomorphism in (e) with the previous exercise and Problem 2 from the previous problem set to conclude that the number $\#I$ of isomorphism classes $[(V_i, \rho_i)]$ of irreducible $\mathbb{C}$-linear Γ-representations equals the number of conjugacy classes in Γ, that every $\mathbb{C}$-bilinear pairing below is a perfect pairing of $\mathbb{C}$-vector spaces,

$$H_i \times V_i \to \mathbb{C}[\Gamma] \xrightarrow{\mathrm{Tr}_\Gamma} \mathbb{C}, \quad H_i := \mathrm{Hom}_{\mathrm{Rep}_\Gamma^{\mathbb{C}}}((V_i, \rho_i), \mathbb{C}[\Gamma]),$$

that each multiplicity m_i equals the $\mathbb{C}$-vector space dimensions $n_i = \dim_{\mathbb{C}}(V_i)$, and that there is an equality of positive integers,

$$\#\Gamma = \sum_{i \in I} n_i^2.$$

Problem 6. (**Central idempotents of the group algebra give the irreducible representations.**) Inside the center $Z(\mathbb{C}[\Gamma])$ considered as a $\mathbb{C}$-algebra, an element e is an **idempotent** if e^2 equals e. For each idempotent e, the **annihilator** of e is

$$\mathrm{Ann}(e) := \{b \in \mathbb{C}[\Gamma] | eb = 0\}.$$

(a) With respect to the $\mathbb{C}$-algebra isomorphism

$$Z(\widetilde{a}) : Z(\mathbb{C}[\Gamma]) = Z(\mathbb{C}[\Gamma]^{\mathrm{opp}}) \xrightarrow{\cong} \prod_{i \in I} \mathbb{C} \cdot \mathrm{Id}_{H_i},$$

check that the idempotent elements correspond to those elements $(a_i \mathrm{Id}_{H_i})_{i \in I}$ such that every a_i^2 equals a_i, i.e., such that a_i equals 1 or 0.

(b) For an idempotent e, define the **support**, $\mathrm{supp}(e)$, to be the subset of I such that a_i equals 1. Check that the annihilator of e maps isomorphically to the left-right ideal,

$$\{(M_i)_{i\in I} \in \prod_{i\in I} \mathrm{Hom}_{\mathbb{C}}(H_i, H_i) | \forall i \in \mathrm{supp}(e),\ M_i = 0\}.$$

(c) In particular, check that the unique nonzero idempotents with codimension-1 annihilator in $Z(\mathbb{C}[\Gamma])$ are the **primitive idempotents** $\mathbf{e}_i$ for each $i \in I$,

$$\mathbf{e}_i \mapsto (a_j \mathrm{Id}_{H_j})_{j\in I}, \quad a_i = 1, \quad a_j = 0, \forall j \neq i.$$

Moreover, for each $i \in I$, check that the common annihilator in $\mathbb{C}[\Gamma]$ of $\mathbf{e}_j$ for all $j \neq i$ maps isomorphically to the left-right ideal that is the factor $\mathrm{Hom}_{\mathbb{C}}(H_i, H_i)$. As a $\mathbb{C}$-linear left Γ-representation, this $\mathbb{C}$-algebra is a direct sum of n_i copies of the irreducible representation (V_i, ρ_i). Thus, we can construct the irreducible, $\mathbb{C}$-linear, Γ-representations from the full list of idempotents in $Z(\mathbb{C}[\Gamma])$ having codimension-1 annihilator in $Z(\mathbb{C}[\Gamma])$.

Problem 7 (**Schur's Orthogonality Relations and idempotents in the group algebra**.) This exercise reconstructs the primitive idempotents (and thus the irreducible representations) in the group algebra from the information of the irreducible characters. The key is a natural Hermitian inner product on the center of the group algebra, together with Schur's Orthogonality Relations.

(a) For every $\mathbb{C}$-linear (left) Γ-representation (V, ρ) and for every conjugacy class C in Γ, check that the following $\mathbb{C}$-linear operator on V is a morphism of $\mathbb{C}$-linear (left) Γ-representations,

$$\sum_{g\in C} \rho(g) \in \mathrm{Hom}_{\mathbb{C}}(V, V).$$

(b) If (V, ρ) if a finite dimensional, irreducible representation, use Schur's Lemma to conclude that this morphism is a multiple of the identity, say $\rho_C \mathrm{Id}_V$. Taking traces, deduce the identity,

$$\rho_C \cdot \dim_{\mathbb{C}}(V) = \sum_{g\in C} \mathrm{Tr}_V(\rho(g)).$$

(c) Similarly, conclude that the image of the following $\mathbb{C}$-linear operator is contained in the $\mathbb{C}$-linear subrepresentation of invariant elements,

$$\sum_{g\in\Gamma} \rho(g) \in \mathrm{Hom}_{\mathbb{C}}(V, V).$$

If (V, ρ) is a finite dimensional representation whose invariant subspace is zero, conclude that

$$\sum_C \rho_C = 0, \quad \text{i.e.,} \sum_{g\in\Gamma} \mathrm{Tr}_V(\rho(g)) = 0.$$

In particular, this holds if (V, ρ) is a nontrivial irreducible representation. Conversely, if (V, ρ) is a trivial representation, show that the sum equals $\dim_{\mathbb{C}}(V)\#\Gamma$. Thus, since trace is additive for direct sum decomposition, for a general (V, ρ), conclude the identity

$$\frac{1}{\#\Gamma} \sum_{g\in\Gamma} \mathrm{Tr}_V(\rho(g)) = \dim_{\mathbb{C}}(V^\Gamma).$$

(d) For every finite dimensional, $\mathbb{C}$-linear, (left) Γ-representation (V,ρ), the **character** of this representation is the class function,

$$\chi_{(V,\rho)} : \Gamma \to \mathbb{C}, \quad g \mapsto \mathrm{Tr}_V(\rho(g)).$$

