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MAT 536 Problem Set 9

Homework Policy. Please read through all the problems. Please solve 5 of the problems. I will be happy to discuss the solutions during office hours.

Problems.

Problem 0.(Commutation of Cohomology with Filtered Colimits) Let $\mathcal{B}$ be a cocomplete Abelian category satisfying Grothendieck's condition (AB5). Let I be a small filtering category. Let $C^\bullet : I \to \mathrm{Ch}^\bullet(\mathcal{B})$ be a functor.

(a) For every $n \in \mathbb{Z}$, prove that the natural $\mathcal{B}$-morphism,

$$\operatorname*{colim}_{i\in I} H^n(C^\bullet(i)) \to H^n(\operatorname*{colim}_{i\in I} C^\bullet(i)),$$

is an isomorphism. **Prove** that this extends to a natural isomorphism of cohomological δ-functors. This is "commutation of cohomology with filtered colimits".

(b) Let $\mathcal{A}$ be an Abelian category with enough injective objects. Let $F : I \times \mathcal{A} \to \mathcal{B}$ be a bifuncto such that for every object i of I, the functor $F_i : \mathcal{A} \to \mathcal{B}$ is additive and left-exact. Prove that $F_\infty(-) := \operatorname{colim}_{i\in I} F_i(-)$ also forms an additive functor that is left-exact. Also prove that the natural map

$$\operatorname*{colim}_{i\in I} R^n(F_i) \to R^n(F_\infty)$$

is an isomorphism. This is "commutation of right derived functors with filtered colimits".

Problem 1.(The Topological Space of a Presheaf and an Alternative Description of Sheafification for Sets) Let (X, τ_X) be a topological space. A *space over* X is a continuous map of topological spaces, $f : (Y, \tau_Y) \to (X, \tau_X)$. For spaces over X, $f : (Y, \tau_Y) \to (X, \tau_X)$ and $g : (Z, \tau_Z) \to (X, \tau_X)$, a *morphism of spaces over* X from f to g is a continuous map $u : (Y, \tau_Y) \to (Z, \tau_X)$ such that $g \circ u$ equals f.

(a)(The Category of Spaces over X) For every space over X, $f : (Y, \tau_Y) \to (X, \tau_X)$, prove that $\mathrm{Id}_Y : (Y, \tau_Y) \to (Y, \tau_Y)$ is a morphism from f to f. For spaces over X, $f : (Y, \tau_Y) \to (X, \tau_X)$, $g : (Z, \tau_Z) \to (X, \tau_X)$ and $h : (W, \tau_W) \to (X, \tau_X)$, for every morphism from f to g, $u : (Y, \tau_Y) \to (Z, \tau_Z)$, and for every morphism from g to h, $v : (Z, \tau_Z) \to (W, \tau_W)$, prove that the composition $v \circ u : (Y, \tau_Y) \to (W, \tau_W)$ is a morphism from f to h. Conclude that these notions form a category, denoted $\mathbf{Top}_{(X,\tau_X)}$.

(b)(The Sheaf of Sections) For every space over X, $f:(Y,\tau_Y)\to(X,\tau_X)$, for every open U of τ_X, define $\mathrm{Sec}_f(U)$ to be the set of continuous functions $s:(U,\tau_U)\to(Y,\tau_Y)$ such that $f\circ s$ is the inclusion morphism $(U,\tau_Y)\to(X,\tau_X)$. For every inclusion of τ_X-open subsets, $U\supseteq V$, for every s in $\mathrm{Sec}_f(U)$, define $s|_V$ to be the restriction of s to the open subset V. **Prove** that $s|_V$ is an element of $\mathrm{Sec}_f(V)$. **Prove** that these rules define a functor

$$\mathrm{Sec}_f:\tau_X\to\mathbf{Sets}.$$

Prove that this functor is a sheaf of sets on (X,τ_X).

(c)(The Sections Functor) For spaces over X, $f(Y,\tau_Y)\to(X,\tau_X)$ and $g:(Z,\tau_Z)\to(X,\tau_X)$, for every morphism from f to g, $u:(Y,\tau_Y)\to(Z,\tau_Z)$, for every τ_X-open set U, for every s in $\mathrm{Sec}_f(U)$, **prove** that $u\circ s$ is an element of $\mathrm{Sec}_g(U)$. For every inclusion of τ_X-open sets, $U\supseteq V$, **prove** that $u\circ(s|_V)$ equals $(u\circ s)|_V$. Conclude that these rules define a morphism of sheaves of sets,

$$\mathrm{Sec}_u:\mathrm{Sec}_f\to\mathrm{Sec}_g.$$

Prove that $\mathrm{Sec}_{\mathrm{Id}_Y}$ is the identity morphism of Sec_f. For spaces over X, $f:(Y,\tau_Y)\to(X,\tau_X)$, $g:(Z,\tau_Z)\to(X,\tau_X)$ and $h:(W,\tau_W)\to(X,\tau_X)$, for every morphism from f to g, $u:(Y,\tau_Y)\to(Z,\tau_Z)$, and for every morphism from g to h, $v:(Z,\tau_Z)\to(W,\tau_W)$, **prove** that $\mathrm{Sec}_{v\circ u}$ equals $\mathrm{Sec}_v\circ\mathrm{Sec}_u$. Conclude that these rules define a functor,

$$\mathrm{Sec}:\mathbf{Top}_{(X,\tau_X)}\to\mathbf{Sets}-\mathrm{Sh}_{(X,\tau_X)}.$$

(d)(The Éspace Étalè) For every presheaf of sets over X, $\mathcal{F}$, define $\mathrm{Esp}_{\mathcal{F}}$ to be the set of pairs (x,ϕ_x) of an element x of X and an element ϕ_x of the stalk $\mathcal{F}_x=\mathrm{colim}_{x\in U}\mathcal{F}(U)$; such an element is called a *germ* of $\mathcal{F}$ at x. Denote by

$$\pi_{\mathcal{F}}:\mathrm{Esp}_{\mathcal{F}}\to X,$$

the set map sending (x,ϕ_x) to x. For every open subset U of X and for every element ϕ of $\mathcal{F}(U)$, define $B(U,\phi)\subset\mathrm{Esp}_{\mathcal{F}}$ to be the image of the morphism,

$$\widetilde{\phi}:U\to\mathrm{Esp}_{\mathcal{F}},\ x\mapsto\phi_x.$$

Let (U,ψ) and (V,χ) be two such pairs. Let (x,ϕ_x) be an element of both $B(U,\psi)$ and $B(V,\chi)$. **Prove** that there exists an open subset W of $U\cap V$ containing x such that $\psi|_W$ equals $\chi|_W$. Denote this common restriction by $\phi\in\mathcal{F}(W)$. Conclude that (x,ϕ_x) is contained in $B(W,\phi)$, and this is contained in $B(U,\psi)\cap B(V,\chi)$. Conclude that the collection of all subset $B(U,\phi)$ of $\mathrm{Esp}_{\mathcal{F}}$ is a topological basis. Denote by $\tau_{\mathcal{F}}$ the associated topology on $\mathrm{Esp}_{\mathcal{F}}$. **Prove** that $\tau_{\mathcal{F}}$ is the finest topology on $\mathrm{Esp}_{\mathcal{F}}$ such that for every τ_X-open set U and for every $\phi\in\mathcal{F}(U)$, the set map $\widetilde{\phi}$ is a continuous map $(U,\tau_U)\to(\mathrm{Esp}_{\mathcal{F}},\tau_{\mathcal{F}})$. In particular, since every composition $\pi_{\mathcal{F}}\circ\widetilde{\phi}$ is the continuous inclusion of (U,τ_U) in (X,τ_X), conclude that every $\widetilde{\phi}$ is continuous for the topology $\pi_{\mathcal{F}}^{-1}(\tau_X)$ on $\mathrm{Esp}_{\mathcal{F}}$. Since $\tau_{\mathcal{F}}$ refines this topology, **prove** that

$$\pi_{\mathcal{F}}:(\mathrm{Esp}_{\mathcal{F}},\tau_{\mathcal{F}})\to(X,\tau_X)$$

is a continuous map, i.e., $\pi_{\mathcal{F}}$ is a space over X.

(e)(The Éspace Functor) For every morphism of presheaves of sets over X, $\alpha : \mathcal{F} \to \mathcal{G}$, for every (x, ϕ_x) in $\mathrm{Esp}_{\mathcal{F}}$, define $\mathrm{Esp}_\alpha(x, \phi_x)$ to be $(x, \alpha_x(\phi_x))$, where $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is the induced morphism of stalks. For every τ_X-open set U and every $\phi \in \mathcal{F}(U)$, **prove** tht the composition $\mathrm{Esp}_\alpha \circ \widetilde{\phi}$ equals $\widetilde{\alpha_U(\phi)}$ as set maps $U \to \mathrm{Esp}_{\mathcal{G}}$. By construction, $\widetilde{\alpha_U(\phi)}$ is continuous for the topology $\tau_{\mathcal{G}}$. Conclude that $\widetilde{\phi}$ is continuous for the topology $(\mathrm{Esp}_\alpha)^{-1}(\tau_{\mathcal{G}})$ on $\mathrm{Esp}_{\mathcal{F}}$. Conclude that $\tau_{\mathcal{F}}$ refines this topology, and thus Esp_α is a continuous function,

$$\mathrm{Esp}_\alpha : (\mathrm{Esp}_{\mathcal{F}}, \tau_{\mathcal{F}}) \to (\mathrm{Esp}_{\mathcal{G}}, \tau_{\mathcal{G}}).$$

Prove that $\mathrm{Esp}_{\mathrm{Id}_{\mathcal{F}}}$ equals the identity map on $\mathrm{Esp}_{\mathcal{F}}$. For morphisms of presheaves of sets over X, $\alpha : \mathcal{F} \to \mathcal{G}$ and $\beta : \mathcal{G} \to \mathcal{H}$, **prove** that $\mathrm{Esp}_{\beta\circ\alpha}$ equals $\mathrm{Esp}_\beta \circ \mathrm{Esp}_\alpha$. Conclude that these rules define a functor,

$$\mathrm{Esp} : \mathbf{Sets} - \mathrm{Presh}_{(X,\tau_X)} \to \mathbf{Top}_{(X,\tau_X)}.$$

