MAT 552 PROBLEM SET 2

Problem 0.(**The Rank Theorem.**) Recall the following two important theorems about morphisms of manifolds (true both in the setting of C^∞ maps between smooth manifolds and in the setting of holomorphic maps between complex manifolds).

Theorem 0.1 (Implicit Function Theorem)**.** *For every morphism of manifolds $f : M \to N$, for every point p of M such that the derivative map d_pf is a surjective morphism of vector spaces, there exists an open neighborhood V of $f(p)$ in N, and open neighborhood U of p in $f^{-1}(V)$, an open neighborhood W of the origin in $\mathbb{A}^d$ for some nonnegative integer d, and an isomorphism of manifolds $g : U \to V \times W$ such that $pr_V \circ g$ equals the restriction of f to U. In particular, the fiber $f^{-1}(f(p)) \cap U$ is a closed embedded submanifold of U of dimension d.*

(a) In the special case that d_pf is an isomorphism of vector spaces, conclude that there exists an open neighborhood of V of $f(p)$ and an open neighborhood U of p in $f^{-1}(V)$ such that the restriction of f to U is an isomorphism of manifolds from U to V. This is the **Inverse Function Theorem**.

(b) Next, assume the Inverse Function Theorem, and let $f : M \to N$ be a morphism of manifolds such that the derivative map at a point, d_pf, is a surjective morphism of vector spaces. Deduce the Implicit Function Theorem as follows. Let $T_pM \to \mathbb{A}^d$ be a surjective linear transformation that restricts on the kernel of d_pf to an isomorphism of vector spaces. Using coordinate charts, show that there exists an open neighborhood of p in M and a morphism of manifolds from that neighborhood to $\mathbb{A}^d$ such that the induced derivative map at p is the given surjective linear transformation. Apply the Inverse Function Theorem to the product morphism from the neighborhood of p in M to the product manifold $N \times \mathbb{A}^d$.

(c) Next, assume the Inverse Function Theorem / Implicit Function Theorem and let $f : M \to N$ be a morphism of manifolds such that the derivative map at a point, d_pf, is an injective morphism of vector spaces. Let V be an open neighborhood of $f(p)$ in N that admits an isomorphism of manifolds, $g : V \to \mathbb{A}^n$, to an open neighborhood of the origin in $\mathbb{A}^n$. Let $W \subset \mathbb{A}^n$ be a vector subspace that is complementary to the image of the derivative map $d_p(g \circ f)$. Consider the induced morphism h from $f^{-1}(V) \times W \to \mathbb{A}^n$ given by $h(q, w) = g(f(q)) + w$. The inverse image of $g(V)$ is an open neighborhood of $(p, 0)$ in $f^{-1}(V) \times W$. Apply the Inverse Function Theorem to the restriction of h to this open neighborhood to conclude that, up to shrinking V, there exists an open neighborhood U of p in $f^{-1}(V)$ and a submersion to an open neighborhood of the origin in W, $s : V \to s(V) \subset W$, such that $s \circ f$ is constant on U, and the induced morphism from U to the fiber of s over p is an isomorphism from U to a closed submanifold of V (namely, the fiber of the submersion, which is a closed submanifold using the Implicit Function Theorem).

(d) Finally, assume the Inverse Function Theorem / Implicit Function Theorem and prove the (Constant) Rank Theorem. Let r be a nonnegative integer and let $f : M \to N$ be a morphism of manifolds such that the rank of the derivative map equals r for every point p of M. Let V be an open neighborhood of $f(p)$ in N that is isomorphic to an open neighborhood of the origin in $\mathbb{A}^n$. For an appropriate surjective linear projection $\mathbb{A}^n \to \mathbb{A}^r$, conclude that the induced morphism from an open neighborhood of p in M to $\mathbb{A}^r$ is a submersion. Combine this with the arguments above to conclude that, up to shrinking the neighborhood V of $f(p)$, there exists an open neighborhood of the origin in $\mathbb{A}^r$, and an open neighborhood U of p in the preimage in M such that the restriction of f to U factors as a composition of a submersion from U to the open neighborhood in $\mathbb{A}^r$ followed by a closed embedding of this open neighborhood as a closed submanifold of the open neighborhood V of $f(p)$ in N. This is the **(Constant) Rank Theorem**.

(e) In particular, let G be a connected Lie group, let M and N be manifolds with specified actions of G on M and N. Let $f : M \to N$ be a morphism of manifolds that is compatible with the G-actions, i.e., f is a G-**equivariant** morphism of manifolds. If the action of G on M is transitive, conclude that f satisfies the hypotheses of the (Constant) Rank Theorem. Hence, locally on M, the morphism f factors as a composition of a submersion followed by an immersion.

(f) Also use the Implcit Function Theorem to prove "submersive descent" for morphisms. Let M and N be manifolds, and let $f : M \to N$ be a function. Let $g : P \to M$ be a surjective submersion of manifolds. Prove that f is a morphism of manifolds if and only if the composite function $f \circ g : P \to N$ is a morphism of manifolds. (Hint: The Implict Function Theorem guarantees that locally on M there exist morphisms $M \to P$ that are "sections" of g. Now compose $f \circ g$ with these local sections.)

Problem 1. (**Standard parabolics, partial flag manifolds, and Bruhat decomposition for A_{n-1}-type.**) In this problem, work through the various parts for the simply connected, simple, complex Lie group $\mathbf{SL}_n(\mathbb{C})$ for a representative collection of pairs (n, Γ).

As in the previous problem set, let $(\mathbf{SL}_n(\mathbb{C}), T'_n, B'_n)$ be the special linear complex Lie group with its standard maximal torus and standard Borel. Thus, for the usual direct sum decomposition into 1-dimensional subspaces,

$$V = \mathbb{C}^n = \bigoplus_{i=1}^{n} \mathbb{C} \cdot \mathbf{e}_i = \bigoplus_{i=1}^{n} L_i,$$

the torus T'_n is the subgroup of $G = \mathbf{SL}_n(\mathbb{C})$ mapping every L_i back to itself, and B'_n is the subgroup of $\mathbf{SL}_n(\mathbb{C})$ that preserves every subspace in the standard flag,

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_j \subset \cdots \subset F_{n-1} \subset F_n = V, \quad F_j = \bigoplus_{i=1}^{j} L_i.$$

Denote by $T_n[2]$ the elementary Abelian 2-group of 2-torsion elements in T_n. One lift $W_n \subset N_G(T'_n)/T_n[2]$ of the Weyl group $N_G(T'_n)/T'_n$ is determined by the standard pinning $(\mathbf{e}_1, \dots, \mathbf{e}_n)$, i.e., the ordered bases of the 1-dimensional subspaces $(L_1, \dots, L_n)$ up to simultaneous (invertible) scalar. The inverse image of W_n in $N_G(T'_n)$ consists of matrices A such that for every basis vector $\mathbf{e}_i$, the product $A \cdot \mathbf{e}_i$ equals $\pm \mathbf{e}_j$ for some basis vector $\mathbf{e}_j$. This group is generated by the positive simple reflections $\{s_i : 1 \leq i \leq n-1\}$, where s_i maps $\mathbf{e}_i$ to $\mathbf{e}_{i+1}$ and maps $\mathbf{e}_{i+1}$ to $-\mathbf{e}_i$, leaving fixed all other basis vectors.

