# MAT 543 Problem Set 6

**Homework Policy.** Read through and carefully consider all of the following problems. Please write up and hand-in solutions to **five** of the problems.

Each student is encouraged to work with other students, but submitted problem sets must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource.

**Textbook Problems.**

**Problem 1.** Problem 13.1, p. 108, Forster.

**Problem 2.** Problem 13.2, p. 108, Forster.

**Problem 3.** Problem 13.3, p. 108, Forster.

**Problem 4.** Problem 14.1, p. 118, Forster.

**Problem 5.** Problem 14.2, p. 118, Forster.

**Problem 6.** Let $X$ be a topological space. An *Abelian group space over* $X$ is a datum

$$(\pi : A \to X, z : X \to A, s : A \times_X A \to A, i : A \to A),$$

consisting of a continuous map of topological spaces, $\pi : A \to X$, a section of $\pi$, $z : X \to Z$, called the *zero section*, a continuous map, $s : A \times_X A \to A$, that commutes with projection to $X$, called the *addition map*, and a homeomorphism, $i : A \to A$, that commutes with projection to $X$, called the *group inverse map*, such that for every $x \in X$, the fiber $A_x := \pi^{-1}(\{x\})$ with the zero section $z(x) \in A_x$, with the binary operation $s_x : A_x \times A_x \to A_x$, and the unary operation $i_x : A_x \to A_x$ is an Abelian group. For every Abelian group space over $X$, show that $z$, $s$ and $i$ induce a structure of Abelian group on every set $\Pi_{A/S}(U)$ such that $\Pi_{A/S}$ is a sheaf of Abelian groups. Conversely, for every sheaf of Abelian groups $\mathcal{F}$ on $X$, show that the espace étalé has a natural structure of Abelian group space over $X$. Extend to Abelian group spaces / sheaves of Abelian groups the earlier adjointness theorem for spaces over $X$ / sheaves of sets on $X$.

**Problem 7.** Continuing the previous exercise, let $X$ be a connected, paracompact, Hausdorff topological space with a neighborhood basis of open subsets that are path connected and simply connected (so that our construction of the universal cover makes sense). Let $A$ be an Abelian group space over $X$ such that the continuous map $\pi : A \to X$ is a covering map (with $A$ not necessarily connected). For every $x \in X$, prove that there exists a neighborhood $U$ of $x$ such that the restricted sheaf $\Pi_{A/X}|_U$ is isomorphic to the sheaf of locally constant functions to the Abelian group $A_x$ with its discrete topology. Conversely, prove that the espace étalé is a covering space of $X$ for every sheaf $\mathcal{F}$ of Abelian groups on $X$ such that for every $x \in X$, the restriction $\mathcal{F}|_U$ is isomorphic to the sheaf of locally constant functions to the stalk $\mathcal{F}_x$ with its discrete topology.

**Problem 8.** Continuing the previous two exercises, let $A$ be $X \times A_0$ for an Abelian group $A_0$ with its discrete topology and the projection $\mathrm{pr}_1 : X \times A_0 \to X$. For every open covering $\mathfrak{U} = (U_i)_{i \in I}$ of $X$ by open sets that are path connected and simply connected, use the *analytic continuation method* of Section 10.A to construct an isomorphism of Abelian groups,

$$\check{H}^1(\mathfrak{U}, \Pi_{A/X}) \to \mathrm{Hom}_{\mathrm{Groups}}(\pi_1(X, x_0), A_0).$$

Use this to give an alternative proof of Theorem 12.7. Think about the following, but you need not write a solution. If $A$ is as in Problem 7, but it is not necessarily a product $X \times A_0$, how could you extend this isomorphism? (Hint: Group cohomology for nontrivial $\pi_1(X, x_0)$-modules.)

**Problem 9.** Modify the proof of Theorem 12.7 to prove the following: for a (connected, paracompact, Hausdorff) Riemann surface $X$, there is a sequence of Abelian groups that is exact in the middle,

$$H^0(X, \Omega_X^{1,0}) \to \check{H}^1(X, \underline{\mathbb{C}}_X) \to \check{H}^1(X, \mathcal{O}_X),$$

where the first map is the periods / summands of automorphy homomorphism from Section 10.A. Later we will see that for a compact Riemann surface $X$, this is a short exact sequence of Abelian groups (equivalent to the weight one Hodge structure of $X$).

**Problem 10.** For the Riemann sphere, work out explicitly the exact sequence from Problem 9. Can you also do this for the elliptic curves from Exercise 1.5, p. 9? What about for the hyperelliptic curves from Problems 8 and 9 of Problem Set 5?