(f)(The Adjointness Natural Transformations) For every presheaf of sets over X, $\mathcal{F}$, for every τ_X-open set U, for every $\phi \in \mathcal{F}(U)$, **prove** that $\widetilde{\phi}$ is an element of $\mathrm{Sec}_{\pi_{\mathcal{F}}}(U)$. For every τ_X-open subset $U \supseteq V$, **prove** that $\widetilde{\phi}|_V$ equals $\widetilde{\phi|_V}$. Conclude that $\phi \mapsto \widetilde{\phi}$ is a morphism of presheaves of sets over X,

$$\theta_{\mathcal{F}} : \mathcal{F} \to \mathrm{Sec} \circ \mathrm{Esp}(\mathcal{F}).$$

For every morphism of presheaves of sets over X, $\alpha : \mathcal{F} \to \mathcal{G}$, for every τ_X-open set U, for every $\phi \in \mathcal{F}(U)$, **prove** that $\mathrm{Esp}_\alpha \circ \theta_{\mathcal{F},U}(\phi)$ equals $\widetilde{\alpha_U(\phi)}$, and this in turn equals $\theta_{\mathcal{G},U} \circ \alpha_U(\phi)$. Conclude that $\mathrm{Sec} \circ \mathrm{Esp}(\alpha) \circ \theta_{\mathcal{F}}$ equals $\theta_{\mathcal{G}} \circ \alpha$. Therefore θ is a natural transformation of functors,

$$\theta : \mathrm{Id}_{\mathbf{Sets}-\mathrm{Presh}_{(X,\tau_X)}} \Rightarrow \mathrm{Sec} \circ \mathrm{Esp}.$$

(g)(Alternative Description of Sheafification) Since $\mathrm{Sec} \circ \mathrm{Esp}(\mathcal{F})$ is a sheaf, **prove** that there exists a unique morphism

$$\widetilde{\theta}_{\mathcal{F}} : \mathrm{Sh}(\mathcal{F}) \to \mathrm{Sec} \circ \mathrm{Esp}(\mathcal{F})$$

factoring $\theta_{\mathcal{F}}$. For every element $t \in \mathrm{Sec} \circ \mathrm{Esp}(\mathcal{F})(U)$, a *t-pair* is a pair (U_0, s_0) of a τ_X-open subset $U \supseteq U_0$ and an element $s_0 \in \mathcal{F}(U_0)$ such that $t^{-1}(B(U_0, s_0))$ equals U_0. Define $\mathfrak{U}$ to be the set of t-pairs, and define $\iota : \mathfrak{U} \to \tau_U$ to be the set map $(U_0, s_0) \mapsto U_0$. **Prove** that $(U, \iota : \mathfrak{U} \to \tau_U)$ is an open covering. For every pair of t-pairs, (U_0, s_0) and (U_1, s_1), for every $x \in U_0 \cap U_1$, prove that there exists a τ_X-open subset $U_{0,1} \subset U_0 \cap U_1$ containing x such that $s_0|_{U_{0,1}}$ equals $s_1|_{U_{0,1}}$. **Prove** that this data gives rise to a section $s \in \mathrm{Sh}(\mathcal{F})(U)$ such that $\widetilde{\theta}_{\mathcal{F}}(s)$ equals t. Conclude that $\widetilde{\theta}$ is an epimorphism. On the other hand, for every $r, s \in \mathcal{F}(U)$, if $\theta_{\mathcal{F},x}(r_x)$ equals $\theta_{\mathcal{F},x}(s_x)$, **prove** that $\widetilde{r}(x)$ equals $\widetilde{s}(x)$, i.e., r_x equals s_x. Conclude that every morphism $\widetilde{\theta}_x$ is a monomorphism, and hence $\widetilde{\theta}$ is a monomorphism of sheaves. Thus, finally **prove** that $\widetilde{\theta}_{\mathcal{F}}$ is an isomorphism of sheaves. Conclude that $\widetilde{\theta}$ is a natural isomorphism of functors,

$$\widetilde{\theta} : \mathrm{Sh} \Rightarrow \mathrm{Sec} \circ \mathrm{Esp}.$$

(h) For every space over X, $f:(Y,\tau_Y)\to(X,\tau_X)$, for every τ_X-open U, for every $s\in\mathrm{Sec}_f(U)$, and for every $x\in U$, define a set map,

$$\eta_{f,U,x}:\mathrm{Sec}_f(U)\to Y,\ s\mapsto s(x).$$

Prove that for every τ_X-open subset $U\supseteq V$ that contains x, $\eta_{f,V,x}(s|_V)$ equals $\eta_{f,U,x}(s)$. Conclude that the morphisms $\eta_{f,U,x}$ factor through set maps,

$$\eta_{f,x}:(\mathrm{Sec}_f)_x\to Y,\ s_x\mapsto s(x).$$

Define a set map,

$$\eta_f:\mathrm{Esp}_{\mathrm{Sec}_f}\to Y,\ (x,s_x)\mapsto\eta_{f,x}(s_x).$$

Prove that $\eta_f\circ\widetilde{s}$ equals s as set maps $U\to Y$. Since s is continuous for τ_Y, conclude that $\widetilde{s}$ is continuous for the inverse image topology $(\eta_f)^{-1}(\tau_Y)$ on $\mathrm{Esp}_{\mathrm{Sec}_f}$. Conclude that τ_{Sec_f} refines this topology, and thus η_f is a continuous map,

$$\eta_f:(\mathrm{Esp}_{\mathrm{Sec}_f},\tau_{\mathrm{Sec}_f})\to(Y,\tau_Y).$$

Also **prove** that $f\circ\eta_f$ equals π_{Sec_f}. Conclude that η_f is a morphism of spaces over X. Finally, for spaces over X, $f:(Y,\tau_Y)\to(X,\tau_X)$ and $g:(Z,\tau_Z)\to(X,\tau_X)$, and for every morphism from f to g, $u:(Y,\tau_Y)\to(Z,\tau_Z)$, **prove** that $u\circ\eta_f$ equals $\eta_g\circ\mathrm{Esp}\circ\mathrm{Sec}(u)$. Conclude that $f\mapsto\eta_f$ defines a natural transformation of functors,

$$\eta:\mathrm{Esp}\circ\mathrm{Sec}\Rightarrow\mathrm{Id}_{\mathbf{Top}_{(X,\tau_X)}}.$$

(i)(The Adjoint Pair) **Prove** that $(\mathrm{Esp},\mathrm{Sec},\theta,\eta)$ is an adjoint pair of functors.

Problem 2.(Alternative Description of Inverse Image) Let $f:(Y,\tau_Y)\to(X,\tau_X)$ be a continuous function of topological spaces. Since the category of topological spaces is a Cartesian category (by Problem 2(e) on Problem Set 8), for every space over X, $g:(Z,\tau_Z)\to(X,\tau_X)$, there is a fiber product diagram in **Top**,

$$\begin{array}{ccc}(Z,\tau_Z)\times_{(X,\tau_X)}(Y,\tau_Y) & \xrightarrow{g^*f} & (Z,\tau_Z)\\ {\scriptstyle f^*g}\downarrow & & \downarrow{\scriptstyle g}\\ (Y,\tau_Y) & \xrightarrow[f]{} & (X,\tau_X)\end{array}.$$

Denote the fiber product by $f^*(Z,\tau_Z)$.

(a) For spaces over X, $g:(Z,\tau_Z)\to(X,\tau_X)$ and $h:(W,\tau_W)\to(X,\tau_X)$, for every morphism of spaces over X, $u:(Z,\tau_Z)\to(W,\tau_W)$, **prove** that there is a unique morphism of topological spaces,

$$f^*u:f^*(Z,\tau_Z)\to f^*(W,\tau_W),$$

such that $f^*h\circ f^*u$ equals f^*g and $h^*f\circ f^*u$ equals $u\circ g^*f$. **Prove** that $f^*\mathrm{Id}_Z$ is the identity morphism of $f^*(Z,\tau_Z)$. For spaces over X, $g:(Z,\tau_Z)\to(X,\tau_X)$, $h:(W,\tau_W)\to(X,\tau_X)$ and

$i : (M, \tau_M) \to (X, \tau_X)$, for every morphism from g to h, $u : (Z, \tau_Z) \to (W, \tau_W)$, and for every morphism from h to i, $v : (W, \tau_W) \to (M, \tau_M)$, **prove** that $f^*(v \circ u)$ equals $f^*v \circ f^*u$. Conclude that these rules define a functor,

$$f^*_{\mathrm{Sp}} : \mathbf{Top}_{(X,\tau_X)} \to \mathbf{Top}_{(Y,\tau_Y)}.$$

Prove that this functor is contravariant in f. In particular, there is a composite functor,

$$f^*_{\mathrm{Sp}} \circ \mathrm{Esp}_{(X,\tau_X)} : \mathbf{Sets} - \mathrm{Sh}_{(X,\tau_X)} \to \mathbf{Top}_{(Y,\tau_Y)}.$$

(b) Consider the composite functor,

$$f_* \circ \mathrm{Sec}_{(Y,\tau_Y)} : \mathbf{Top}_{(Y,\tau_Y)} \to \mathbf{Sets} - \mathrm{Sh}_{(Y,\tau_Y)} \to \mathbf{Sets} - \mathrm{Sh}_{(X,\tau_X)}.$$

Prove directly (without using the inverse image functor on sheaves) that $(f^*_{\mathrm{Sp}} \circ \mathrm{Esp}_{(X,\tau_X)}, f_* \circ \mathrm{Sec}_{(Y,\tau_Y)})$ extends to an adjoint pair of functors. Use this to conclude that the composite $\mathrm{Sec}_{(Y,\tau_Y)} \circ f^*_{\mathrm{Sp}} \circ \mathrm{Esp}_{(X,\tau_X)}$ is naturally isomorphic to the inverse image functor on sheaves of sets.