The set $\Delta = \Delta(\mathbf{SL}_n(\mathbb{C}), T'_n, B'_n)$ of positive simple roots is in natural bijection with this set of positive simple reflections, namely $\{\alpha_i : 1 \leq i \leq n-1\}$,

$$\alpha_i = \chi_{n,i} - \chi_{n,i+1}, \quad s_i(\beta) = \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i, \quad i \in \{1, \dots, n-1\}.$$

For simplicity, identify Δ with the set of integers $\{1, \dots, n-1\}$. **Nota bene.** Denote by $w_0 \in W_n$ the unique element of maximal length, i.e., the unique element with $w_0(\mathbf{e}_j) = \mathbf{e}_{n-j}$ for every $j \in \Delta$. The outer automorphism $\iota(g) = w_0(g^\dagger)^{-1}w_0^{-1}$ of $\mathbf{SL}_n(\mathbb{C})$ preserves B'_n and T'_n, yet permutes Δ by the rule $\iota(j) = n - j$. Thus, this enumeration of the roots depends on an additional choice.

Let Γ be a subset of $\Delta = \{1, \dots, n-1\}$,

$$\Gamma \subset \{1, \dots, n-1\}, \quad \Gamma = \{j_1, \dots, j_d\}, \quad 0 < j_1 < \cdots < j_d < n.$$

By convention, set $j_0 = 0$ and $j_{d+1} = n$. Denote by P_Γ the subgroup of $\mathbf{SL}_n(\mathbb{C})$ that preserves every subspace F_j for $j \in \Gamma$.

(a) For the standard enumeration $\alpha = \alpha_i$, $i = 1, \dots, n-1$, of positive simple roots in $\Delta(\mathbf{SL}_n(\mathbb{C}), T'_n, B'_n)$, check that the root group

$$U_{-\alpha} = f_{n,-\alpha}(U_+) = s_\alpha U_\alpha s_\alpha^{-1},$$

is contained in P_Γ if and only if i is contained in Γ. Similarly, check that $s_i P_\Gamma s_i^{-1}$ equals P_Γ if and only if $i \in \Gamma$. These reflections give generators for the subgroup $W_{n,\Gamma} \subset W_n$ of all elements w that preserve P_Γ, i.e., $wP_\Gamma w^{-1}$ equals P_Γ. Finally, check that P_Γ is generated as a closed Lie subgroup of $\mathbf{SL}_n(\mathbb{C})$ by B'_n and by the root groups $U_{-\alpha}$ for $i \in \Gamma$, and also it is generated by the conjugates $s_i B s_i^{-1}$ for $i \in \Gamma$.

Thus, for a general triple (G, T, B) of a connected, reductive, complex Lie group G, a maximal torus T, and a Borel containing B, for every subset Γ of $\Delta(G, T, B)$, we could define P_Γ to be the closed Lie subgroup generated by B and the root groups $U_{-\alpha}$ for $\alpha \in \Gamma$, and this also equals the closed Lie subgroup generated by all conjugates sBs^{-1} for all positive simple reflections $s \in W = N_G(T)/T$ associated to $\alpha \in \Gamma$.

(b) For the standard transitive action of $\mathbf{SL}_n(\mathbb{C}) = \mathbf{SL}(V)$ on the following partial flag manifold, check that the stabilizer of the partial flag $(F_j)_{j \in \Gamma}$ equals P_Γ, and thus the partial flag variety is $\mathbf{SL}_n(\mathbb{C})$-equivariantly biholomorphic to $\mathbf{SL}_n(\mathbb{C})/P_\Gamma$,

$$\text{Flag}(\Gamma; V) = \{(E_j)_{j \in \Gamma} \in \prod_{j \in \Gamma} \text{Grass}_{\mathbb{C}}(j, V) : \forall (j,k) \in \Gamma \times \Gamma \text{ s.t. } j \leq k, \quad E_j \subset E_k\}.$$

Also check that the complex dimension of the partial flag manifold equals

$$m = \sum_{e=1}^{d} (j_{e+1} - j_e) j_e.$$

(c) For every $w \in W_n$, check that the double coset $C(w) := P_\Gamma w P_\Gamma$ in $\mathbf{SL}_n(\mathbb{C})$ depends only on the double coset $[w] \in W_{n,P} \backslash W_n / W_{n,P}$. Check that the image $E(w) := C(w)/P_\Gamma$ in the partial flag manifold $\mathbf{SL}_n(\mathbb{C})/P_\Gamma$ contains the following flag,

$$(F_j^w)_{j \in \Gamma}, \quad F_j^w := \sum_{i=1}^{j} L_{w(i)}.$$

Moreover, check that $E(w)$ is the following subset of the partial flag manifold,

$$E(w) = \{(E_j)_{j\in\Gamma} \in \mathrm{Flag}(\Gamma;V) | \forall j \in \Gamma, \forall i = 1,\dots,n,\ \dim(E_j \cap F_i) = \dim(F_j^w \cap F_i)\}.$$

Check that the closure $\overline{E}(w)$ is the following (Zariski) closed subset of $\mathrm{Flag}(\Gamma;V)$,

$$\overline{E}(w) := \{(E_j)_{j\in\Gamma} \in \mathrm{Flag}(\Gamma;V) | \forall j \in \Gamma, \forall i = 1,\dots,n,\ \dim(E_j \cap F_i) \geq \dim(F_j^w \cap F_i)\}.$$

Check that the relative complement $\overline{E}(w) \setminus E(w)$ is a (Zariski) closed subset of $\overline{E}(w)$, so that $E(w)$ is a (Zariski) open subset of $\overline{E}(w)$. Thus, altogether, $E(w)$ is a (Zariski) locally closed subset of $\mathrm{Flag}(\Gamma;V)$. Finally, as a complex manifold with an action of the root groups $U_\alpha \cong \mathbb{C}$, check that $E(w)$ is isomorphic to a product of additive groups $\mathbb{C}\times\cdots\times\mathbb{C} = \mathbb{C}^\ell$, for some integer $\ell = \ell_\Gamma([w]) \geq 0$. Thus, the collection of locally closed subsets $(E(w))_{[w]}$ give a partition of $\mathrm{Flag}(\Gamma;V)$ into cells of (real) dimension $2\ell_\Gamma([w])$ for the finitely many elements $[w] \in W_{n,\Gamma}\backslash W_n/W_{n,\Gamma}$.

(d) For a representative set of choices of (n,Γ), check for every $[w] \in W_{n,\Gamma}\backslash W_n/W_{n,\Gamma}$ that the nonnegative integer $\ell_\Gamma([w])$ from the previous part equals the Coxeter length $\ell(w)$ of a double coset representative w having least length. As usual, the Coxeter length is the least word length of a representation of w as a word in the simple reflections $(s_i)_{i=1,\dots,n-1}$ generating W_n. Also, determine the partial order on double cosets determined by the inclusion partial order on closures, $\overline{E}([w]) \subset \overline{E}([w'])$. This is the **Bruhat order** on $W_{n,\Gamma}\backslash W_n/W_{n,\Gamma}$ relative to Δ.

