MAT 543 Problem Set 4

Homework Policy. Read through and carefully consider all of the following problems. Please write up and hand-in solutions to **five** of the problems.

Each student is encouraged to work with other students, but submitted problem sets must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource.

Textbook Problems.

Problem 1. Problem 10.1, p. 81, Forster.

Problem 2. Problem 10.2, p. 81, Forster.

Problem 3. Problem 10.3, p. 81, Forster.

Problem 4. Problem 10.4, p. 81, Forster.

Problem 5. For every $\mathbb{R}$-vector space $V_{\mathbb{R}}$, a *complex extension* of $V_{\mathbb{R}}$ is a pair $(W, T : V_{\mathbb{R}} \to W)$ of a $\mathbb{C}$-vector space W and an $\mathbb{R}$-linear transformation $T : V_{\mathbb{R}} \to W$. The complex extension is *initial* if some $\mathbb{R}$-basis $\mathcal{B}$ for $V_{\mathbb{R}}$ maps injectively to a $\mathbb{C}$-basis for W.

(a) Prove that there exists an initial complex extension $(V_{\mathbb{C}}, r : V_{\mathbb{R}} \to V_{\mathbb{C}})$.

(b) For an initial complex extension $(V_{\mathbb{C}}, r : V_{\mathbb{R}} \to V_{\mathbb{C}})$, for every complex extension $(W, T : V_{\mathbb{R}} \to W)$, prove that there exists a unique $\mathbb{C}$-linear transformation $T_{\mathbb{C}} : V_{\mathbb{C}} \to W$ such that $T_{\mathbb{C}} \circ r$ equals T. In particular, conclude that $(V_{\mathbb{C}}, r : V_{\mathbb{R}} \to V_{\mathbb{C}})$ is unique up to unique $\mathbb{C}$-linear isomorphism, and every $\mathbb{R}$-linear map of $\mathbb{R}$-vector spaces, $f : V_{\mathbb{R}} \to V'_{\mathbb{R}}$, extends uniquely to a $\mathbb{C}$-linear map, $f_{\mathbb{C}} : V_{\mathbb{C}} \to V'_{\mathbb{C}}$ such that $f_{\mathbb{C}} \circ r$ equals $r' \circ f$. For these reasons, $V_{\mathbb{C}}$ is often called a *complexification* of $V_{\mathbb{R}}$.

(c) For the field automorphism $\sigma : \mathbb{C} \to \mathbb{C}$ defined by $\sigma(x + iy) = \sigma(z) = \overline{z} = x - iy$, for every $\mathbb{C}$-vector space W, define $W^{\sigma} = \overline{W}$ to be the $\mathbb{C}$-vector space whose underlying $\mathbb{R}$-vector space equals W, but with new scalar multiplication $*$ defined by $z * \vec{w} := \sigma(z) \cdot \vec{w}$. Prove that the identity map $W \to \overline{\overline{W}}$ is a $\mathbb{C}$-linear map. Prove that for every complex extension $(W, T : V_{\mathbb{R}} \to W)$, also $(\overline{W}, T : V_{\mathbb{R}} \to \overline{W})$ is a complex extension. Conclude that there exists a unique $\mathbb{C}$-linear isomorphism $c_V : V_{\mathbb{C}} \to \overline{V}_{\mathbb{C}}$ such that $c_V \circ r$ equals r. Prove that this same function considered as a map $\overline{V}_{\mathbb{C}} \to V_{\mathbb{C}}$ is $\mathbb{C}$-linear. Show that $c_V \circ c_V$ equals the identity.

(d) A *descent datum* is a pair (W, c) of a $\mathbb{C}$-vector space W and a $\mathbb{C}$-linear isomorphism $c : W \to \overline{W}$ such that $c \circ c$ equals the identity. Stated differently, $c(z \cdot \vec{w}) = \overline{z} \cdot c(\vec{w})$, and $c(c(\vec{w})) = \vec{w}$ for every

$z \in \mathbb{C}$ and for every $\vec{w} \in W$. For every descent datum, define $r : V_{\mathbb{R}} \hookrightarrow W$ to be the fixed set of c. Prove that $V_{\mathbb{R}}$ is an $\mathbb{R}$-linear subspace of W. Prove that $(W, r : V_{\mathbb{R}} \to W)$ is a complexification of $V_{\mathbb{R}}$. Also prove that $c_V : W \to \overline{W}$ equals c. Thus, every descent datum is *effective*, i.e., it is a complexification.