Problem 3.(An Epimorphism of Sheaves that is not Epimorphic on Global Sections) Let $\mathbb{R}$ be the usual real line with coordinate θ and with the Euclidean topology and standard structure of differentiable manifold. **Prove** that translations of $\mathbb{R}$ by elements of $\mathbb{Z}$ are diffeomorphisms, and conclude that there is a unique structure of differentiable manifold on the quotient, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, such that the quotient set map,

$$q : \mathbb{R} \to \mathbb{S}^1,$$

is a C^∞ map that is a local diffeomorphism. Let $A^0_{\mathbb{S}^1}$ be the presheaf of $\mathbb{R}$-vector spaces that associates to every open subset U the collection of all C^∞ functions $f : U \to \mathbb{R}$ and with the usual notion of restriction. **Prove** that $A^0_{\mathbb{S}^1}$ is a sheaf of $\mathbb{R}$-vector spaces. Let $A^1_{\mathbb{S}^1}$ be the presheaf of $\mathbb{R}$-vector spaces that associates to every open subset U the collection of all C^∞ differential 1-forms on U with the usual notion of restriction, i.e., locally these differential forms are isomorphic to $f(\theta)d\theta$ for a C^∞ function $f(\theta)$. **Prove** that $A^1_{\mathbb{S}^1}$ is a sheaf of $\mathbb{R}$-vector spaces. For every open set U, for every C^∞ function $f : U \to \mathbb{R}$, **prove** that the differential df is an element of $A^1_{\mathbb{S}^1}(U)$. **Prove** that

$$d_U : A^0_{\mathbb{S}^1}(U) \to A^1_{\mathbb{S}^1}(U),$$

is an $\mathbb{R}$-linear transformation. For every open subset $V \subset U$, **prove** that $d_V(f|_V)$ equals $(d_U(f))|_V$. Conclude that these $\mathbb{R}$-linear transformations define a morphism of sheaves of $\mathbb{R}$-vector spaces,

$$d : A^0_{\mathbb{S}^1} \to A^1_{\mathbb{S}^1}.$$

Let U be an open subset of $\mathbb{S}^1$ such that there exists a $s : U \to \mathbb{R}$ of q over U. **Prove** that d_U is surjective. Conclude that d is an epimorphism in the category of sheaves of $\mathbb{R}$-vector spaces on $\mathbb{S}^1$. **Prove** that the differential 1-form $d\theta$ on $\mathbb{R}$ is invariant under translations by $\mathbb{Z}$. Conclude that there exists a unique differential 1-form α in $A^1_{\mathbb{S}^1}(\mathbb{S}^1)$ such that $q^*\alpha$ equals $d\theta$. **Prove** that there exists no f in $A^0_{\mathbb{S}^1}(\mathbb{S}^1)$ such that α equals df. **Hint.** Prove that, up to constant, the coordinate function θ is the unique C^∞ function on $\mathbb{R}$ with $d\theta = q^*\alpha$. If there were f with $df = \alpha$, conclude that $f \circ q = \theta + C$, and this is not invariant under translation by $\mathbb{Z}$.

Problem 4.(Flasque Sheaves) Let (X, τ_X) be a topological space, and let $\mathcal{C}$ be a category. A $\mathcal{C}$-presheaf F on (X, τ_X) is *flasque* (or *flabby*) if for every inclusion of τ_X-open sets, $U \supseteq V$, the restriction morphism $A^U_V : A(U) \to A(V)$ is an epimorphism.

(a)(Pushforward Preserves Flasque Sheaves) For every continuous function $f : (X, \tau_X) \to (Y, \tau_Y)$, for every flasque $\mathcal{C}$-presheaf F on (X, τ_X), **prove** that f_*F is a flasque $\mathcal{C}$-presheaf on (Y, τ_Y).

(b)(Restriction to Opens Preserves Flasque Sheaves) For every τ_X-open subset U, for the continuous inclusion $i : (U, \tau_U) \to (X, \tau_X)$, for every flasque $\mathcal{C}$-presheaf F on (X, τ_X), **prove** that $i^{-1}F$ is a flasque $\mathcal{C}$-presheaf. Also, for every $\mathcal{C}$-sheaf F on (X, τ_X), **prove** that the presheaf inverse image $i^{-1}F$ is already a sheaf, so that the sheaf inverse image agrees with the presheaf inverse image.

(c)(H^1-Acyclicity of Flasque Sheaves) Let $\mathcal{A}$ be an Abelian category realized as a full subcategory of the category of left R-modules (via the embedding theorem). Let

$$0 \longrightarrow A' \xrightarrow{q} A \xrightarrow{p} A'' \longrightarrow 0$$

be a short exact sequence of $\mathcal{A}$-sheaves on (X, τ_X). Let U be a τ_X-open set. Let $t : A''(U) \to T$ be a morphism in $\mathcal{A}$ such that $t \circ p(U)$ is the zero morphism. Assume that A' is flasque. **Prove** that t is the zero morphism as follows. Let $a'' \in A''(U)$ be any element. Let $\mathcal{S}$ be the set of pairs (V, a) of a τ_X-open subset $V \subseteq U$ and an element $a \in A(V)$ such that $p(V)(a)$ equals $a''|_V$. For elements (V, a) and $(\widetilde{V}, \widetilde{a})$ of $\mathcal{S}$, define $(V, a) \le (\widetilde{V}, \widetilde{a})$ if $V \subseteq V'$ and $\widetilde{a}|_V$ equals a. **Prove** that this defines a partial order on $\mathcal{S}$. Use the sheaf axiom for A to **prove** that every totally ordered subset of $\mathcal{S}$ has a least upper bound in $\mathcal{S}$. Use Zorn's Lemma to conclude that there exists a maximal element (V, a) in $\mathcal{S}$. For every x in U, since p is an epimorphism of sheaves, **prove** that there exists (W, b) in $\mathcal{S}$ such that $x \in W$. Conclude that on $V \cap W$, $a|_{V\cap W} - b|_{V \cap W}$ is in the kernel of $p(V \cap W)$. Since the sequence above is exact, **prove** that there exists unique $a' \in A'(V \cap W)$ such that $q(V \cap W)(a')$ equals $a|_{V\cap W} - b|_{V\cap W}$. Since A' is flasque, **prove** that there exists $a'_W \in A'(W)$ such that $a'_W|_{V\cap W}$ equals a'. Define $a_W = b + q(W)(a'_W)$. **Prove** that (W, a_W) is in $\mathcal{S}$ and $a|_{V\cap W}$ equals $a_W|_{V\cap W}$. Use the sheaf axiom for A once more to **prove** that there exists unique $(V \cap W, a_{V\cap W})$ in $\mathcal{S}$ with $a_{V\cap W}|_V$ equals a and $a_{V\cap W}|_W$ equals a_W. Since (V, a) is maximal, conclude that $W \subset V$, and thus x is in V. Conclude that V equals U. Thus, a'' equals $p(U)(a)$. Conclude that $t(a'')$ equals 0, and thus t is the zero morphism. (For a real challenge, modify this argument to avoid any use of the embedding theorem.)

(d)(H^r-Acyclicity of Flasque Sheaves) Let $C^\bullet = (C^q, d^q_C)_{q\ge 0}$ be a complex of $\mathcal{A}$-sheaves on (X, τ_X). Assume that every C^q is flasque. Let $r \ge 0$ be an integer, and assume that the cohomology sheaves $h^q(C^\bullet)$ are zero for $q = 0, \ldots, r$. Use (c) and induction on r to prove that for the associated complex in $\mathcal{C}$,

$$C^\bullet(U) = (C^q(U), d^q_C(U))_{q\ge 0}$$

also $h^q(C^\bullet(U))$ is zero for $q = 0, \ldots, r$.

Problem 5.($\Lambda - \Pi$-modules) Let (X, τ_X) be a topological space. Let Λ and Π be presheaves of associative, unital rings on (X, τ_X). The most common case is to take both Λ and Π to be the constant presheaf with values $\mathbb{Z}$. Assume, for simplicity, that $\Lambda(\varnothing)$ and $\Pi(\varnothing)$ are the zero ring. A *presheaf of* $\Lambda - \Pi$*-bimodules* on (X, τ_X) is a presheaf M of Abelian groups on (X, τ_X) together with a structure of $\Lambda(U) - \Pi(U)$-bimodule on every Abelian group $M(U)$ such that for every open subset $U \supseteq V$, relative to the restriction homomorphisms of associative, unital rings,

$$\Lambda^U_V : \Lambda(U) \to \Lambda(V), \quad \Pi^U_V : \Pi(U) \to \Pi(V),$$

every restriction homomorphism of Abelian groups,

$$M^U_V : M(U) \to M(V),$$

is a homomorphism of $\Lambda(U) - \Pi(U)$-bimodules. For presheaves of $\Lambda - \Pi$-bimodules on (X, τ_X), M and N, a *morphism of presheaves of* $\Lambda - Pi$*-bimodules* is a morphism of presheaves of Abelian groups $\alpha : M \to N$ such that for every open U, the Abelian group homomorphism,

$$\alpha(U) : M(U) \to N(U),$$

is a homomorphism of $\Lambda(U) - \Pi(U)$-bimodules.

(a)(The Category of Presheaves of $\Lambda - \Pi$-Bimodules) **Prove** that these notions form a category $\Lambda - \Pi - \text{Presh}_{(X,\tau_X)}$. Prove that this is an Abelian category that satisfies Grothendieck's axioms (AB1), (AB2), (AB3), (AB3*), (AB4) and (AB5).

(b)(Discontinuous $\Lambda - \Pi$-Bimodules) A *discontinuous* $\Lambda - \Pi$-bimodule is a specification K for every nonempty τ_X-open U of a $\Lambda(U) - \Pi(U)$-bimodule $K(U)$, but without any specification of restriction morphisms. For discontinuous $\Lambda - \Pi$-bimodules K and L, a *morphism* of discontinuous $\Lambda - \Pi$-bimodules $\alpha : K \to L$ is a specification for every nonempty τ_X-open U of a homomorphism $\alpha(U) : K(U) \to L(U)$ of $\Lambda(U) - \Pi(U)$-bimodules. **Prove** that with these notions, there is a category $\Lambda - \Pi - \text{Disc}_{(X,\tau_X)}$ of discontinuous $\Lambda - \Pi$-bimodules. Prove that this is an Abelian category that satisfies Grothendieck's axioms (AB1), (AB2), (AB3), (AB3*), (AB4), (AB4*) and (AB5).