(e) Since the cells in the cellular decomposition all have even dimension, conclude that the differentials in the cellular chain complex are all zero. Thus, the elements $[E(w)]$, as $[w]$ ranges over $W_{n,\Gamma}\backslash W_n/W_{n,\Gamma}$, give a $\mathbb{Z}$-basis of the cellular homology $H_*(\mathrm{Flag}(\Gamma;\mathbb{C}^n);\mathbb{Z})$. Each basis element $[E(w)]$ is homogeneous of degree $2\ell_\Gamma([w])$. For a representative set of choices of (n,Γ), use this to compute the even Betti numbers of $\mathrm{Flag}(\Gamma;\mathbb{C}^n)$ (the odd Betti numbers are all zero). Check that the Betti numbers are unimodular and symmetric, as implied by Poincaré duality for the connected, closed, complex manifold $\mathrm{Flag}(\Gamma;\mathbb{C}^n)$.

(f) For the Poincaré dual classes in cohomology,

$$PD[E(w)] \in H^{2m-2\ell}(\mathrm{Flag}(\Gamma;\mathbb{C}^n)),$$

for every triple $[w],[w'],[w''] \in W_{n,\Gamma}\backslash W_n/W_{n,\Gamma}$, the **Littlewood-Richardson coefficient** $c^{[w]}_{[w'],[w'']}$ is defined as the structure constants of the cup product pairing in cohomology,

$$PD[E(w')] \smile PD[E(w'')] = \sum_{[w]} c^{[w]}_{[w'],[w'']} PD[E(w)].$$

For a few cases of (n,Γ), write out these structure constants.

Problem 2. (**Borel's Theorem on the cohomology ring of complete flag manifolds.**) The graded polynomial $\mathbb{Z}$-algebra $\mathbb{Z}[x_1,\dots,x_n]$ on n homogeneous variables x_i of degree 1 is the graded symmetric algebra on its degree 1 part,

$$\mathbb{Z}[x_1,\dots,x_n]_1 = X_n = \mathbb{Z}\cdot x_1 \oplus \cdots \oplus \mathbb{Z}\cdot x_n,\ \ \mathbb{Z}[x_1,\dots,x_n] = \mathrm{Sym}^\bullet_{\mathbb{Z}}(X_n).$$

Denote by $X_n^0 \subset X_n$ the free Abelian subgroup of those linear combinations $a_1x_1 + \cdots + a_nx_n$ such that $a_1+\cdots+a_n$ equals 0. Denote by $\mathbb{Z}[x_1,\dots,x_n]^0$ the graded polynomial $\mathbb{Z}$-subalgebra of $\mathbb{Z}[x_1, \dots, x_n]$ that is the symmetric algebra on X_n^0. In particular, the following composite $\mathbb{Q}$-algebra homomorphism is an isomorphism,

$$\mathbb{Q}[x_1, \dots, x_n]^0 \hookrightarrow \mathbb{Q}[x_1, \dots, x_n] \twoheadrightarrow \mathbb{Q}[x_1, \dots, x_n]/\langle \sigma_1 \rangle,$$

where for every $r = 1, \dots, n$, the r^{th} **symmetric polynomial** is defined by

$$\sigma_r(x_1, \dots, x_n) = \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} \cdots x_{i_r}.$$

(a) Review the statement of the first fundamental theorem of $\mathfrak{S}_n$-invariants that for the standard action of the symmetric group $\mathfrak{S}_n$ on the graded polynomial $\mathbb{Q}$-algebra $\mathbb{Q}[x_1, \dots, x_n]$, the ring of $\mathfrak{S}_n$-invariants is the graded polynomial $\mathbb{Q}$-subalgebra,

$$\mathbb{Q}[x_1, \dots, x_n]^{\mathfrak{S}_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n],$$

where each σ_i is homogeneous of degree i.

(b) Identifying X_n as the character lattice $X^*(T_n)$ of the standard maximal torus T_n in $\mathbf{GL}_n(\mathbb{C})$, check that the induced map of symmetric $\mathbb{Z}$-algebras,

$$\mathrm{Sym}^\bullet_{\mathbb{Z}} X^*(T_n) \to \mathrm{Sym}^\bullet_{\mathbb{Z}} X^*(T_n')$$

is identified with the following homomorphism of graded $\mathbb{Z}$-algebras, compatibly with the natural action of $W_n \cong \mathfrak{S}_n$,

$$\mathbb{Z}[x_1, \dots, x_n] \to \mathbb{Z}[x_1, \dots, x_n]/\langle \sigma_1 \rangle.$$

Thus, up to tensoring with $\mathbb{Q}$, identify the symmetric $\mathbb{Q}$-algebra on $X^*(T_n)$, resp. on $X^*(T_n')$, with the graded polynomial $\mathbb{Q}$-algebra $\mathbb{Q}[x_1, \dots, x_n]$, resp. with $\mathbb{Q}[x_1, \dots, x_n]^0$, compatibly with the W_n-action.

(c) For every $r = 1, \dots, n-1$, show that there exists a unique homogeneous element

$$f_r \in \mathrm{Sym}^\bullet_{\mathbb{Q}}(X^*(T_n'))^{W_n},$$

of degree $1+r$ corresponding to σ_{1+r} in $\mathbb{Q}[x_1, \dots, x_n]/\langle \sigma_1 \rangle$. The elements $f_1, \dots, f_{n-1}$ are the **fundamental invariants** with degree $\deg(f_i) = 1 + i$.

(d) For a few small values of n, check that each Betti number b_{2e} of the complete flag variety $\mathbf{SL}_n(\mathbb{C})/B_n'$ equals the Betti number b_e of the graded $\mathbb{Q}$-algebra,

$$\mathrm{Sym}^\bullet_{\mathbb{Q}}(X^*(T_n'))/\langle f_1, \dots, f_{n-1} \rangle \cong \mathbb{Q}[x_1, \dots, x_n]/\langle \sigma_1, \sigma_2, \dots, \sigma_2 \rangle.$$

Theorem 0.2 (Chevalley-Shephard-Todd)**.** *For every finite dimensional vector space V over a characteristic 0 field k together with a k-linear action of a finite group G, inside the graded k-algebra $k[V]$ of polynomial functions on V, the graded k-subalgebra of G-invariant polynomials, $k[V]^G$, is regular if and only if the group G is generated by elements g that act on V by quasi-reflections, i.e., the G-invariant subspace of V has codimension ≤ 1. In this case, $k[V]^G$ is a polynomial k-algebra whose generators are homogeneous elements, and $k[V]$ is a finite, free module over $k[V]^G$.*

(e) By definition, every Weyl group is generated by reflections. Conclude that for every triple (G, T, B) of a connected, semisimple complex Lie group G, a rank-r maximal torus T, and a Borel subgroup B containing T, for the natural action of the Weyl group $W = N_G(T)/T$ on the graded $\mathbb{Q}$-algebra $\mathrm{Sym}^\bullet_{\mathbb{Q}}(X^*(T) \otimes \mathbb{Q})$, the graded $\mathbb{Q}$-subalgebra of W-invariants is itself a graded polynomial $\mathbb{Q}$-algebra

$$\mathrm{Sym}^\bullet_{\mathbb{Q}}(X^*(T) \otimes \mathbb{Q})^W \cong \mathbb{Q}[f_1, \dots, f_r]$$

for homogeneous elements f_i of degrees $1+d_i$, with $1 = d_1 \leq \cdots \leq d_r$. These are the **fundamental invariants**. They will recur at the end of the semester when we prove the **Harish-Chandra isomorphism**.