(e) Now let V be a $\mathbb{C}$-vector space considered as an $\mathbb{R}$-vector space. Show that $r : V \to V_{\mathbb{C}}$ need not be $\mathbb{C}$-linear for the specified structures of $\mathbb{C}$-vector space on the source and target. For the $\mathbb{C}$-linear map $f : V \to V$ by $f(\vec{v}) = i \cdot \vec{v}$, considering f as an $\mathbb{R}$-linear map, the associated $\mathbb{C}$-linear map $f_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ is not necessarily equal to multiplication by i. This $\mathbb{C}$-linear map is usually denoted by J. Define $V_{1,0} \subset V_{\mathbb{C}}$, resp. $V_{0,1} \subset V_{\mathbb{C}}$, to be the $+i$-eigenspace, resp. $-i$-eigenspace, for J. Prove that these two $\mathbb{C}$-vector subspaces give a direct sum decomposition of V. For the corresponding $\mathbb{C}$-linear projections, $\pi_{1,0} : V_{\mathbb{C}} \to V_{1,0}$, resp. $\pi_{0,1} : V_{\mathbb{C}} \to V_{0,1}$, prove that the compositions $\pi_{1,0} \circ r : V \to V_{1,0}$, resp. $\pi_{0,1} \circ r : V \to V_{0,1}$, are both $\mathbb{R}$-linear isomorphisms. Show that the first composition is even $\mathbb{C}$-linear, but the second composition defines a $\mathbb{C}$-linear isomorphism $\overline{V} \to V_{0,1}$. In this sense, $V_{\mathbb{C}}$ equals $V \oplus \overline{V}$. Finally, check that the map $c : V_{\mathbb{C}} \to \overline{V}_{\mathbb{C}}$ restricts to a $\mathbb{C}$-linear isomorphism $V_{0,1} \to \overline{V}_{1,0}$ that intertwines the two compositions above.

Problem 6. Continue the previous problem by explaining how complexification behaves with respect to the dual real vector space $V^{\vee}$ for finite-dimensional $\mathbb{R}$-vector spaces $V_{\mathbb{R}}$. Also explain how complexification behaves with respect to the real exterior powers $\bigwedge_{\mathbb{R}}^m V_{\mathbb{R}}$. For a finite-dimensional $\mathbb{C}$-vector space V considered as a $\mathbb{R}$-vector space, describe how the complexification and its J-eigenspace decomposition behave with respect to the dual **complex** vector space $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Finally, and most importantly, describe how the complexification and its J-eigenspace decomposition behave with respect to the **complex** exterior powers $\bigwedge_{\mathbb{C}}^m V$.

Problem 7. Let $d \geq 1$ be an integer. Let X, resp. Y, each be a copy of $\mathbb{P}^1$ with holomorphic coordinate z, resp. w, on the open complement of ∞. Define $\pi : Y \to X$, resp. $\rho : Y \to X$, by $z = \pi(w) = w^d$, resp. by $z = \rho(w) = w^d + w^{-d}$. Prove that each of these is a proper holomorphic map that is Galois, determine the isomorphism type of the Galois group, and explicitly determine the linear fractional transformations of Y giving the deck transformations.

Problem 8. Let X be a compact Riemann surface. Prove the assertion from lecture that the $\mathbb{C}$-vector space of global sections $(\mathcal{M}_X/\mathcal{O}_X)(X)$ the sheaf $\mathcal{M}_X/\mathcal{O}_X$ of Laurent tails is canonically identified (via the $\mathbb{C}$-linear map sending a section to its germs) with the direct sum over all points $p \in X$ of the stalk $\mathcal{M}_{X,p}/\mathcal{O}_{X,p}$. In terms of a local holomorphic coordinate $z : (U, p) \to (V, 0)$ at p, also prove that $\mathcal{M}_{X,p}/\mathcal{O}_{X,p}$ is the free $\mathbb{C}$-vector space with countable basis $\{z^-k | k \in \mathbb{Z}_{\geq 1}\}$. Altogether, conclude that $(\mathcal{M}_X/\mathcal{O}_X)(X)$ is an infinite-dimensional $\mathbb{C}$-vector space.

Problem 9. For the compact Riemann surface $X = \mathbb{P}^1$, prove that the complex of $\mathbb{C}$-vector spaces from lecture,

$$0 \to \mathcal{O}_X(X) \to \mathcal{M}_X(X) \xrightarrow{q} (\mathcal{M}_X/\mathcal{O}_X)(X),$$

is *right exact*, i.e., q is surjective. Conversely, prove that for every compact Riemann surface X that has this property, there exists a meromorphic function $\phi \in \mathcal{M}_X(X)$ with a single pole at a single point $p \in X$ and with pole order precisely 1. Conclude that $\phi : (X, p) \to (\mathbb{P}^1, \infty)$ is a biholomorphism, so that X is isomorphic to $\mathbb{P}^1$.