(c)(The Presheaf Associated to a Discontinuous $\Lambda - \Pi$-Bimodule) For every discontinuous $\Lambda - \Pi$-bimodule K, for every nonempty τ_X-open subset U, define

$$\widetilde{K}(U) = \prod_{W \subseteq U} K(W)$$

as a $\Lambda(U) - \Pi(U)$-bimodule, where the product is over nonempty open subsets $W \subseteq U$ (in particular also $W = U$ is allowed), together with its natural projections $\pi^U_W : \widetilde{K}(U) \to K(W)$. Also define $\widetilde{K}(\varnothing)$ to be a zero object. For every inclusion of τ_X-open subsets $U \supseteq V$, define

$$\widetilde{K}^U_V : \prod_{W \subseteq U} K(W) \to \prod_{W \subseteq V} K(W),$$

to be the unique morphism of $\Lambda(U)-\Pi(U)$-bimodules such that for every $W \subset V$, $\pi_W^V \circ \widetilde{K}_V^U$ equals π_W^U. **Prove** that $\widetilde{K}$ is a presheaf of $\Lambda-\Pi$-bimodules. For discontinuous $\Lambda-\Pi$-bimodules K and L, for every morphism of discontinuous $\Lambda-\Pi$-bimodules, $\alpha : K \to L$, for every τ_X-open set U, define

$$\widetilde{\alpha}(U) : \prod_{W \subseteq U} K(W) \to \prod_{W \subseteq U} L(W)$$

to be the unique morphism of $\Lambda(U)-\Pi(U)$-bimodules such that for every $W \subseteq U$, $\pi_{L,W}^U \circ \widetilde{\alpha}(U)$ equals $\pi_{K,W}^U$. **Prove** that $\widetilde{\alpha}$ is a morphism of presheaves of $\Lambda-\Pi$-bimodules. **Prove** that these notions define a functor,

$$\widetilde{*} : \Lambda-\Pi-\mathrm{Disc}_{(X,\tau_X)} \to \Lambda-\Pi-\mathrm{Presh}_{(X,\tau_X)}.$$

Prove that this is an exact functor that preserves arbitrary limits and finite colimits.

(d)(The Čech Object of a Discontinuous $\Lambda-\Pi$-Bimodule is Acyclic) For every open covering $(U, \iota : \mathfrak{U} \to \tau_U)$, define

$$\tau_{\mathfrak{U}} = \bigcup_{U_0 \in \mathfrak{U}} \tau_{\iota(U_0)} = \{W \in \tau_U | \exists U_0 \in \mathfrak{U}, W \subset \iota(U_0)\}.$$

For every discontinuous $\Lambda-\Pi$-bimodule K, define

$$\widetilde{K}(\mathfrak{U}) := \prod_{W \in \tau_{\mathfrak{U}}} K(W)$$

together with its projections $\pi_W : \widetilde{K}(\mathfrak{U}) \to K(W)$. In particular, define

$$\pi_{\mathfrak{U}}^U : \widetilde{K}(\mathfrak{U}) \to \widetilde{K}(\mathfrak{U})$$

to be the unique $\Lambda(U)-\Pi(U)$-morphism such that for every $W \in \tau_{\mathfrak{U}}$, $\pi_W \circ \pi_{\mathfrak{U}}^U$ equals π_W.

For every nonempty $W \in \tau_{\mathfrak{U}}$, define

$$\mathfrak{U}^W := \{U_0 \in \mathfrak{U} | W \subset \iota(U_0)\}.$$

Prove that

$$\check{C}^r(\mathfrak{U}, \widetilde{K}) = \prod_{(U_0,\dots,U_r)\in\mathfrak{U}^{r+1}} \prod_{W \subseteq \iota(U_0,\dots,U_r)} K(W)$$

together with its projection $\pi_{(U_0,\dots,U_r;W)} : \check{C}^r(\mathfrak{U}, \widetilde{K}) \to K(W)$ for every nonempty $W \subset \iota(U_0, \dots, U_r)$; if $\iota(U_0, \dots, U_r)$ is empty, the corresponding factor is a zero object. For every integer $r \geq 0$, for every $i = 0, \dots, r+1$, **prove** that the morphism

$$\partial_r^i : \check{C}^r(\mathfrak{U}, \widetilde{K}) \to \check{C}^{r+1}(\mathfrak{U}, \widetilde{K}),$$

is the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every nonempty $W \in \tau_{\mathfrak{U}}$ and for every $(U_0, \dots, U_r, U_{r+1}) \in (\mathfrak{U}^W)^{r+2}$, $\pi_{U_0,\dots,U_r,U_{r+1};W} \circ \partial_r^i$ equals $\pi_{U_0,\dots,U_{i-1},U_{i+1},\dots,U_{r+1};W}$. For every integer $r \geq 0$ and for every $i = 0, \dots, r$, **prove** that the morphism

$$\sigma_{r+1}^i : \check{C}^{r+1}(\mathfrak{U}, \widetilde{K}) \to \check{C}^r(\mathfrak{U}, \widetilde{K}),$$

is the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every nonempty $W \in \tau_{\mathfrak{U}}$ and for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r;W} \circ \sigma^i_{r+1}$ equals equals $\pi_{U_0,\dots,U_{i-1},U_i,U_i,U_{i+1},\dots,U_{r+1};W}$. For every integer $r \geq 0$, prove that the morphism

$$g^r_{\widetilde{K},\mathfrak{U}} : \widetilde{K}(\mathfrak{U}) \to \check{C}^r(\mathfrak{U}, \widetilde{K})$$

is the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every nonempty $W \in \tau_{\mathfrak{U}}$ and for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r;W} \circ g^r$ equals π_W.

For every nonempty $W \in \tau_{\mathfrak{U}}$, for every $r \geq 0$, define

$$\check{C}^r(\mathfrak{U}, \widetilde{K})^W := \prod_{(U_0,\dots,U_r)\in(\mathfrak{U}^W)^{r+1}} K(W),$$

with its projections

$$\pi_{U_0,\dots,U_r|W} : \check{C}^r(\mathfrak{U}, \widetilde{K})^W \to K(W).$$

Define

$$\pi^r_{-;W} : \check{C}^r(\mathfrak{U}, \widetilde{K}) \to \check{C}^r(\mathfrak{U}, \widetilde{K})^W$$

to be the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r|W} \circ \pi^r_{-;W}$ equals $\pi_{U_0,\dots,U_r;W}$. For every integer $r \geq 0$ and for every $i = 0, \dots, r+1$, **prove** that there exists a unique $\Lambda(U) - \Pi(U)$-morphism

$$\partial^i_r : \check{C}^r(\mathfrak{U}, \widetilde{K})^W \to \check{C}^{r+1}(\mathfrak{U}, \widetilde{K})^W,$$

such that $\partial^i_r \circ \pi^r_{-;W}$ equals $\pi^{r+1}_{-;W} \circ \partial^i_r$, and **prove** that for every $(U_0, \dots, U_r, U_{r+1}) \in (\mathfrak{U}^W)^{r+2}$, $\pi_{U_0,\dots,U_r,U_{r+1}|W} \circ \partial^i_r$ equals $\pi_{U_0,\dots,U_{i-1},U_{i+1},\dots,U_{r+1}|W}$. For every integer $r \geq 0$ and for every $i = 0, \dots, r$, **prove** that there exists a unique $\Lambda(U) - \Pi(U)$-morphism

$$\sigma^i_{r+1} : \check{C}^{r+1}(\mathfrak{U}, \widetilde{K})^W \to \check{C}^r(\mathfrak{U}, \widetilde{K})^W,$$

such that $\sigma^i_{r+1} \circ \pi^{r+1}_{-;W}$ equals $\pi^r_{-;W} \circ \sigma^i_{r+1}$, and **prove** that for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r|W} \circ \sigma^i_{r+1}$ equals equals $\pi_{U_0,\dots,U_{i-1},U_i,U_i,U_{i+1},\dots,U_{r+1}|W}$. For every integer $r \geq 0$, **prove** that there exists a unique $\Lambda(U) - \Pi(U)$-morphism

$$g^r : K(W) \to \check{C}^r(\mathfrak{U}, \widetilde{K})^W$$

such that $\pi^r_{-;W} \circ g^r$ equals $g^r \circ \pi_W$, and **prove** that for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r|W} \circ g^r$ equals $\mathrm{Id}_{K(W)}$. Conclude that

$$\pi^\bullet_{-;W} : \check{C}^\bullet(\mathfrak{U}, \widetilde{K}) \to \check{C}^\bullet(\mathfrak{U}, \widetilde{K})^W$$

is a morphism of cosimplicial $\Lambda(U)$–$\Pi(U)$-bimodules that is compatible with the coaugmentations $g^\bullet$. **Prove** that these morphisms realize $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})$ in the category $S^\bullet\Lambda(U) - \Pi(U) - \text{Bimod}$ as a product,

$$\check{C}^\bullet(\mathfrak{U}, \widetilde{K}) = \prod_{W\in\tau_{\mathfrak{U}}} \check{C}^\bullet(\mathfrak{U}, \widetilde{K})^W.$$

Using the Axiom of Choice, prove that there exists a set map

$$\phi : \tau_{\mathfrak{U}} \smallsetminus \{\emptyset\} \to \mathfrak{U}$$

such that for every nonempty $W \in \tau_{\mathfrak{U}}$, $\phi(W)$ is an element in $\mathfrak{U}^W$. For every integer $r \geq 0$, define

$$\check{C}^r(\phi, \widetilde{K})^W : \check{C}^r(\mathfrak{U}, \widetilde{K})^W \to K(W)$$

to be $\pi_{\phi(W),\dots,\phi(W)|W}$. **Prove** that for every integer $r \geq 0$ and for every $i = 0, \dots, r+1$, $\check{C}^{r+1}(\phi, \widetilde{K})^W \circ \partial_r^i$ equals $\check{C}^r(\phi, \widetilde{K})^W$. **Prove** that for every integer $r \geq 0$ and for every $i = 0, \dots, r$, $\check{C}^r(\phi, \widetilde{K})^W \circ \sigma_{r+1}^i$ equals $\check{C}^{r+1}(\phi, \widetilde{K})^W$. Conclude that

$$\check{C}^\bullet(\phi, \widetilde{K})^W \to \text{const}_{K(W)}$$

is a morphism of cosimplicial $\Lambda(U) - \Pi(U)$-bimodules. **Prove** that $\check{C}^\bullet(\phi, \widetilde{K})^W \circ g^\bullet$ equals the identity morphism of $\text{const}_{K(W)}$. For every nonempty $W \in \tau_{\mathfrak{U}}$, for every integer $r \geq 0$, for every integer $i = 0, \dots, r$, define

$$g^i_{\phi,r+1} : \check{C}^{r+1}(\mathfrak{U}, \widetilde{K})^W \to \check{C}^r(\mathfrak{U}, \widetilde{K})^W$$

to be the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r|W} \circ g^i_{\phi,r+1}$ equals $\pi_{U_0,\dots,U_i,\phi(W),\dots,\phi(W)|W}$. **Prove** the following identities (cosimplicial homotopy identities),

$$g^0_{\phi,r+1} \circ \partial^0_r = g^r \circ \check{C}^r(\phi, \widetilde{K})^W, \quad g^r_{\phi,r+1} \circ \partial^{r+1}_r = \text{Id}_{\check{C}^r(\mathfrak{U},\widetilde{K})^W},$$

$$g^j_{\phi,r+1} \circ \partial^i_r = \begin{cases} \partial^i_{r-1} \circ g^{j-1}_{\phi,r}, & 0 \leq i < j \leq r, \\ g^{i-1}_{\phi,r+1} \circ \partial^i_r, & 0 < i = j \leq r, \\ \partial^{i-1}_{r-1} \circ g^j_{\phi,r}, & 1 \leq j+1 < i \leq r+1. \end{cases}$$

$$g^j_{\phi,r} \circ \sigma^i_{r+1} = \begin{cases} \sigma^i_r \circ g^{j+1}_{\phi,r+1}, & 0 \leq i \leq j \leq r-1, \\ \sigma^{i-1}_r \circ g^j_{\phi,r+1}, & 0 \leq j < i \leq r. \end{cases}$$