Theorem 0.3 (Borel's Theorem)**.** *Let R be a commutative, unital ring such that the graded R-algeba $H^*(G;R)$ is a free exterior R-algebra. Associated to the natural isomorphism $X^*(T) \to H^2(G/B;\mathbb{Z})$, the induced map of graded R-algebras,*

$$Sym^{\bullet}_R(X^*(T)\otimes R) \to H^*(G/B;R) = H^*(G/T;R),$$

is surjective with kernel ideal equal to the complete intersection ideal generated by $f_1,\dots,f_r$. In particular, if G is simple, then the kernel of the following cup-product map has $\mathbb{Q}$-dimension 1 and is spanned by the W-invariant f_1,

$$Sym^2_{\mathbb{Q}}(X^*(T)\otimes\mathbb{Q}) \to H^4(G/B;\mathbb{Q}).$$

In particular, the complex dimension of G/B equals $d_1+\cdots+d_r$, since that is the top degree of a nonzero graded component modulo a complete intersection ideal with generators of degrees $(1+d_1,\dots,1+d_r)$. Also, since the Bruhat cells indexed by the Weyl group W also form a $\mathbb{Z}$-module basis for $H^*(G/B;\mathbb{Z})$, the order of W equals $(1+d_1)\cdots(1+d_r)$. This also follows from the fact that the action of W on $X^*(T)\otimes\mathbb{R}$ is properly discontinuous and generically free, so that flatness implies that the length, $(1+d_1)\cdots(1+d_r)$, of the fiber ring $\mathrm{Sym}^{\bullet}_{\mathbb{R}}(X^*(T)\otimes\mathbb{R})^G \to \mathrm{Sym}^{\bullet}_{\mathbb{R}}(X^*(T)\otimes\mathbb{R})$ over the origin equals the length of the generic fiber ring, namely $\#W$.

Conclude that the Euler characteristic of G/T equals $\#W$. Since the action of W on G/T by $w\bullet(gT) = (gw^{-1})T$ is well-defined and free, conclude that the quotient space $G/N_G(T)$ has Euler characteristic equal to 1.

(f) For two or three choices of n, check Borel's theorem in A_{n-1}-type, and identify each cohomology class $PD[E(w)]$ as an element in the ring $\mathbb{Q}[x_1,\dots,x_n]/\langle\sigma_1,\dots,\sigma_n\rangle$.

Problem 3. (**Abelian fundamental groups of Lie groups and exponential maps of Abelian Lie groups.**) For every connected Lie group $\widetilde{T}$, for every closed Lie subgroup N that is normal, but not necessarily connected, consider the conjugation action of $\widetilde{T}$ on the discrete group N/N^0.

(a) Prove that the conjugation action of $\widetilde{T}$ on N/N^0 is trivial. In particular, if N^0 equals $\{e\}$, so that N is a discrete, closed, normal subgroup of $\widetilde{T}$, prove that N is contained in the center of $\widetilde{T}$.

(b) Next, when (T,e) is a connected Lie group, for the universal covering group,

$$\phi : (\widetilde{T},\widetilde{e}) \to (T,e),$$

apply the previous part to $\pi_1 T := \mathrm{Ker}(\phi)$ to conclude that $\pi_1 T$ is a subgroup of the center of $\widetilde{G}$. In particular, conclude that the fundamental group $\pi_1 T$ of every connected Lie group (T,e) is an Abelian group, i.e., the following Hurewicz map is an isomorphism,

$$\pi_1 T \to H_1(T;\mathbb{Z}).$$

Theorem 0.4 (Hopf)**.** *Not only for connected Lie groups, but also for every path connected, homotopy-associative H-space G, the fundamental group is Abelian. Moreover, for every commutative, unital ring R such that every cohomology group $H^*(G;R)$ is a finitely generated, free R-module, the graded cohomology R-algebra*

$H^*(G;R)$ is a graded commutative Hopf R-algebra. The dual Hopf R-algebra is the Pontryagin algebra structure on the homology.

There is a classification of graded commutative Hopf algebras in characteristic 0 due to Hopf and Leray. The extension over perfect characteristic p fields is due to Borel.

Theorem 0.5 (Hopf, Leray)**.** *For a characteristic* 0 *field k, every graded commutative Hopf algebra with finite dimensional graded components and that is connected (i.e., the negatively graded components are zero) is isomorphic to the tensor product k-algebra of a symmetric k-algebra on finitely many generators in even degrees and an exterior k-algebra on finitely many generators in odd degrees.*

Corollary 0.6 (Hopf)**.** *For every connected Lie group G, the graded cohomology $\mathbb{Q}$-algebra $H^*(G;\mathbb{Q})$ is isomorphic to an exterior algebra on finitely many generators in odd degrees $(2d_1+1,\dots,2d_r+1)$, i.e., G is $\mathbb{Q}$-cohomologically a product of spheres of odd dimensions $(2d_1+1,\dots,2d_r+1)$. In fact, there exists a finite set of "bad primes" for each G such that this holds for an arbitrary coefficient ring in which each bad prime is invertible.*

As a particular instance of the corollary, if K is a connected Lie group that is compact, so that Poincaré duality holds, then the real dimension of G equals $r+2(d_1+\dots+d_r)$, since that is the top degree such that the corresponding component of the exterior algebra is nonzero. Also, the sum of the Betti numbers equals 2^r. If K is the compact real form of a simply connected, semisimple, complex Lie group G, and if for a Borel subgroup B of G the intersection $B\cap K$ is a rank-r maximal torus in K, then the quotient $K/(B\cap K)$ equals G/B. Thus, the complex dimension of G/B equals $d_1+\dots+d_r$ (for the second time).

The corollary follows from the two theorems: since $H^*(G;\mathbb{Q})$ has nonzero graded components for only finitely many degrees (all nonnegative), this is a connected, graded commutative $\mathbb{Q}$-Hopf algebra, and the symmetric $\mathbb{Q}$-algebra factor must equal $\mathbb{Q}$ (or else there would be nonzero components in infinitely many degrees).

Problem 4 (**Exponential maps in the Abelian case**.) Let T be a connected Lie group that is Abelian. Denote the Lie algebra by $\mathfrak{h}$. Denote the universal covering of T by

$$\phi : (\widetilde{T},\widetilde{e}) \to (T,e).$$

(a) Prove that the Lie group exponential map,

$$\mathrm{Exp}_{\widetilde{T}} : (\mathfrak{h},0) \to (\widetilde{T},\widetilde{e}),$$

is an isomorphism. Conclude that there is a canonical isomorphism of the discrete group $H_1(T;\mathbb{Z})$ with a closed Abelian subgroup of the Lie algebra $\mathfrak{h}$.

(b) When T is a linear complex torus $\cong \mathbb{G}_m(\mathbb{C})^r$, the canonical isomorphism above defines an isomorphism of $\mathbb{C}$-vector spaces,

$$\mathrm{Sym}^d_{\mathbb{C}}(\mathfrak{t}^\vee) \to \mathrm{Sym}^d_{\mathbb{Q}}(X^*(T)\otimes\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}.$$

Check that all of the maps are compatible with the natural actions of the automorphism group of the Lie group, $\mathrm{Aut}(T,e)$. In particular, when a Weyl group W acts on (T, e) by conjugation, the isomorphism above is W-equivariant, hence defines an isomorphism of $\mathbb{C}$-algebras of W-invariants,

$$\text{Sym}^\bullet_{\mathbb{C}}(\mathfrak{t}^\vee)^W \to \text{Sym}^d_{\mathbb{Q}}(X^*(T)\otimes\mathbb{Q})^W \otimes_{\mathbb{Q}} \mathbb{C}.$$

This simplifies the computation of the fundamental invariants.

