MAT 312/AMS 351 Problem Set 1

Homework Policy. Write up solutions of the required problems. Read and attempt the extra problems, but do not write up those solutions for grading. Many of these problems are taken from or inspired by problems in the textbook.

Each student is encouraged to work with other students, but submitted problem sets must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource.

Required Problems.

Problem 1.(Atiyah and MacDonald, p. 11, 1.1) For every nilpotent element x of a commutative unital ring A, prove that $1+x$ is a unit. Prove that for every unit u of A, also $u+x$ is a unit.

Problem 2.(Atiyah and MacDonald, p. 11, 1.2) For every ring A, for every element $f = a_0 + a_1x + \cdots + a_nx^n$ in the polynomial ring $A[x]$, prove that f is a unit in $A[x]$ if and only if a_0 is a unit in A and every $a_1, \dots, a_n$ is nilpotent. Similarly, prove that f is nilpotent if and only if every $a_0, \dots, a_n$ is nilpotent. Prove that f is a zerodivisor if and only if there exists $b \in A \setminus \{0\}$ such that bf equals 0. Finally, if f and g are elements of $A[x]$ such that for every maximal ideal $\mathfrak{m} \subset A$, the images of f and g in $(A/\mathfrak{m})[x]$ are both nonzero, then prove that also the image of fg is also nonzero.

Problem 3.(Atiyah and MacDonald, p. 11, 1.4) For every commutative, unital ring A, prove that the Jacobson radical of $A[x]$ equals the nilradical of $A[x]$.

Problem 4.(Atiyah and MacDonald, p. 11, 1.8) For every commutative, unital ring A and for every nonempty totally ordered collection of prime ideals, prove that the intersection of these prime ideals is again a prime ideal. Conclude that every prime ideal contains a minimal prime ideal.

Problem 5.(Atiyah and MacDonald, p. 11, 1.9) For every ideal $\mathfrak{a}$ of A that is properly contained in A, prove that the radical ideal of $\mathfrak{a}$ equals the intersection of all prime ideals of A that contain $\mathfrak{a}$.

Problem 6.(Atiyah and MacDonald, p. 32, 2.3) For every local ring A, for nonzero finitely generated A-modules M and N, prove that also $M \otimes_A N$ is nonzero.

Problem 7.(Atiyah and MacDonald, p. 32, 2.8) For A-flat A-modules M and N, prove that also $M \otimes_A N$ is A-flat. For every flat A-algebra B, and for every B-flat B-module P, prove that also P is A-flat.

Problem 8.(Atiyah and MacDonald, p. 32, 2.9) For a short exact sequence of A-modules, if the first and third modules are finitely generated, then so is the middle module. Then use the Snake Lemma to prove that if the first and third modules are finitely presented, then so is the middle module.

Problem 9.(Atiyah and MacDonald, p. 32, 2.19) Prove that for every filtering partially ordered set I, prove exactness of the functor from the category of I-compatible systems of A-modules to the category of A-modules that sends each I-system to its colimit.

Problem 10.(Atiyah and MacDonald, p. 32, 2.20) Prove that tensor product commutes with colimits.

Extra Problems. Atiyah and MacDonald, p. 11, Exercise 1.6. p. 32, Exercises 2.2, 2.4, 2.10, 2.14, 2.15, 2.16, 2.17, 2.18, 2.21, 2.24, 2.25, 2.26.