Conclude that $g^\bullet$ and $\check{C}^\bullet(\phi, \widetilde{K})^W$ are homotopy equivalences between $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})^W$ and $\text{const}_{K(W)}$. Conclude that $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})$ is homotopy equivalent to $\text{const}_{\widetilde{K}(\mathfrak{U})}$. In particular, **prove** that the associated cochain complex of $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})^W$ is acyclic with $\check{H}^0(\mathfrak{U}, \widetilde{K})^W$ equal to $K(W)$. Similarly, **prove** that the associated cochain complex of $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})$ is acyclic with $\check{H}^0(\mathfrak{U}, \widetilde{K})$ equal to $K(\mathfrak{U})$.

(e)(The Forgetful Functor to Discontinuous $\Lambda - \Pi$-Bimodules; Preservation of Injectives) For every presheaf M of $\Lambda - \Pi$-bimodules on (X, τ_X), define $\Phi(M)$ to be the discontinuous $\Lambda - \Pi$-bimodule $U \mapsto M(U)$. For presheaves of $\Lambda - \Pi$-bimodules, M and N, for every morphism of presheaves of $\Lambda - \Pi$-bimodules, $\alpha : M \to N$, define $\Phi(\alpha) : \Phi(M) \to \Phi(N)$ to be the assignment $U \mapsto \alpha(U)$. **Prove** that these rules define a functor

$$\Phi : \Lambda - \Pi - \text{Presh}_{(X,\tau_X)} \to \Lambda - \Pi - \text{Disc}_{(X,\tau_X)}.$$

Prove that this is a faithful exact functor that preserves arbitrary limits and finite colimits. For every presheaf M of $\Lambda-\Pi$-bimodules, for every τ_X-open U, define

$$\theta_{M,U} : M(U) \to \prod_{W \subseteq U} M(W)$$

to be the unique homomorphism of $\Lambda(U)-\Pi(U)$-bimodules such that for every τ_X-open subset $W \subset U$, $\pi^U_W \circ \theta_{M,U}$ equals M^U_W. **Prove** that $U \mapsto \theta_{M,U}$ is a morphism of presheaves of $\Lambda-\Pi$-bimodules,

$$\theta_M : M \to \widetilde{\Phi(M)}.$$

For every morphism of presheaves of $\Lambda-\Pi$-bimodules, $\alpha : M \to N$, for every τ_X-open set U, **prove** that $\widetilde{\Phi(\alpha)} \circ \theta_M$ equals $\theta_N \circ \alpha$. Conclude that θ is a natural transformation of functors,

$$\theta : \mathrm{Id}_{\Lambda-\Pi-\mathrm{Presh}_{(X,\tau_X)}} \Rightarrow \widetilde{*} \circ \Phi.$$

For every discontinuous $\Lambda-\Pi$-bimodule K, for every τ_X-open U, define

$$\eta_{K,U} : \prod_{W \subseteq U} K(W) \to K(U)$$

to be π^U_W. **Prove** that $U \mapsto \eta_{K,U}$ is a morphism of discontinuous $\Lambda-\Pi$-bimodules. For every pair of discontinuous $\Lambda-\Pi$-bimodules, K and L, for every morphism of discontinuous $\Lambda-\Pi$-bimodules, $\beta : K \to L$, **prove** that $\eta_L \circ \Phi(\widetilde{\beta})$ equals $\beta \circ \eta_L$. Conclude that η is a natural transformation of functors,

$$\eta : \Phi \circ \widetilde{*} \Rightarrow \mathrm{Id}_{\Lambda-\Pi-\mathrm{Disc}_{(X,\tau_X)}}.$$

Prove that $(\Phi, \widetilde{*}, \theta, \eta)$ is an adjoint pair of functors. Since Φ preserves monomorphisms, use Problem 3(d), Problem Set 5 to **prove** that $\widetilde{*}$ sends injective objects to injective objects. Since the forgetful morphism from sheaves to presheaves preserves monomorphisms, **prove** that the sheafification functor Sh sends injective objects to injective objects. Conclude that $\mathrm{Sh} \circ \widetilde{*}$ sends injective objects to injective objects.

(f)(Enough Injectives) Recall from Problems 3 and 4 of Problem Set 5 that for every τ_X-open set U, there are enough injective $\Lambda(U)-\Pi(U)$-bimodules. Using the Axiom of Choice, conclude that $\Lambda-\Pi-\mathrm{Disc}_{(X,\tau_X)}$ has enough injective objects. In particular, for every presheaf M of $\Lambda-\Pi$-bimodules, for every open set U, let there be given a monomorphism of $\Lambda(U)-\Pi(U)$-bimodules,

$$\epsilon_U : M(U) \to I(U),$$

with $I(U)$ an injective $\Lambda(U)-\Pi(U)$-bimodule. Conclude that $\widetilde{I}$ is an injective presheaf of $\Lambda-\Pi$-bimodules, and the composition

$$M \xrightarrow{\theta_M} \widetilde{\Phi(M)} \xrightarrow{\widetilde{\epsilon}} \widetilde{I}$$

is a monomorphism of presheaves of $\Lambda-\Pi$-bimodules. If M is a sheaf, conclude that $\mathrm{Sh}(\widetilde{I})$ is an injective sheaf of $\Lambda-\Pi$-bimodules. Also, use (d) to prove that the composition

$$M \xrightarrow{\theta_M} \widetilde{\Phi(M)} \xrightarrow{\widetilde{\epsilon}} \widetilde{I} \xrightarrow{sh} \mathrm{Sh}(\widetilde{I})$$

is a monomorphism of sheaves of $\Lambda-\Pi$-bimodules. (**Hint:** Since $\sigma_{x,U}$ is a filtering small category, use Problem 0 to reduce to the statement that for every open covering $(U,\mathfrak{U})$, the morphism $M(U) \to \widetilde{M}(\mathfrak{U})$ is a monomorphism. Realize this a part of the Sheaf Axiom for M.) Conclude that both the category $\Lambda-\Pi-\text{Presh}_{(X,\tau_X)}$ and $\Lambda-\Pi-\text{Sh}_{(X,\tau_X)}$ have enough injective objects. In particular, for an additive, left-exact functor F, resp. G, on the category of presheaves of $\Lambda-\Pi$-bimodules, resp. the category of sheaves of $\Lambda-\Pi$-bimodules, there are right derived functors $((R^nF)_n,(\delta^n)_n)$, resp. $((R^nG)_n,(\delta^n)_n)$. Finally, since $\widetilde{*}$ is exact and sends injective objects to injective objects, use the Grothendieck Spectral Sequence (or universality of the cohomological δ-functor) to **prove** that $(R^nF)\circ\widetilde{*}$ is $R^n(F\circ\widetilde{*})$.

(g)(Enough Flasque Sheaves; Injectives are Flasque) Let K be a discontinuous $\Lambda-\Pi$-bimodule on X. For every τ_X-open set U, **prove** that $\widetilde{K}(U)\to \text{Sh}(\widetilde{K})(U)$ is the colimit over all open coverings $\mathfrak{U}\subset\tau_U$ (ordered by refinement as usual) of the morphism

$$\pi^U_{\mathfrak{U}}:\widetilde{K}(U)\to\widetilde{K}(\mathfrak{U}).$$

In particular, since every morphism $\widetilde{K}(U)\to\widetilde{K}(\mathfrak{U})$ is surjective (by the Axiom of Choice), conclude that also

$$\text{sh}(U):\widetilde{K}(U)\to\text{Sh}(\widetilde{K})(U)$$

is surjective. Use this to **prove** that $\text{Sh}(\widetilde{K})$ is a flasque sheaf.

For every injective $\Lambda-\Pi$-sheaf I, for the monomorphism $\theta_I: I\to\text{Sh}(\widetilde{\Phi(I)})$, there exists a retraction $\rho:\text{Sh}(\widetilde{\Phi(I)})\to I$. Also $\text{Sh}(\widetilde{\Phi(I)})$ is flasque. Use this to **prove** that also I is flasque.

(h)(Sheaf Cohomology; Flasque Sheaves are Acyclic) For every τ_X-open set U, prove that the functor

$$\Gamma(U,-):\Lambda-\Pi-\text{Presh}_{(X,\tau_X)}\to\Lambda(U)-\Pi(U)-\text{Bimod},\quad M\mapsto M(U)$$

is an exact functor. Also prove that the functor

$$\Gamma(U,-):\Lambda-\Pi-\text{Sh}_{(X,\tau_X)}\to\Lambda(U)-\Pi(U)-\text{Bimod}$$

is an additive, left-exact functor. Use (g) to conclude that every sheaf M of $\Lambda-\Pi$-modules admits a resolution, $\epsilon: M\to I^\bullet$ by injective sheaves of $\Lambda-\Pi$-modules that are also flasque. Conclude that $\Gamma(U,-)$ extends to a universal cohomological δ-functor formed by the right derived functors, $((H^n(U,-))_n,(\delta^n)_n)$. Finally, assume that M is flasque. Use Problem 4(d) to **prove** that $I^\bullet(U)$ is an acyclic complex of $\Lambda(U)-\Pi(U)$-bimodules. Conclude that for every flasque sheaf M of $\Lambda-\Pi$-bimodules, for every $n\geq 0$, $H^n(U,M)$ is zero, i.e., flasque sheaves of $\Lambda-\Pi$-bimodules are acyclic for the right derived functors of $\Gamma(U,-)$.