Problem 5 (**Low homotopy groups of special linear groups.**) Let V be a real or complex vector space of dimension $n+1 \geq 2$. Let $\mathbf{SL}(V) \cong \mathbf{SL}_{n+1}$ be the corresponding real or complex Lie group, let $v \in V \setminus \{0\}$ be an element, and consider the induced standard left action,

$$L_v : \mathbf{SL}(V) \to V \setminus \{0\}.$$

(a) Show that this action is surjective, i.e., the orbit of v equals all of $V\setminus\{0\}$. Show that the stabilizer subgroup H of v is isomorphic, as a Lie group, to a semidirect product of a special linear group $\mathbf{SL}(W) \cong \mathbf{SL}_n$ with $W \cong V/\text{span}(v)$ and a product of copies of the additive group (which are contractible). Thus, the long exact sequence of homotopy groups is

$$\cdots \to \pi_{k+1}(V\setminus\{0\}) \to \pi_k(\mathbf{SL}(W)) \to \pi_k(\mathbf{SL}(V)) \to \pi_k(V\setminus\{0\}) \to \dots$$
$$\to \pi_3(V\setminus\{0\}) \to \pi_2(\mathbf{SL}(W)) \to \pi_2(\mathbf{SL}(V)) \to \pi_2(V\setminus\{0\}) \to \pi_1(\mathbf{SL}(W)) \to \pi_1(\mathbf{SL}(V)) \to \pi_1(V\setminus\{0\}).$$

(b) When the field is $\mathbb{R}$, so that $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to a sphere $\mathbb{S}^{n-1}$, use the long exact sequence of homotopy sequences of a fiber bundle to conclude that for every $k \leq n-2$, the following induced map of homotopy groups is an isomorphism,

$$\pi_k(\mathbf{SL}_n(\mathbb{R})) \to \pi_k(\mathbf{SL}_{n+1}(\mathbb{R}).$$

Thus, the relative homotopy group $\pi_k(\mathbf{SL}_{n+1}(\mathbb{R}), \mathbf{SL}_n(\mathbb{R}))$ is zero for $k \leq n-2$. If you know about relative homotopy groups, also conclude that the following induced map of homotopy groups is surjective,

$$\pi_{n-1}(\mathbf{SL}_{n+1}(\mathbb{R})) \to \pi_{n-1}(\mathbf{SL}_{n+1}(\mathbb{R}), \mathbf{SL}_n(\mathbb{R})).$$

In fact, Bott periodicity states that in this "stable range", the homotopy groups $\pi_k(\mathbf{SL}(\mathbb{R}))$ are periodic in k with period 8,

$$\pi_k(\mathbf{SL}(\mathbb{R})) = \begin{cases} \{0\}, & k \equiv 2,4,5,6 \ (\text{ mod } 8), \\ \mathbb{Z}/2\mathbb{Z}, & k \equiv 0,1 \ (\text{ mod } 8), \\ \mathbb{Z}, & k \equiv 3,7 \ (\text{ mod } 8) \end{cases}$$

(c) If you know the Hurewicz theorem, use the previous part to conclude that the relative homology groups $H_k(\mathbf{SL}_{n+1}(\mathbb{R}), \mathbf{SL}_n(\mathbb{R}); \mathbb{Z})$ are zero for $k \leq n-2$ and the following map of homology groups is surjective,

$$H_{n-1}(\mathbf{SL}_{n+1}(\mathbb{R}); \mathbb{Z}) \to H_{n-1}(\mathbf{SL}_{n+1}(\mathbb{R}), \mathbf{SL}_n(\mathbb{R}); \mathbb{Z}).$$

Use the long exact sequence of relative homology to conclude that the pushforward maps

$$H_k(\mathbf{SL}_n(\mathbb{R}); \mathbb{Z}) \to H_k(\mathbf{SL}_{n+1}(\mathbb{R}); \mathbb{Z})$$

are isomorphisms for every $k \leq n-2$.

(d) For $n = 2$, conclude that $\mathbf{SL}_2(\mathbb{R})$ has the homotopy type of a circle. Thus, $\pi_1\mathbf{SL}_2(\mathbb{R})$ equals $\mathbb{Z}$, and all higher homotopy groups are zero. Similarly, when $n = 3$, compute that $\pi_2(\mathbb{R}^3 \setminus \{0\}) \to \pi_1(\mathbf{SL}_2(\mathbb{R}))$ is nonzero. Conclude that $\pi_1(\mathbf{SL}_3(\mathbb{R}))$ is a finite cyclic group (possibly zero) and $\pi_2(\mathbf{SL}_3(\mathbb{R}))$ is zero. In fact, $\pi_1(\mathbf{SL}_3(\mathbb{R}))$ is cyclic of order 2 (see the next problem).

(e) For every $n \geq 4$, conclude that $\pi_2(\mathbf{SL}_n(\mathbb{R}))$ is zero and $\pi_1(\mathbf{SL}_n(\mathbb{R}))$ is cyclic of order 2. For the universal cover $\widetilde{\mathbf{SL}_n}(\mathbb{R})$, use the Hurewicz theorem to conclude that the homology groups H_r vanish for $r = 1$ and $r = 2$. Conclude that the following Hurewicz map is an isomorphism,

$$\pi_3(\mathbf{SL}_n(\mathbb{R})) \to H_3(\mathbf{SL}_n(\mathbb{R}); \mathbb{Z}).$$

(f) Repeat these parts for the field $\mathbb{C}$. Conclude that for every $k \leq 2n - 1$, both of the following pushforward maps are isomorphisms,

$$\pi_k(\mathbf{SL}_n(\mathbb{C})) \to \pi_k(\mathbf{SL}_{n+1}(\mathbb{C})), \quad H_k(\mathbf{SL}_n(\mathbb{C})) \to H_k(\mathbf{SL}_{n+1}(\mathbb{C})).$$

In fact, by Bott periodicity, in this stable range the homotopy groups are periodic in $k \geq 2$ with period 2,

$$\pi_k(\mathbf{SL}(\mathbb{C})) = \begin{cases} \{0\}, & k \equiv 0 \pmod 2, \\ \mathbb{Z}, & k \equiv 1 \pmod 2) \end{cases}$$

Explicitly compute that $\mathbf{SL}_2(\mathbb{C})$ has the homotopy of the 3-sphere $\mathbb{S}^3$. For every $n \geq 2$, prove that $\pi_1(\mathbf{SL}_n(\mathbb{C}))$ and $\pi_2(\mathbf{SL}_n(\mathbb{C}))$ are zero and the following Hurewicz map is an isomorphism,

$$\pi_3(\mathbf{SL}_n(\mathbb{C})) \to H_3(\mathbf{SL}_n(\mathbb{C}); \mathbb{Z}).$$

