MAT 536 Problem Set 6

Homework Policy. Please read through all the problems. Please solve 5 of the problems. I will be happy to discuss the solutions during office hours.

Problems.

Problem 1.(Effaceable monomorphisms). Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. Let

$$R = ((R^n : \mathcal{A} \to \mathcal{B})_{n\in\mathbb{Z}}, (\delta^n_{R,\Sigma})_{n\in\mathbb{Z}}),$$

be a cohomological δ-functor (typically we assume that the functors are zero for $n < 0$). Recall that a monomorphism in $\mathcal{A}$,

$$u : A \hookrightarrow I,$$

is R**-effaceable in degree** n if the associated morphism in $\mathcal{B}$,

$$R^n(u) : R^n(A) \to R^n(I),$$

is the zero morphism.

(a) For objects A, I and J of $\mathcal{A}$, for monomorphism,

$$u : A \hookrightarrow I, \quad v : A \hookrightarrow J,$$

prove that the associated monomorphism,

$$(u, v) : A \hookrightarrow I \oplus J$$

is R-effaceable in degree n if and only if both u and v are R-effaceable in degree n.

(b) For objects A, I and J of $\mathcal{A}$, for every pair of monomorphisms,

$$u : A \hookrightarrow I, \quad v : A \hookrightarrow J,$$

that are both R-effaceable in degree n, prove that there are commutative diagrams of short exact sequences whose rows are quotients of monomorphisms that are R-effaceable in degree n,

$$\begin{array}{ccccccccccc}
\Sigma : & 0 & \longrightarrow & A & \xrightarrow{u} & I & \xrightarrow{\overline{u}} & \mathrm{Coker}(u) & \longrightarrow & 0 \\
{\scriptstyle q_1}\downarrow & & & \downarrow{\scriptstyle \mathrm{Id}_A} & & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle \overline{q}_1} & & & , \\
\Sigma \oplus T : & 0 & \longrightarrow & A & \xrightarrow{(u,v)} & I \oplus J & \xrightarrow{\overline{(u,v)}} & \mathrm{Coker}(u,v) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccccc}
\Sigma: & 0 & \longrightarrow & A & \xrightarrow{v} & J & \xrightarrow{\overline{u}} & \mathrm{Coker}(v) & \longrightarrow & 0 \\
{\scriptstyle q_2}\downarrow & & & \downarrow{\scriptstyle \mathrm{Id}_A} & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle \overline{q}_2} & & & . \\
\Sigma \oplus T: & 0 & \longrightarrow & A & \xrightarrow{(u,v)} & I \oplus J & \xrightarrow{\overline{(u,v)}} & \mathrm{Coker}(u,v) & \longrightarrow & 0
\end{array}$$

(c) For a cohomological δ-functor,

$$G = ((G^n : \mathcal{A} \to \mathcal{B})_{n\in\mathbb{Z}}, (\delta^n_{G,\Sigma})_{n\in\mathbb{Z}}),$$

let

$$(\gamma^m : R^m \Rightarrow G^m)_{m<n}$$

be a sequence of natural transformations that commute with the morphisms $(\delta^m_{R,\Sigma})_{m<n-1}$ and $(\delta^m_{G,\Sigma})_{m<n-1}$ in the usual way. For every short exact sequence in $\mathcal{A}$,

$$\Sigma: \ 0 \longrightarrow A \xrightarrow{u} I \xrightarrow{\overline{u}} C \longrightarrow 0,$$

such that u is R-effaceable in degree n, define

$$\gamma^n_\Sigma(A) : R^n(A) \to G^n(A),$$

to be the unique morphism such that the following diagram commutes,

$$\begin{array}{ccccccc}
R^{n-1}(I) & \xrightarrow{R^{n-1}(\overline{u})} & R^{n-1}(C) & \xrightarrow{\delta^{n-1}_{R,\Sigma}} & R^n(A) & \xrightarrow{0} & R^n(I) \\
{\scriptstyle \gamma^{n-1}(I)}\downarrow & & {\scriptstyle \gamma^{n-1}(C)}\downarrow & & \downarrow{\scriptstyle \gamma^n_\Sigma(A)} & & & . \\
G^{n-1}(I) & \xrightarrow[G^{n-1}(\overline{u})]{} & G^{n-1}(C) & \xrightarrow[\delta^{n-1}_{G,\Sigma}]{} & G^n(A) & \xrightarrow[G^n(u)]{} & R^n(I)
\end{array}$$