(i)(Computation of Sheaf Cohomology via Flasque Resolutions; Canonical Resolutions; Independence of $\Lambda-\Pi$) Use (h) and the hypercohomlogy spectral sequence to **prove** that for every sheaf M of $\Lambda-\Pi$-bimodules, for every acyclic resolution $\epsilon_M: M\to M^\bullet$ of M by sheaves of $\Lambda-\Pi$-bimodules that are flasque, for every integer $n\geq 0$, there is a canonical isomorphism of $H^n(U,M)$

with $h^n(M^\bullet(U))$. In particular, the functor $\top = \mathrm{Sh}\circ\widetilde{\divideontimes}\circ\Phi$, the natural transformation $\theta : \mathrm{Id} \Rightarrow \top$, and the natural transformation

$$\mathrm{Sh}\circ\widetilde{\divideontimes}\circ\eta\circ\Phi : \top\top \Rightarrow \top,$$

form a *triple* on the category $\Lambda-\Pi-\mathrm{Sh}_{(X,\tau_X)}$. There is an associated cosimplicial functor,

$$L_\top : \Lambda-\Pi-\mathrm{Sh}_{(X,\tau_X)} \to S^\bullet\Lambda-\Pi-\mathrm{Sh}_{(X,\tau_X)}$$

and a functorial coaugmentation,

$$\theta_M : \mathrm{const}^\bullet_M \to L^\bullet_\top(M).$$

The associated (unnormalized) cochain complex of this cosimplicial object is an acyclic resolution of M by flasque sheaves of $\Lambda-\Pi$-bimodules, and it is *canonical*, depending on no choices of injective resolutions.

Finally, let $\widehat{\Lambda}\to\Lambda$ and $\widehat{\Pi}\to\Pi$ be morphisms of presheaves of associative, unital rings. This induces a functor,

$$\Lambda-\Pi-\mathrm{Sh}_{(X,\tau_X)} \to \widehat{\Lambda}-\widehat{\Pi}-\mathrm{Sh}_{(X,\tau_X)}.$$

For every sheaf M of $\Lambda-\Pi$-bimodules, and for every acyclic resolution $\epsilon : M \to M^\bullet$ of M by flasque sheaves of $\Lambda-\Pi$-bimodules, this is also an acyclic, flasque resolution of M with the associated structure of sheaves of $\widehat{\Lambda}-\widehat{\Pi}$-bimodules. For the natural map of cohomological δ-functors from the derived functors of $\Gamma(U,-)$ on $\Lambda-\Pi-\mathrm{Sh}_{(X,\tau_X)}$ to the derived functors of $\Gamma(U,-)$ on $\widehat{\Lambda}-\widehat{\Pi}-\mathrm{Sh}_{(X,\tau_X)}$, **prove** that this natural map is a natural isomorphism of cohomological δ-functors. This justifies the notation $H^n(U,-)$ that makes no reference to the underlying presheaves Λ and Π, and yet is naturally a functor to $\Lambda(U)-\Pi(U)-\mathrm{Bimod}$ whenever M is a sheaf of $\Lambda-\Pi$-bimodules.

Problem 6.(Flasque Sheaves are Čech-Acyclic) Let (X,τ_X) be a topological space. Let M be a presheaf of $\Lambda-\Pi$-bimodules on (X,τ_X). Let U be a τ_X-open set. Let $(U,\iota:\mathfrak{U}\to\tau_U)$ be an open covering. For every τ_X-open subset V, define $(V,\iota_V:\mathfrak{U}\to\tau_V)$ to be the open covering $\iota_V(U_0) = V\cap\iota(U_0)$. For simplicity, denote this by $(V,\mathfrak{U}_V)$. For every integer $r\geq 0$, define $\underline{\check{C}}^r(\mathfrak{U},M)(V)$ to be the $\Lambda(V)-\Pi(V)$-bimodule $\check{C}^r(\mathfrak{U}_V,M)$. Moreover, define

$$\partial^i_{V,r} : \underline{\check{C}}^r(\mathfrak{U},M)(V) \to \underline{\check{C}}^{r+1}(\mathfrak{U},M)(V),\quad \sigma^i_{V,r+1} : \underline{\check{C}}^{r+1}(\mathfrak{U},M)(V) \to \underline{\check{C}}^r(\mathfrak{U},M)(V),$$

to be the face and degeneracy maps on $\check{C}^\bullet(\mathfrak{U}_V,M)$. Finally, let $\eta^r_V : M(V)\to\underline{\check{C}}^r(\mathfrak{U},M)(V)$ be the coadjunction of sections from Problem 5(e), Problem Set 8. For every inclusion of τ_X-open subsets $W\cap V\cap U$, the identity map $\mathrm{Id}_{\mathfrak{U}}$ is a refinement of open coverings,

$$\phi^V_W : (V,\iota_V:\mathfrak{U}\to\tau_V) \to (W,\iota_W:\mathfrak{U}\to\tau_W).$$

By Problem 5(f) from Problem Set 8, $\check{C}^r(\phi^V_W,M)$ is an associated morphism of $\Lambda(V)-\Pi(V)$-bimodules, denoted

$$\underline{\check{C}}^r(\mathfrak{U},M)^V_W : \underline{\check{C}}^r(\mathfrak{U},M)(V) \to \underline{\check{C}}^r(\mathfrak{U},M)(W).$$

(a)(The Presheaf of Čech Objects) **Prove** that the rules $V \mapsto \underline{\check{C}}^r(\mathfrak{U}, M)(V)$ and $\underline{\check{C}}^r(\mathfrak{U}, M)^V_W$ define a presheaf $\underline{\check{C}}^r(\mathfrak{U}, M)$ of $\Pi - \Lambda$-bimodules on U. Moreover, **prove** that the rules $V \mapsto \partial^i_{V,r}$, resp. $V \mapsto \sigma^i_{V,r+1}$, $V \mapsto \eta^r_V$, define morphisms of presheaves of $\Lambda - \Pi$-bimodules,

$$\partial^i_r : \underline{\check{C}}^r(\mathfrak{U}, M) \to \underline{\check{C}}^{r+1}(\mathfrak{U}, M), \quad \sigma^i_{r+1} : \underline{\check{C}}^{r+1}(\mathfrak{U}, M) \to \underline{\check{C}}^r(\mathfrak{U}, M), \quad \eta^r : M|_U \to \underline{\check{C}}^r(\mathfrak{U}, M).$$

Use Problem 5(f) from Problem Set 8 again to prove that these morphisms define a functor,

$$\underline{\check{C}}^\bullet : \sigma \times \Lambda - \Pi - \mathrm{Presh}_{(X,\tau_X)} \to S^\bullet\Lambda - \Pi - \mathrm{Presh}_{(U,\tau_U)},$$

compatible with cosimplicial homotopies for pairs of refinements and together with a natural transformation of cosimplicial objects,

$$\eta^\bullet : \mathrm{const}^\bullet_{M|_U} \to \underline{\check{C}}^\bullet(\mathfrak{U}, M).$$

(b)(The Čech Resolution Preserves Sheaves and Flasques) For every $(U_0, \dots, U_r)$ in $\mathfrak{U}^{r+1}$, denote by $i_{U_0,\dots,U_r} : (\iota(U_0, \dots, U_r), \tau_{\iota(U_0,\dots,U_r)}) \to (U, \tau_U)$ the continuous inclusion map. **Prove** that $\underline{\check{C}}^r(\mathfrak{U}, M)$ is isomorphic as a presheaf of $\Lambda - \Pi$-bimodules to

$$\prod_{(U_0,\dots,U_r)} (\iota_{U_0,\dots,U_r})_* \iota^{-1}_{U_0,\dots,U_r} M.$$

Use Problem 4(a) and (b) to **prove** that $\underline{\check{C}}^r(\mathfrak{U}, M)$ is a sheaf whenever M is a sheaf, and it is flasque whenever M is flasque.

(c)(Localy Acyclicity of the Čech Resolution) Assume now that M is a sheaf. For every τ_X-open subset $V \subset U$ such that there exists $* \in \mathfrak{U}$ with $V \subset \iota(*)$, conclude that $(V, \mathfrak{U}_V)$ refines to $(V, \{V\})$. Using Problem 5(h), Problem Set 8, **prove** that

$$\eta^\bullet_V : \mathrm{const}^\bullet_{M(V)} \to \underline{\check{C}}^\bullet(\mathfrak{U}, M)(V)$$

is a homotopy equivalence. Conclude that for the cochain differential associated to this cosimplicial object,

$$d^r = \sum_{i=0}^r (-1)^i \partial^i_r,$$

the coaugmentation

$$\eta_V : M(V) \to \underline{\check{C}}^\bullet(\mathfrak{U}, M)(V)$$

is an acyclic resolution. Conclude that the coaugmentation of complexes of sheaves of $\Pi - \Lambda$-bimodules,

$$\eta : M|_U \to \underline{\check{C}}^\bullet(\mathfrak{U}, M)$$

is an acyclic resolution.

Now assume that M is flasque. **Prove** that η is a flasque resolution of the flasque sheaf $M|_U$. Using Problem 5(i), **prove** that the cohomology of the complex of $\Lambda(U)-\Pi(U)$-bimodules,

$$\check{H}^n(\mathfrak{U}, M) := h^n(\check{C}^\bullet(\mathfrak{U}, M), d^\bullet)$$

equals $H^\bullet(U, M)$. Using Problem 5(h), **prove** that $H^0(U, M)$ equals $M(U)$ and $H^n(U, M)$ is zero for every integer $n > 0$. Conclude that for every flasque sheaf M of $\Lambda-\Pi$-bimodules, for every open covering $(U, \mathfrak{U})$, $M(U) \to \check{H}^0(\mathfrak{U}, M)$ is an isomorphism and $\check{H}^n(\mathfrak{U}, M)$ is zero for every integer $n > 0$.