(g) If you know about Leray spectral sequences, write the Leray spectral sequence in homology or cohomology associated to the fibration $\mathbf{SL}_{n+1}(\mathbb{C}) \to \mathbb{C}^{n+1} \setminus \{0\} \sim \mathbb{S}^{2n+1}$. Since the homology of $\mathbb{S}^{2n+1}$ is concentrated in just 2 degrees, and since the sum of the Betti numbers for $\mathbf{SL}_{n+1}(\mathbb{C})$ equals 2^n, which is 2 times the sum for $\mathbf{SL}_n(\mathbb{C})$, conclude that the spectral sequence degenerates (all differentials are zero). Thus, there is an isomorphism,

$$H^*(\mathbf{SL}_{n+1}(\mathbb{C}); \mathbb{Z}) \cong H^*(\mathbf{SL}_n(\mathbb{C}); \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(\mathbb{S}^{2n+1}; \mathbb{Z}).$$

By induction, conclude that

$$H^*(\mathbf{SL}_{n+1}(\mathbb{C}); \mathbb{Z}) \cong H^*(\mathbb{S}^3; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(\mathbb{S}^5; \mathbb{Z}) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} H^*(\mathbb{S}^{2n+1}; \mathbb{Z}).$$

Thus, by Hopf's theorem, the degrees $1 + d_i$ for the fundamental invariants of $\mathbf{SL}_{n+1}(\mathbb{C})$ are $2, 3, \dots, n, n+1$, i.e., $(d_1, \dots, d_n) = (1, 2, \dots, n-1, n)$.

Problem 6 (Gram-Schmidt and retractions to maximal compact subgroups.) For a (positive definite) real inner product space $(V_{\mathbb{R}}, \langle \bullet, \bullet \rangle_{\mathbb{R}})$, resp. for a (positive definite) complex Hermitian inner product space $(V_{\mathbb{C}}, \langle \bullet, \bullet \rangle_{\mathbb{C}})$, review the statement of the Gram-Schmidt theorem.

(a) In the two respective cases, inside the Borel subgroup $B_{\mathbb{R}} \subset \mathbf{GL}(V_{\mathbb{R}})$, resp. inside the Borel subgroup $B_{\mathbb{C}} \subset \mathbf{GL}(V_{\mathbb{C}})$, define B_+ to be the closed real Lie subgroup of upper triangular matrices that have only positive entries on the main diagonal. Define SB_+ to be the intersection of B_+ with the special linear group, i.e., the subset of B_+ of elements with determinant equal to 1. In the respective cases, interpret Gram-Schmidt as saying that the following multiplication map is a diffeomorphism,

$$\mathbf{O}(V_{\mathbb{R}}, \langle \bullet, \bullet \rangle_{\mathbb{R}}) \times B_+ \to \mathbf{GL}(V_{\mathbb{R}}),$$

$$\mathbf{SO}(V_{\mathbb{R}}, \langle \bullet, \bullet \rangle_{\mathbb{R}}) \times SB_+ \to \mathbf{SL}(V_{\mathbb{R}}),$$

$$\mathbf{U}(V_{\mathbb{C}}, \langle \bullet, \bullet \rangle_{\mathbb{C}}) \times B_+ \to \mathbf{GL}(V_{\mathbb{C}}),$$
$$\mathbf{SU}(V_{\mathbb{C}}, \langle \bullet, \bullet \rangle_{\mathbb{C}}) \times SB_+ \to \mathbf{SL}(V_{\mathbb{C}}).$$

(b) Show that B_+ and SB_+ are contractible. Conclude that the diffeomorphisms above define retractions of $\mathbf{GL}_n(\mathbb{R})$ to the (compact) orthogonal group $\mathbf{O}(n, \mathbb{R})$, of $\mathbf{SL}_n(\mathbb{R})$ to the (compact) special orthogonal group $\mathbf{SO}(n, \mathbb{R})$, of $\mathbf{GL}_n(\mathbb{C})$ to the (compact) unitary group $\mathbf{U}(n, \mathbb{R})$, and of $\mathbf{SL}_n(\mathbb{C})$ to the (compact) special unitary group $\mathbf{SU}(n, \mathbb{R})$. Combine this with the previous exercise to write down the low degree homotopy groups and homology groups of $\mathbf{SU}(n, \mathbb{R})$.

(c) Finally, modify the argument from lecture to prove that the conjugation action of $\mathbf{PSU}(2, \mathbb{R}) = \mathbf{SU}(2, \mathbb{R})/\mu_2 \cdot \mathrm{Id}_{2\times 2}$ on the Lie algebra $\mathfrak{su}(2, \mathbb{R})$ with its (minus) determinant inner product defines an isomorphism $\mathbf{SU}(2, \mathbb{R})/\mu_2 \to \mathbf{SO}(3, \mathbb{R})$. From this, conclude that $\pi_1\mathbf{SO}(3, \mathbb{R})$ is cyclic of order 2, and thus also $\pi_1\mathbf{SL}_2(\mathbb{R})$ is cyclic of order 2. This completes the missing step of the previous exercise. Use this to complete the computation of the low degree homotopy and homology groups of the real Lie groups $\mathbf{SL}_n(\mathbb{R})$. Via the Gram-Schmidt retractions, also complete the computation of the low degree homotopy and homology groups of the real Lie groups $\mathbf{SO}(n, \mathbb{R})$.

(d) Read through Exercises 2.14 – 2.16 from the textbook. This explains the modifications of the above necessary to compute the low degree homotopy groups and homology groups of the symplectic groups $\mathbf{Sp}_{2n}(\mathbb{C})$ and their real compact forms, $\mathbf{Sp}(2n, \mathbb{R})$. Show that all of these have vanishing π_1, π_2, H_1 and H_2. Thus, the Hurewicz map from π_3 to H_3 is an isomorphism.

Problem 7 (**Triviality of the second homotopy group of a connected Lie group**.) Read through the following, and work out the details for one or two connected Lie groups. If you are not comfortable with higher homotopy groups, consider only the case that G is simply connected. In this case, by the Hurewicz theorem, π_2 equals the homology group H_2.

All of the connected Lie groups in the previous examples had vanishing π_2. In fact that is always true. One proof reduces to the case of a compact, connected Lie group. Via the Abelianization homomorphism, every such Lie group surjects onto a product of copies of the circle (which has vanishing π_2), where the kernel is a compact, connected Lie group with finite fundamental group. Using the long exact sequence of a fibration, it suffices to prove the result for a compact, connected Lie group with finite fundamental group. Up to passing to the (finite) universal covering, it suffices to prove the result for compact, connected, simply connected Lie groups.

The proof then uses the remarkable fact (proved by Borel) that every such Lie group is the compact form of a connected, simply connected, semisimple, complex Lie group G, which retracts onto the original Lie group. Thus the two Lie groups are homotopy equivalent, and it suffices to prove that $\pi_2(G)$ is trivial.

For a Borel subgroup B of G (which Borel proved exists), the quotient complex manifold G/B is a complex projective manifold as in the first problem. Since B is homotopy equivalent to a product of circles, which has vanishing π_2, the long exact sequence gives

$$0 \to \pi_2(G) \to \pi_2(G/B) \to \pi_1(B).$$

By the Hurewicz theorem, $\pi_2(G/B)$ equals $H_2(G/B)$, and this is a free Abelian group by the Bruhat decomposition of the cellular homology of G/B.