Every $\rho(g)$ is diagonalizable with eigenvalues ζ that are roots of unity with $\zeta^{-1} = \overline{\zeta}$. Conclude the identity

$$\chi_{(V^\vee,\rho^\vee)}(g) = \chi_{(V,\rho)}(g^{-1}) = \overline{\chi_{(V,\rho)}(g)}.$$

Similarly, for representations (U,σ) and (V,ρ), for the tensor product of the eigendecompositions to prove the identity,

$$\chi_{(U\otimes V,\sigma\otimes\rho)}(g) = \chi_{(U,\sigma)}(g)\chi_{(V,\rho)}(g).$$

Consequently, conclude the identity,

$$\chi_{\mathrm{Hom}_{\mathbb{C}}(U,V)}(g) = \overline{\chi_{(U,\sigma)}(g)}\chi_{(V,\rho)}(g).$$

Sum over g and use (c) to deduce the identity,

$$\dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Rep}_\Gamma^{\mathbb{C}}}((U,\sigma),(V,\rho)) = \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\overline{\chi_{(U,\sigma)}(g)}\cdot\chi_{(V,\rho)}(g).$$

(e) Now assume that (V,ρ) is irreducible. For every class function,

$$\alpha : \Gamma \to \mathbb{C},$$

with **associated central element**,

$$\frac{1}{\#\Gamma}\sum_{g\in\Gamma}\alpha(g)\mathbf{b}_{g^{-1}},$$

conclude that the associated $\mathbb{C}$-linear operator on V,

$$\sum_{g\in\Gamma}\alpha(g)\rho(g^{-1}),$$

equals $\lambda\mathrm{Id}_V$ where λ satisfies the identity

$$\lambda\dim_{\mathbb{C}}(V) = \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\alpha(g)\chi_{(V,\rho)}(g^{-1}) = \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\alpha(g)\overline{\chi_{(V,\rho)}(g)}.$$

In particular, this central element annihilates the primitive idempotent corresponding to (V,ρ) if and only if the class function α is orthogonal to the class function $\chi_{(V,\rho)}$ with respect to the Hermitian inner product on the $\mathbb{C}$-vector space of class functions defined by

$$\langle \alpha,\beta\rangle := \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\alpha(g)\overline{\beta(g)}.$$

(f) Let (U,σ) and (V,ρ) be finite dimensional, $\mathbb{C}$-linear, (left) Γ-representations. Reinterpret (d) as an identity,

$$\dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{Rep}_\Gamma^{\mathbb{C}}}((U,\sigma),(V,\rho)) = \langle \chi_{(V,\rho)},\chi_{(U,\sigma)}\rangle.$$

In particular, if (U,σ) and (V,ρ) are irreducible representations, conclude that $\langle \chi_{(V,\rho)},\chi_{(U,\sigma)}\rangle$ equals 0 unless the representations are isomorphic, in which case $\langle \chi_{(V,\rho)},\chi_{(U,\sigma)}\rangle$ equals 1. Thus, the characters of irreducible representations form an orthonormal subset of the $\mathbb{C}$-vector space of $\mathbb{C}$-valued class functions with respect to the Hermitian inner product defined above.

(g) Finally, since the dimension of the space of class functions equals the dimension of $Z(\mathbb{C}[\Gamma])$, and since this equals the number $\#I$ of isomorphism classes of irreducible representations, conclude that the characters of irreducible representations form an orthonormal basis for the $\mathbb{C}$-vector space of $\mathbb{C}$-valued class functions with respect to the Hermitian inner product defined above. Altogether this is the **Schur orthogonality relations**.

(h) Deduce that for the irreducible representations (V_i, ρ_i) for $i \in I$ with character $\chi_i = \chi_{(V_i,\rho_i)}$, the corresponding central elements,

$$\mathbf{e}_i := \frac{\chi_i(e)}{\#\Gamma} \sum_{g \in \Gamma} \chi_i(g) \mathbf{b}_{g^{-1}},$$

are the primitive idempotents. (**Hint.** First use (e) and (f) to show that the primitive idempotent is a scalar multiple of this sum. Then square the element and examine the coefficient of $\mathbf{b}_e$ to conclude that the idempotent equals the sum.)

Combined with the previous problem, the data of the characters of the irreducible representations as a subset of the $\mathbb{C}$-vector space of $\mathbb{C}$-valued class functions on Γ is sufficient to reconstruct all irreducible, $\mathbb{C}$-linear, (left) Γ-representations via the corresponding isotypic factor of $\mathbb{C}[\Gamma]$.

Problem 8. For a finite Abelian group Γ, interpret the previous problems as an eigendecomposition of the group $\mathbb{C}$-algebra as a direct sum of 1-dimensional subrepresentations indexed by the elements of the Pontrjagin dual group.

Problem 9. Also explicitly work out the decomposition of the group $\mathbb{C}$-algebra $\mathbb{C}[\Gamma]$ for the symmetric groups $\Gamma = \mathfrak{S}_3$ and $\Gamma = \mathfrak{S}_4$. In fact, there is a different method for reconstructing the irreducible, $\mathbb{C}$-linear, (left) $\mathfrak{S}_n$-representations of $\mathfrak{S}_n$ as left $\mathbb{C}[\mathfrak{S}_n]$-submodules of $\mathbb{C}[\mathfrak{S}_n]$. These submodules are cyclic submodules $V_\lambda := \mathbb{C}[\mathfrak{S}_n]c_\lambda$ generated by the **Young symmetrizer** c_λ associated to each partition λ of n. There are combinatorial formulas for the character of V_λ, but often the Young symmetrizer provides a more direct approach.