Problem 7.(Čech Cohomology is a Derived Functor on Presheaves) Let (X, τ_X) be a topological space. Let U be a τ_X-open set. Let $(U, \iota : \mathfrak{U} \to \tau_U)$ be an open covering. For every presheaf A of $\Lambda-\Pi$-bimodules, denote by $\check{C}^\bullet(\mathfrak{U}, A)$ the object in $\mathbf{Ch}^{\geq 0}(\Lambda-\Pi-\text{Bimod})$ associated to the cosimplicial object.

(a)(Exactness of the Functor of Čech Complexes; The δ-Functor of Čech Cohomologies) Use Problem 5 of Problem Set 8 to **prove** that this is an additive functor

$$\check{C}^\bullet(\mathfrak{U}, -) : \Lambda-\Pi-\text{Presh}_{(X,\tau_X)} \to \text{Ch}^{\geq 0}(\Lambda-\Pi-\text{Bimod}).$$

Prove that for every short exact sequence of presheaves of $\Lambda-\Pi$-bimodules,

$$0 \longrightarrow A' \xrightarrow{q} A \xrightarrow{p} A'' \longrightarrow 0,$$

the associated sequence of cochain complexes,

$$0 \longrightarrow \check{C}^\bullet(\mathfrak{U}, A') \xrightarrow{\check{C}^\bullet(\mathfrak{U},q)} \check{C}^\bullet(\mathfrak{U}, A) \xrightarrow{\check{C}^\bullet(\mathfrak{U},p)} \check{C}^\bullet(\mathfrak{U}, A'') \longrightarrow 0,$$

is a short exact sequence. Use this to prove that the Čech cohomology functor $\check{H}^0(\mathfrak{U}, A) = h^0(\check{C}^\bullet(\mathfrak{U}, A))$ is an additive, left-exact functor, and the sequence of Čech cohomologies,

$$\check{H}^r(\mathfrak{U}, A) = h^r(\check{C}^\bullet(\mathfrak{U}, A)),$$

extend to a cohomological δ-functor from $\Lambda-\Pi-\text{Presh}_{(X,\tau_X)}$ to $\Lambda(U)-\Pi(U)-\text{Bimod}$.

(b)(Effaceability of Čech Cohomology) For every presheaf A of $\Lambda-\Pi$-bimodules, use Problem 5(e) and 5(f) to **prove** that $\theta_A : A \to \widetilde{\Phi(A)}$ is a natural monomorphism of presheaves of $\Lambda-\Pi$-bimodules. Use Problem 5(d) to prove that for every $r \geq 0$, $\check{H}^r(\mathfrak{U}, \widetilde{\Phi(A)})$ is zero. Conclude that $\check{H}^r(\mathfrak{U}, -)$ is effaceable. **Prove** that the cohomological δ-functor $((\check{H}^r(\mathfrak{U}, A))_r, (\delta^r)_r)$ is universal. Conclude that the natural transformation of cohomological δ-functors from the right derived functor of $\check{H}^0(\mathfrak{U}, -)$ to the Čech cohomology δ-functor is a natural isomorphism of cohomological δ-functors.

(c)(Hypotheses of the Grothendieck Spectral Sequence) Denote by

$$\Psi : \Lambda-\Pi-\text{Sh}_{(X,\tau_X)} \to \Lambda-\Pi-\text{Presh}_{(X,\tau_X)},$$

the additive, fully faithful embedding (since we are already using Φ for the forgetful morphism to discontinuous $\Lambda-\Pi$-bimodules). Recall from Problem 6(c) on Problem Set 8 that this extends to an adjoint pair of functors (Sh, Φ). Recall the construction of Sh as a filtering colimit of Čech cohomologies $\check{H}^0(\mathfrak{U}, -)$. Since $\check{H}^0(\mathfrak{U}, -)$ is left-exact, and since $\Lambda - \Pi - \text{Presh}_{(X,\tau_X)}$ satisfies Grothendieck's condition (AB5), **prove** that Sh is left-exact. Use Problem 3(d), Problem Set 5 to **prove** that Ψ sends injective objects to injective objects. Use Problem 5(g) to **prove** that every injective sheaf I of $\Lambda-\Pi$-bimodules is flasque. Use Problem 6(c) to **prove** that $\Psi(I)$ is acyclic for $\check{H}^\bullet(\mathfrak{U}, -)$. **Prove** that the pair of functors Ψ and $\check{H}^0(\mathfrak{U}, -)$ satisfy the hypotheses for the Grothendieck Spectral Sequence. Conclude that there is a convergent, first quadrant cohomological spectral sequence,

$$^I E_2^{p,q} = \check{H}^p(\mathfrak{U}, R^q\Psi(A)) \Rightarrow H^{p+q}(U, A).$$

(d)(The Derived Functors of Ψ are the Presheaves of Sheaf Cohomologies) For every sheaf A of $\Lambda-\Pi$-bimodules, for every integer $r \geq 0$, for every τ_X-open set U, denote $\mathcal{H}^r(A)(U)$ the additive functor $H^r(U, A)$. In particular, $\mathcal{H}^0(A)(U)$ is canonically isomorphic to $A(U)$. Thus, for all τ_X-open sets, $V \subset U$, there is a natural transformation

$$*|_V^U : \mathcal{H}^0(-)(U) \to \mathcal{H}^0(-)(V).$$

Use universality to **prove** that this uniquely extends to a morphism of cohomological δ-functors,

$$*|_V^U : ((\mathcal{H}^r(-)(U))_r, (\delta^r)_r) \to ((\mathcal{H}^r(-)(V))_r, (\delta^r)_r).$$

Prove that for all τ_X-open sets, $W \subset V \subset U$, both the composite morphism of cohomological δ-functors,

$$*|_W^V \circ *|_V^U : ((\mathcal{H}^r(-)(U))_r, (\delta^r)_r) \to ((\mathcal{H}^r(-)(V))_r, (\delta^r)_r) \to ((\mathcal{H}^r(-)(W))_r, (\delta^r)_r),$$

and the morphism of cohomological δ-functors,

$$*|_W^U : ((\mathcal{H}^r(-)(U))_r, (\delta^r)_r) \to ((\mathcal{H}^r(-)(W))_r, (\delta^r)_r),$$

extend the functor $*|_W^V \circ *|_V^U = *|_W^U$ from $\mathcal{H}^0(-)(U)$ to $\mathcal{H}^0(-)(W)$. Use the uniqueness in the universality to conclude that these two morphisms of cohomological δ-functors are equal. **Prove** that $((\mathcal{H}^r(-))_r, (\delta^r)_r)$ is a cohomological δ-functor from $\Lambda - \Pi - \text{Sh}_{(X,\tau_X)}$ to $\Lambda - \Pi - \text{Presh}_{(X,\tau_X)}$. Use Problem 5(h) to **prove** that every flasque sheaf is acyclic for this cohomological δ-functor. Combined with Problem 5(i), **prove** that the higher functors are effaceable, and thus this cohomological δ-functor is universal. Conclude that this the canonical morphism of cohomological δ-functors from the right derived functors of Ψ to this cohomological δ-functor is a natural isomorphism of cohomological δ-functors. In particular, combined with the last part, this gives a convergent, first quadrant spectral sequence,

$$^I E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathcal{H}^q(A)) \Rightarrow H^{p+q}(U, A).$$

This is the *Čech-to-Sheaf Cohomology Spectral Sequence.* In particular, conclude the existence of monomorphic abutment maps,

$$\check{H}^r(\mathfrak{U}, A) \to H^r(U, A).$$

as well as abutment maps,

$$H^r(U, A) \to H^0(\mathfrak{U}, \mathcal{H}^r(A)).$$

(e)(The Colimit of Čech Cohomology with Respect to Refinement) Since Čech complexes are compatible with refinement, and the refinement maps are well-defined up to cosimplicial homotopy, the induced refinement maps on Čech cohomology are independent of the choice of refinement. Use this to define a directed system of Čech cohomologies. Denote the colimit of this direct system as follows,

$$\check{H}^\bullet(U, -) = \operatorname*{colim}_{\mathfrak{U} \in \sigma_{x,U}} \check{H}^\bullet(\mathfrak{U}, -).$$

Prove that this extends uniquely to a cohomological δ-functor such that for every open covering $(U, \mathfrak{U})$, the induced sequence of natural transformations,

$$* |_U^{\mathfrak{U}} : ((\check{H}^r(\mathfrak{U}, -))_r, (\delta^r)_r) \to ((\check{H}^r(U, -))_r, (\delta^r)_r),$$

is a natural transformation of cohomological δ-functors. Repeat the steps above to deduce the existence of a unique convergent, first quadrant spectral sequence,

$${}^I E_2^{p,q} = \check{H}^p(U, \mathcal{H}^q(A)) \Rightarrow H^{p+q}(U, A),$$

such that for every open covering $(U, \mathfrak{U})$, the natural maps

$$* |_U^{\mathfrak{U}} : \check{H}^p(\mathfrak{U}, \mathcal{H}^q(A)) \to \check{H}^p(U, \mathcal{H}^q(A))$$

extend uniquely to a morphism of spectral sequences. In particular, conclude the existence of monomorphic abutment maps

$$\check{H}^r(U, A) \to H^r(U, A)$$

as well as abutment maps

$$H^r(U, A) \to \check{H}^0(U, \mathcal{H}^r(A)).$$

Use the first abutment maps to define subpresheaves $\check{\mathcal{H}}^r(A)$ of $\mathcal{H}^r(A)$ by $V \mapsto \check{H}^r(V, A)$.