For every commutative diagram of short exact sequences in $\mathcal{A}$,

$$\begin{array}{ccccccccccc}
\Sigma: & 0 & \longrightarrow & A & \xrightarrow{u} & I & \xrightarrow{\overline{u}} & C & \longrightarrow & 0, \\
{\scriptstyle u}\downarrow & & & {\scriptstyle u_A}\downarrow & & \downarrow{\scriptstyle u_I} & & \downarrow{\scriptstyle u_C} & & \\
\widetilde{\Sigma}: & 0 & \longrightarrow & \widetilde{A} & \xrightarrow{\widetilde{u}} & \widetilde{I} & \xrightarrow{\widetilde{\overline{u}}} & \widetilde{C} & \longrightarrow & 0,
\end{array}$$

such that both u and $\widetilde{u}$ are R-effaceable in degree n, carefully chase through the "commuting cube" argument from lecture to prove that the following diagram commutes,

$$\begin{array}{ccc}
R^n(A) & \xrightarrow{R^n(u_A)} & R^n(\widetilde{A}) \\
{\scriptstyle \gamma^n_\Sigma(A)}\downarrow & & \downarrow{\scriptstyle \gamma^n_{\widetilde{\Sigma}}(\widetilde{A})} \\
G^n(A) & \xrightarrow[G^n(u_A)]{} & G^n(\widetilde{A})
\end{array}$$

(d) Now, applying (c) to the two commutative diagrams from (b), prove that the morphisms $\gamma^n_\Sigma(A)$ and $\gamma^n_T(A)$ are equal.

(e) Finally, for every short exact sequence in $\mathcal{A}$,

$$\Pi :\ 0 \longrightarrow K \xrightarrow{q} A \xrightarrow{p} Q \longrightarrow 0,$$

prove that also the following composite monomorphism is R-effaceable in degree n,

$$K \xrightarrow{q} A \xrightarrow{u} I.$$

Thus, there is a commutative diagram of short exact sequences in $\mathcal{A}$,

$$\begin{array}{ccccccccccc}
\Pi : & 0 & \longrightarrow & K & \xrightarrow{q} & A & \xrightarrow{p} & Q & \longrightarrow & 0, \\
{\scriptstyle w}\big\downarrow & & & {\scriptstyle \mathrm{Id}_K}\big\downarrow & & \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle w} & & & . \\
\widetilde{\Pi} : & 0 & \longrightarrow & K & \xrightarrow{u\circ q} & I & \xrightarrow{p} & \mathrm{Coker}(u\circ q) & \longrightarrow & 0
\end{array}$$

Mimic the "commuting cube" argument for the cube associated to this diagram to conclude that the following diagram commutes,

$$\begin{array}{ccc}
R^{n-1}(Q) & \xrightarrow{\delta^{n-1}_{R,\Pi}} & R^n(K) \\
{\scriptstyle \gamma^{n-1}(Q)}\big\downarrow & & \big\downarrow{\scriptstyle \gamma^n_\Pi(K)} \\
G^{n-1}(Q) & \xrightarrow[\delta^{n-1}_{G,\Pi}]{} & > G^n(K)
\end{array}$$

(f) Assume that for every object A of $\mathcal{A}$ there exists a monomorphism,

$$u : A \hookrightarrow I,$$

that is R-effaceable in degree n. Conclude that there exists a unique extension of $(\gamma^m)_{m<n}$ to a sequence of natural transformations,

$$(\gamma^m : R^m \Rightarrow G^m)_{m\leq n},$$

that commutes with the morphisms $(\delta^m_{R,\Sigma})_{m<n}$ and $(\delta^m_{G,\Sigma})_{m<n}$ in the usual way. Assuming that for every integer $r \geq n$, for every object A of $\mathcal{A}$ there exists a monomorphism,

$$u : A \hookrightarrow I,$$

that is R-effaceable in degree n. Conclude that there exists a unique exteion of $(\gamma^m)_{m<n}$ to a natural transformation of cohomological δ-functors,

$$\gamma = (\gamma^m : R^m \Rightarrow G^m)_{m\in\mathbb{Z}}.$$

Finally, if also R is concentrated in degrees $\geq m$, conclude that $R^m \to R$ is initial among all natural transformations from R^m to a cohomological δ-functor concentrated in degrees $\geq m$.

Problem 2. For a right-exact additive functor $F : \mathcal{A} \to \mathcal{B}$, for a homological δ-functor

$$L = ((L_m : \mathcal{A} \to \mathcal{B})_{m \geq 0}, (\delta_m^{L,\Sigma})_{m \geq 1}),$$

that is coeffaceable in degrees ≥ 0, for a natural equivalence of functors $\alpha : L_0 \to F$, prove *carefully* that α is final among all natural transformations $\beta : G_0 \to F$ from a homological δ-functor concentrated in degrees ≥ 0,

$$G = ((G_m : \mathcal{A} \to \mathcal{B})_{m \geq 0}, (\delta_m^{G,\Sigma})_{m \geq 1}).$$

It is a good idea to use opposite categories to guide you, but please check the details carefully. Conclude that if $\mathcal{A}$ has enough projective objects, then every right-exact additive functor F has an extension to a universal homological δ-functor (that is essentially unique).