Thus, the subgroup $\pi_2(G)$ of $\pi_2(G/B)$ is also a free Abelian group. By the Hurewicz theorem again, also $H_2(G)$ equals $\pi_2(G)$, and this is a free Abelian group. The rank of this group equals the $\mathbb{Q}$-dimension of the $\mathbb{Q}$-vector space $H_2(G;\mathbb{Q})$. By Hopf's theorem, the homology $H_*(G;\mathbb{Q})$ with its Pontryagin product is a connected, graded commutative, Hopf $\mathbb{Q}$-algebra that is isomorphic to a graded exterior algebra on finitely many generators of odd degree. Since G is simply connected, $H_1(G;\mathbb{Q})$ is zero. Thus, all generators are in odd degree ≥ 3. Therefore $H_2(G;\mathbb{Q})$ is zero.

Problem 8 (**The third homotopy group of a connected, simply connected, simple complex Lie group.**) This exercise requires the use of spectral sequences. Please read through this exercise, but only attempt if you are comfortable with algebraic topology.

Let G be a connected, simply connected, simple complex Lie group. Let $T \subset G$ be a maximal torus. Let $B \subset G$ be a Borel subgroup that contains T.

(a) Prove that the quotient holomorphic map

$$G/T \to G/B$$

is a G-equivariant map of G-homogeneous complex manifolds. Conclude that the map is a fiber bundle. Also, the fiber over the distinguished point B/B is the quotient manifold B/T, which is isomorphic to a product of additive groups $\mathbb{C}$. Thus, the fibers are connected and contractible. Conclude that the quotient holomorphic map is a homotopy equivalence. Thus, the homotopy, homology, and cohomology of G/T equals the same for G/B. In particular, the Bruhat decomposition for G/B describes the homology and cohomology of G/T.

(b) Since T has the homotopy type of a product of circles, all of its higher homotopy groups are zero. Conclude that for every $n \geq 3$, the following maps of homotopy groups are isomorpisms,

$$\pi_n(G) \xrightarrow{\cong} \pi_n(G/T) = \pi_n(G/B), \quad n \geq 3.$$

Of course we also have the sequence

$$0 = \pi_2(G) \to \pi_2(G/T) \xrightarrow{\cong} \pi_1(T) \to \pi_1(G) = 0.$$

(c) Show that the conjugation action of the Weyl group $W \subset N_G(T)$ on G sends left T-cosets to left T-cosets. Conclude that there is an induced action of W on the quotient manifold G/T such that the quotient map

$$q : G \to G/T$$

is W-equivariant. Thus, the induced maps of homotopy groups, homology groups, and cohomology groups are W-equivariant. Moreover, the Leray spectral sequences associated to q converging to the homology, resp. cohomology, of G are W-equivariant.

(d) Since the conjugation action of W on G is the restriction of the conjugation action of the entire group G on G, and since G is path connected, conclude that every conjugation map of G to itself is homotopic to the identity map. Conclude that the W-action on the homotopy groups, homology groups, and cohomology groups of G are all trivial. Thus, the image of the homotopy groups, resp. homology groups, of G in the homotopy groups, resp. homology groups, of G/T are contained in the W-invariant subgroup.

(e) Since T has the homotopy type of a product of circles, conclude that the homology, resp. cohomology, of the linear complex torus T with its W-action is an exterior algebra on its degree 1 part, which equals $X_*(T)$, resp. which equals $X^*(T)$, with its natural W-action.

(f) Write down the low degree terms of the Leray spectral sequence associated to q. Use the vanishing of $H_1(G;\mathbb{Z})$ and $H_2(G;\mathbb{Z})$ to conclude that the following transgression map is a W-equivariant isomorphism,

$$H_2(G/T;\mathbb{Z}) \to H_1(T;\mathbb{Z}) = X_*(T).$$

Using this, conclude that $H_3(G;\mathbb{Z})$ is the free Abelian group that is dual to the kernel K of the cup product map

$$\mathrm{Sym}^2_{\mathbb{Z}} X^*(T) \to H^4(G/T;\mathbb{Z}),$$

and K is W-invariant.

(g) From Borel's theorem, conclude that $K \otimes \mathbb{Q}$ is the $\mathbb{Q}$-span of the fundamental invariant F_1 of degree 2.

(h) If you know about Whitehead products, conclude that the pairing

$$[\bullet, \bullet] : \pi_2(G/T) \times \pi_2(G/T) \to \pi_3(G/T) = \pi_3(G)$$

equals the unique nonzero W-invariant symmetric, bilinear pairing on $X_*(T) = \pi_2(G)$, at least up to nonzero scaling. Since the Weyl group acts by isometries of $X_*(T) \otimes \mathbb{R}$ with respect to this pairing that preserve all of the fundamental invariants, conclude that we can recover the root system of (G,T) from the data of the cohomology algebra $H^*(G/T;\mathbb{Q})$. (If you know the **Formality Theorem** of Deligne-Griffiths-Morgan-Sullivan, this gives a method for constructing the relevant homotopy groups and Whitehead products directly from the cohomology algebra.)

Problem 9 (**Lie algebra of $\mathbf{SL}_2$.**) With the same notation as in Problem 4 of Problem Set 1, let h denote the derivative in $\mathfrak{h}'_2$ at 1 of the cocharacter $z \mapsto \rho_{2,1}(z) \cdot (\rho_{2,2}(z))^{-1}$. Compute explicitly the Lie brackets of the elements h, $E_{1,2}$ and $E_{2,1}$ in the Lie algebra $\mathfrak{sl}_2$.

Problem 10(**More about Borel's theorem and Hopf's theorem.**) This problem is for those students who know about the Leray-Serre spectral sequence. Let G be a connected Lie group and let R be a coefficient ring for cohomology such that Hopf's theorem holds, i.e., $H^*(G;R)$ equals a free exterior algebra on generators $g_1, \dots, g_r$ in odd degrees $1 + 2d_1 \leq \dots 1 + 2d_r$. There is a CW complex BG (well defined up to homotopy) and a principal G-bundle over BG,

$$\pi_G : EG \to BG, \quad \mu_G : G \times EG \xrightarrow{\cong} EG \times_{BG} EG,$$

such that the total space EG of the principal G-bundle is a contractible CW complex.

It follows from the long exact sequence that BG is simply connected and $H_2(BG;\mathbb{Z}) = \pi_2(BG)$ equals the Abelian group $\pi_1(G) = H_1(G;\mathbb{Z})$. The Leray-Serre spectral sequence for π_G is a second-page, cohomological spectral sequence,

$$E_2^{p,q}(\pi_G) = H^p(BG;R) \otimes_R H^q(G;R) \Rightarrow H^{p+q}(EG;R).$$

This is a spectral sequence of graded $H^*(BG;R)$-algebras converging to the cohomology of the contractible space EG, i.e., converging to R concentrated in degree 0. In particular, each element $1\otimes g_i$ in $E_2^{0,1+2d_i}$ must map to a nonzero divisor under some differential of the spectral sequence. Chase through the spectral sequence to conclude the following.