(f)(Reduction of the Spectral Sequence; $\check{H}^1(U, A)$ equals $H^1(U, A)$) For every $r > 0$, **prove** that the associated sheaf of $\mathcal{H}^r(A)$ is a zero sheaf. (**Hint**. Prove the stalks are zero by using commutation of sheaf cohomology with filtered colimits combined with exactness of the stalks functor.) Conclude that $\check{H}^0(U, \mathcal{H}^r(A))$ is zero. In particular, conclude that the natural abutment map,

$$\check{H}^1(U, A) \to H^1(U, A)$$

is an isomorphism. Thus, also $\check{\mathcal{H}}^1(A) \to \mathcal{H}^1(A)$ is an isomorphism. Use this to produce a "long exact sequence of low degree terms" of the spectral sequence,

$$0 \to \check{H}^2(U, A) \to H^2(U, A) \to \check{H}^1(U, \check{\mathcal{H}}^1(A)) \xrightarrow{\delta} \check{H}^3(U, A).$$

(g)(Sheaves that Are Čech-Acyclic for "Enough" Covers are Acyclic for Sheaf Cohomology) Let $\mathcal{B} \subset \tau_X$ be a basis that is stable for finite intersection. For every open U in $\mathcal{B}$, let Cov_U be a collection of open coverings of U by sets in $\mathcal{B}$ such that Cov_U is cofinal with respect to refinement in $\sigma_{x,U}$. Let A be such that for every U in $\mathcal{B}$, for every $(U, \mathfrak{U})$ in Cov_U, for every $r \geq 0$, $\check{H}^r(\mathfrak{U}, A)$ is zero. **Prove** that $\mathcal{H}^r(U, A)$ is zero. Use the spectral sequence to inductively **prove** that for every $r \geq 0$, $\mathcal{H}^r(A)(U)$ is zero, $H^r(U, A)$ is zero and $\mathcal{H}^r(A)(U)$ is zero. Conclude that for every open covering $(X, \iota : \mathfrak{V} \to \mathcal{B})$, the Čech-to-Sheaf Cohomology Spectral Sequence relative to $\mathfrak{V}$ degenerates to isomorphisms

$$\check{H}^r(\mathfrak{V}, A) \to H^r(X, A).$$

If you are an algebraic geometer, let $(X, \mathcal{O}_X)$ be a separated scheme, let $\Lambda = \Pi = \mathcal{O}_X$, let $\mathcal{B}$ be the basis of open affine subsets, let Cov_U be the collection of basic open affine coverings, and let A be a quasi-coherent sheaf. Read the proof that for every basic open affine covering $(U, \mathfrak{U})$ of an affine scheme, for every quasi-coherent sheaf A, $\check{H}^r(\mathfrak{U}, A)$ is zero for all $r \geq 0$ (this is essentially exactness of the Koszul cochain complex for a regular sequence, combined with commutation with colimits). Use this to conclude that quasi-coherent sheaves are acyclic for sheaf cohomology on any affine scheme. Conclude that, on a separated scheme, for every quasi-coherent sheaf, sheaf cohomology is computed as Čech cohomology of any open affine covering.

Problem 8.(The de Rham, Dolbeault and Hodge Theorems) Read about *soft* and *fine* sheaves. In particular, read the proof that soft sheaves are acyclic on paracompact, Hausdorff topological spaces. Read about partitions of unity. For every paracompact, Hausdorff, C^∞ analytic space X, let $\Lambda = \Pi$ equals $\mathcal{E}^0_{\mathbb{R}}$, resp. $\mathcal{E}^0_{\mathbb{C}}$, the sheaf of C^∞ functions to $\mathbb{R}$, resp. $\mathbb{C}$, with its standard real analytic structure. **Prove** that this has partitions of unity, and hence is fine. Conclude that every sheaf M of $\mathcal{E}^0$-modules is also fine.

(a)(de Rham's Theorem) Let X be a C^∞ manifold that is paracompact and Hausdorff (some authors include paracompact and Hausdorff in the definition of manifold). For every integer $n \geq 0$, define $\mathcal{E}^n_{\mathbb{R}}$, resp. $\mathcal{E}^n_{\mathbb{C}}$, to be the sheaf of $\mathcal{E}^0_{\mathbb{R}}$-modules, resp. $\mathcal{E}^0_{\mathbb{C}}$-modules, whose sections on any open are the C^∞ differential n-forms on that open set that are $\mathbb{R}$-valued, resp. $\mathbb{C}$-valued. Let $d^n : \mathcal{E}^n \to \mathcal{E}^{n+1}$ be the morphism of exterior differentiation. **Prove** that this defines a complex $\mathcal{E}^\bullet_{\mathbb{R}}$ in $\text{Ch}^{\geq 0}(\mathbb{R} - \text{Sh}_{(X,\tau_X)})$, the *de Rham complex*, and likewise for $\mathcal{E}^\bullet_{\mathbb{C}}$. The *de Rham cohomology* of X is defined to be the cohomology of the associated complex of global sections,

$$H^n_{\text{dR}}(X, \mathbb{R}) = h^n(\mathcal{E}^\bullet_{\mathbb{R}}(X), d^\bullet), \quad \text{resp. } H^n_{\text{dR}}(X, \mathbb{C}) = h^n(\mathcal{E}^\bullet_{\mathbb{C}}(X), d^\bullet).$$

Let $\epsilon : \underline{\mathbb{R}}_X \to \mathcal{E}^0_{\mathbb{R}}$, resp. $\epsilon : \underline{\mathbb{C}}_X \to \mathcal{E}^0_{\mathbb{C}}$ be the inclusion of the locally constant functions. Read the proof of the Poincaré Lemma. **Prove** that $\epsilon : \underline{\mathbb{R}}_X \to \mathcal{E}^\bullet_{\mathbb{R}}$ is an acyclic resolution of $\underline{\mathbb{R}}_X$ by sheaves that are acyclic for sheaf cohomology, and similarly for $\epsilon : \underline{\mathbb{C}}_X \to \mathcal{E}^\bullet_{\mathbb{C}}$. **Prove** that the hypercohomology spectral sequence degenerates to isomorphisms,

$$H^n_{\text{dR}}(X, \mathbb{R}) \to H^n(X, \underline{\mathbb{R}}_X), \quad \text{resp. } H^n_{\text{dR}}(X, \mathbb{C}) \to H^n(X, \underline{\mathbb{C}}_X).$$

This is the sheaf cohomology version of *de Rham's theorem*.

(b)(The Dolbeault Theorem) Now let X be a paracompact, Hausdorff, complex manifold, and let $\mathcal{O}_X$ be the sheaf of holomorphic functions to $\mathbb{C}$ with its standard complex analytic structure.

This is *not* a fine sheaf, but the sheaf associated to the underlying C^∞ manifold structure, $\mathcal{E}^0_{\mathbb{C}}$, is fine. Now denote $\mathcal{E}^0_{\mathbb{C}}$ by $\mathcal{E}^{0,0}$. For every pair of integers, $p,q \geq 0$, define $\mathcal{E}^{p,q}$ to be the sheaf of $\mathcal{E}^{0,0}$-modules whose sections on each open are the C^∞, $\mathbb{C}$-valued differential forms that can locally on opens U of an open covering be expressed as $\mathcal{E}^{0,0}$-linear combinations of differential forms $dz_1 \wedge \cdots \wedge dz_p \wedge d\overline{z}_{p+1} \wedge \cdots \wedge d\overline{z}_{p+q}$, for a local holomorphic coordinate chart,

$$(z_1, \dots, z_n) : U \to B_1(0) \subset \mathbb{C}^n.$$

Let $\overline{\partial}^{p,q} : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}$ be the usual Dolbeault differential. **Prove** that this defines a complex $\mathcal{E}^{p,\bullet}$ in $\text{Ch}^{\geq 0}(\mathbb{C} - \text{Sh}_{(X,\tau_X)})$, the *Dolbeault complex*. The *Dolbeault cohomology* is defined to be the cohomology of the associated complex of global sections,

$$H^{p,q}_{\text{Dol}}(X) := h^q(\mathcal{E}^{p,\bullet}, \overline{\partial}^{p,\bullet}).$$

For every $p \geq 0$, define $\epsilon^p : \Omega^p_{X,\text{hol}} \to \mathcal{E}^{p,0}$ to be the sheaf of $\mathcal{O}_X$-modules whose sections on an open are the p-forms that are locally $\mathcal{O}_X$-linear combinations of differentials of the form $dz_1 \wedge \cdots \wedge dz_p$. These are the *holomorphic* p-forms. Read the proof of the $\overline{\partial}$-Poincaré Lemma. **Prove** that $\epsilon^p : \Omega^p_{X,\text{hol}} \to \mathcal{E}^{p,\bullet}$ is an acyclic resolution of $\Omega^p_{X,\text{hol}}$ by sheaves that are acyclic for sheaf cohomology. **Prove** that the hypercohomology spectral sequence degenerates to isomorphisms,

$$H^{p,q}_{\text{Dol}}(X) \to H^q(X, \Omega^p_{X,\text{hol}}).$$

This is the *Dolbeault theorem*.

(c)(The Frölicher Spectral Sequence) Continuing the previous part, **prove** that the exterior differential,

$$d^p : \mathcal{E}^{p,0} \to \mathcal{E}^{p+1}_{\mathbb{C}},$$

restricts on $\Omega^p_{X,\text{hol}}$ to a differential

$$d^p : \Omega^p_{X,\text{hol}} \to \Omega^{p+1}_{X,\text{hol}}.$$

Prove that this defines a complex $\Omega^\bullet_{X,\text{hol}}$ in $\text{Ch}^{\geq 0}(\mathbb{C} - \text{Sh}_{(X,\tau_X)})$, the *holomorphic de Rham complex*. **Prove** that the coaugmentation $\epsilon^0 : \underline{\mathbb{C}}_X \to \mathcal{E}^{0,0}$ factors through $\Omega^0_{X,\text{hol}} = \mathcal{O}_X$. Read the proof of the holomorphic Poincaré Lemma. **Prove** that $\epsilon : \underline{\mathbb{C}}_X \to \Omega^\bullet_{X,\text{hol}}$ is an acyclic resolution. **Prove** that this induces an isomorphism of hypercohomology groups (written in the inverse direction),

$$\mathbb{H}^n(X, \Omega^\bullet_{X,\text{hol}}) \to H^n(X, \underline{\mathbb{C}}_X).$$

The corresponding hypercohomology spectral sequence is the *Frölicher spectral sequence* or *Hodge-to-de Rham spectral sequence*,

$$E^{p,q}_2 = H^q(X, \Omega^p_{X,\text{hol}}) \Rightarrow H^{p+q}(X, \underline{\mathbb{C}}_X).$$

In those cases that the dimensions $h^n(X, \underline{\mathbb{C}}_X)$ of $H^n(X, \underline{\mathbb{C}}_X)$ and $h^{p,q}(X)$ of $H^q(X, \Omega^p_{X,\text{hol}})$ are finite, and also

$$h^n(X, \underline{\mathbb{C}}_X) = \sum_{p+q=n} h^{p,q}(X),$$

conclude that this spectral sequence degenerates. In particular, read the proof of the *Hodge theorem* for compact Kähler manifolds.