MAT 543 Problem Set 9

Homework Policy. Read through and carefully consider all of the following problems. Please write up and hand-in solutions to **five** of the problems.

Each student is encouraged to work with other students, but submitted problem sets must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource.

Textbook Problems.

Problem 1. Problem 18.1, p. 152, Forster.

Problem 2. Problem 18.2, p. 152, Forster.

Problem 3. Problem 18.3, p. 152, Forster.

Problem 4. Problem 19.1, p. 158, Forster.

Problem 5. Problem 19.5, p. 158, Forster.

Problem 6. For a compact Riemann surface X of genus $g \geq 2$, if there exists a nonconstant, holomorphic function $f : X \to \mathbb{P}^1$ of degree 2, prove that f is unique up to composing with a holomorphic automorphism $\mathbb{P}^1 \to \mathbb{P}^1$. These Riemann surfaces are precisely the **hyperelliptic curves**.

Problem 7. A compact Riemann surface X of genus $g \geq 2$ is **bielliptic** if there exists an elliptic curve Y and a nonconstant, holomorphic function $f : X \to Y$ of degree 2. Prove that the gonality of X is at most 4.

Problem 8. If $g \geq 4$, prove that X cannot be both bielliptic and hyperelliptic (hint: the $2g+2$ Weierstrass points would map under f to at least $g+1$ points p_i of Y such that $2p_i$ is linear equivalent to $2p_j$ for every i and j). Conversely, for an elliptic curve Y and a nonconstant, holomorphic function $h : Y \to \mathbb{P}^1$ of degree 2, for a degree 2, branched cover $u : C \to \mathbb{P}^1$ with C a genus 0 curve (i.e., C is biholomorphic to $\mathbb{P}^1$), prove that the branched cover of C obtained from $C \times_{\mathbb{P}^1} Y$ is both bielliptic and hyperelliptic of genus $g = 1$, 2, or 3, depending on whether the degree 2 branch divisor of u has 2, 1, or 0 points of intersection with the degree 4 branch divisor of h.

Problem 9. For a bielliptic curve, prove that the ramification divisor D of f in X is in the linear equivalence class of the canonical divisor class K_X, i.e., there exists a meromorphic $(1,0)$-form whose divisor is precisely D. Moreover, for every p with $\mathrm{ord}_p(D) > 0$, prove that the order equals

1, and prove that there exists a meromorphic function g on X with an order-4 pole at p, and with g holomorphic on $X \setminus \{p\}$.

Problem 10. A compact Riemann surface X of genus $g \geq 2$ is **trigonal** if there exists a non-constant, holomorphic function $f : X \to \mathbb{P}^1$ of degree 3. Prove that a trigonal curve cannot be hyperelliptic. (In fact, the only trigonal bielliptic curves have genus $g = 4$.)