Theorem 0.7 (Borel). *The differentials of the spectral sequence map $1\otimes g_i$ to a zero element until the differential $d_{2+2d_i}: E_{2+2d_i}^{0,1+2d_i} \to E_{2+2d_i}^{2+2d_i,0}$, which maps to the image of a nonzero element $F_i\otimes 1 \in H^{2+2d_i}(BG;R)\otimes_R R$. The graded cohomology R-algebra $H^*(BG;R)$ is the free symmetric R-algebra on the elements $F_1,\dots,F_r$ of degrees $2(1+d_i),\dots,2(1+d_r)$.*

Now assume that G is a connected, simply connected, compact real Lie group, and let T be a maximal torus isomorpic to a product of r copies of the circle $U(1)$. The inclusion $i_{G,T}: T\subset G$ induces a continuous map of CW complexes, $Bi_{G,T}: BT\to BG$, well-defined up to homotopy, such that the pullback G-bundle $BT\times_{BG} EG$ is G-equivariantly homotopy equivalent to G/T,

$$\pi_{G,T}: BT\times_{BG} EG \to BT.$$

The action of the finite Weyl group $\mathcal{W}$ on T by conjugation induces an action of $\mathcal{W}$ BT, further inducing an action of $\mathcal{W}$ on the cohomology. The pullback map on cohomology associated to $Bi_{G,T}$ lands in the Weyl-invariant R-submodule.

Since BT is an Eilenberg-MacLane space $K(\pi_1(T),2)$, the cohomology $H^*(BT;\mathbb{Z})$ is equal as a graded $\mathbb{Z}$-algebra to the free polynomial $\mathbb{Z}$-algebra,

$$H^*(BT;\mathbb{Z}) \cong \mathrm{Sym}^\bullet_{\mathbb{Z}}(H^2(BT;\mathbb{Z})) = \mathrm{Sym}^\bullet_{\mathbb{Z}}(X^*(T)),$$

where $X^*(T)$ equals the finite free $\mathbb{Z}$-module of Lie group morphisms from T to $U(1)$. In particular, the graded $\mathbb{R}$-algebra $H^*(BT;\mathbb{R})$ equals the graded $\mathbb{R}$-algebra of $\mathbb{R}$-valued polynomial functions on the Lie algebra $\mathfrak{t}$ of T,

$$H^*(BT;\mathbb{R}) \cong \mathrm{Sym}^\bullet_{\mathbb{R}}(\mathfrak{t}^\vee).$$

In his 1950 ICM address, S.-S. Chern used Chern-Weil theory for principal G-bundles over a manifold to introduce a de Rham theory of differential forms on BG, defining a ring homomorphism from $\mathrm{Sym}^\bullet_{\mathbb{R}}(\mathfrak{g}^\vee)^{\mathrm{Ad}_G}$ to the de Rham cohomology of BG, which is isomorphic as a graded $\mathbb{R}$-algebra to $H^*(BG;\mathbb{R})$. Since the restriction from G to T induces an isomorphism,

$$\mathrm{Sym}^\bullet_{\mathbb{R}}(\mathfrak{g}^\vee)^{\mathrm{Ad}_G} \xrightarrow{\cong} \mathrm{Sym}^\bullet_{\mathbb{R}}(\mathfrak{t}^\vee)^W \subset \mathrm{Sym}^\bullet_{\mathbb{R}}(\mathfrak{t}^\vee),$$

conclude that the pullback map $H^*(BG;\mathbb{Q})\to H^*(BT;\mathbb{Q})^W$ is surjective. Thus, the image of $H^*(BG;\mathbb{Q})$ in $H^*(BT;\mathbb{Q})$ equals the graded $\mathbb{Q}$-subalgebra $H^*(BT;\mathbb{Q})^W$ of W-invariant elements.

Use Chevalley-Shepherd-Todd to conclude that this $\mathbb{Q}$-subalgebra is a polynomial $\mathbb{Q}$-algebra. Use the Leray-Serre Spectral sequence for $\pi_{G,T}$ to conclude Borel's Theorem when R equals $\mathbb{Q}$, i.e., there is an isomorphism,

$$\mathrm{Sym}^\bullet_{\mathbb{Q}}(X^*(T)\otimes\mathbb{Q})/\mathrm{Sym}^{>0}_{\mathbb{Q}}(X^*(T)\otimes\mathbb{Q})^W \xrightarrow{\cong} H^*(G/T;\mathbb{Q}),$$

and the invariant ideal is a complete intersection ideal generated by homogeneous elements $F_1,\dots,F_r$ of degrees $1+d_1,\dots,1+d_r$. Moreover, as a W-representation, this is isomorphic to the $\mathbb{Q}$-group algebra of W.

Since the fiber product $BT \times_{BG} EG$ is G-equivariantly homotopic to G/T, conclude that there is a Leray-Serre Spectral Sequence associated to the morphism $BT \to BG$,

$$E_2^{p,q} = H^p(BG;\mathbb{Q}) \otimes_R H^q(G/T;\mathbb{Q}) \Rightarrow H^{p+q}(BT;\mathbb{Q}).$$

Show that this is equivariant for the W-action on $H^*(G/T;\mathbb{Q})$ and the W-action on $H^*(BT;\mathbb{Q})$. Thus, there is an induced spectral sequence of W-invariant $H^*(BG;\mathbb{Q})$-algebras. Since $H^*(G/T;\mathbb{Q})$ is isomorphic as a W-representation to the group $\mathbb{Q}$-algebra of W, the W-invariants is simply $H^0(G/T;\mathbb{Q}) = \mathbb{Q}$. Conclude that the induced spectral sequence of W-invariants degenerates to an isomorphism,

$$H^*(BG;\mathbb{Q}) \to H^*(BT;\mathbb{Q})^W.$$

In fact, Borel proves this without inverting all integers to get $\mathbb{Q}$.

Corollary 0.8 (Borel)**.** *For a coefficient ring R such that $H^*(G;R)$ is a free exterior algebra, the pullback map $H^*(BG;R) \to H^*(BT;R)^{\mathcal{W}}$ is an isomorphism of graded R-algebras.*

In particular, when R equals $\mathbb{R}$, the field of real numbers, this gives the following.

Corollary 0.9. *In the special case of the coefficient ring $\mathbb{R}$, there is a canonical isomorphism of $H^*(BG;\mathbb{R})$ with the $\mathbb{R}$-subalgebra $Sym^{\bullet}_{\mathbb{R}}(\mathfrak{g}^\vee)^{Ad_G}$ of Ad_G-invariant $\mathbb{R}$-valued polynomials on the Lie algebra $\mathfrak{g}$, and the pullback map $H^*(BG;\mathbb{R}) \to H^*(BT;\mathbb{R})$ is canonically the restriction map of polynomials,*

$$Sym^{\bullet}_{\mathbb{R}}(\mathfrak{g}^\vee)^{Ad_G} \xrightarrow{\cong} Sym^{\bullet}_{\mathbb{R}}(\mathfrak{t}^\vee)^W \subset Sym^{\bullet}_{\mathbb{R}}(\mathfrak{t}^\vee).$$

In his 1950 ICM address, S.-S. Chern gives an interpretation of this via Chern-Weil theory, where he introduces an $\mathbb{R}$-algebra map to the de Rham cohomology of BG, essentially the inverse isomorphism of the map above,

$$\mathrm{Sym}^{\bullet}_{\mathbb{R}}(\mathfrak{g}^\vee)^{\mathrm{Ad}_G} \to H^*(BG;\mathbb{R}).$$