Problem 3.(Balancing derived bifunctors.) Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Abelian categories. Let

$$T : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$$

be a bifunctor, i.e., an assignment to every objects A of $\mathcal{A}$ and to every object B of $\mathcal{B}$ of an object $T(A, B)$ of $\mathcal{C}$, an assignment to every object A of $\mathcal{A}$ and all objects B, $\widetilde{B}$ of $\mathcal{B}$ of set maps,

$$T_{A,-} : \mathrm{Hom}_{\mathcal{B}}(B, \widetilde{B}) \to \mathrm{Hom}_{\mathcal{C}}(T(A, B), T(A, \widetilde{B})),$$

and an assignment to every object B of $\mathcal{B}$ and all objects A, $\widetilde{A}$ of $\mathcal{A}$ of set maps,

$$T_{-,B} : \mathrm{Hom}_{\mathcal{A}}(A, \widetilde{A}) \to \mathrm{Hom}_{\mathcal{C}}(T(A, B), T(\widetilde{A}, B)),$$

making all assignments,

$$T_{A,-} : \mathcal{B} \to \mathcal{C}, \quad T_{-,B} : \mathcal{B} \to \mathcal{C},$$

functors, and such that for every morphism in $\mathcal{A}$,

$$a : A \to \widetilde{A},$$

and for every morphism in $\mathcal{B}$,

$$b : B \to \widetilde{B},$$

the following diagram in $\mathcal{C}$ commutes,

$$\begin{tikzcd} T(A, B) \arrow[r, "T_{A,-}(b)"] \arrow[d, "T_{-,B}(a)"'] & T(A, \widetilde{B}) \arrow[d, "T_{-,\widetilde{B}}(a)"] \\ T(\widetilde{A}, B) \arrow[r, "T_{\widetilde{A},-}(b)"'] & T(\widetilde{A}, \widetilde{B}) \end{tikzcd}$$

(a) For every morphism $a : A \to \widetilde{A}$, prove that the assigment to every object B of $\mathcal{B}$ of the morphism,

$$T_{-,B}(a) : T_{A,-}(B) \to T_{\widetilde{A},-}(B)$$

is a natural transformation of functors,

$$T_{a,-} : T_{A,-} \Rightarrow T_{\widetilde{A},-}.$$

Prove that $T_{\mathrm{Id}_A,-}$ is the identity natural transformation, and prove that $a \mapsto T_{a,-}$ is compatible with composition. Similarly, for every morphism $b : B \to \widetilde{B}$, prove that the assignment to every object A of $\mathcal{A}$ of the morphism,

$$T_{A,-}(b) : T_{-,B}(A) \to T_{-,\widetilde{B}}(A)$$

is a natural transformation of functors,

$$T_{-,b} : T_{-,B} \Rightarrow T_{-,\widetilde{B}}.$$

Prove that the rule $b \mapsto T_{-,b}$ is also compatible with identity and composition.

(b) Now assume, further, that all $T_{A,-}$ and $T_{-,B}$ are right-exact additive functors. For every object A of $\mathcal{A}$, let

$$LT_{A,-} = ((L_nT_{A,-} : \mathcal{B} \to \mathcal{C})_{n\geq 0}, (\delta_n^{LT_{A,-},\Sigma})_{n>0}),$$

be a coeffaceable homological δ-functor extending $T_{A,-}$. For every morphism $a : A \to \widetilde{A}$ in $\mathcal{A}$, prove that the natural transformations $T_{a,-}$ uniquely extend to natural transformations of homological δ-functors,

$$(L_nT_{a,-} : L_nT_{A,-} \Rightarrow L_nT_{\widetilde{A},-})_{n\geq 0}.$$

(c) Assume the following strong hypotheses.

(H1) The category $\mathcal{A}$ has enough projective objects.

(H2) For every projective object P of $\mathcal{A}$, the corresponding functor $T_{P,-}$ is exact.

Thus, for every object B of $\mathcal{B}$, there exists a left derived functor

$$LT_{-,B} = ((L_nT_{-,B} : \mathcal{A} \to \mathcal{C})_{n\geq 0}, (\delta_n^{LT_{-,B},T})_{n>0}),$$

extending $T_{-,B}$. Also, since $T_{P,-}$ is an exact functor, every $L_nT_{P,-}$ is zero for all $n > 0$. For an object A of $\mathcal{A}$, let Σ be a short exact sequence in $\mathcal{A}$,

$$T:\ 0 \longrightarrow K \xrightarrow{\ u\ } P \xrightarrow{\ v\ } A \longrightarrow 0,$$

such that P is a projective object. Denote $T_{A,-}$ by $\widehat{L}^0T_{T,-}$. Prove that the maps $\delta_1^{LT_{-,B},T}$ assemble to give a natural equivalence of functors $\mathcal{B} \to \mathcal{C}$,

$$\delta_1^{LT_{-,B},T} : L_1T_{-,B}(A) \Rightarrow \mathrm{Ker}(T_{a,-} : T_{K,-}(B) \to T_{P,-}(B)).$$

Denote this common functor by $\widehat{L}_1 T_{T,-}(B)$. Prove that this is an additive functor in B.

(d) Next, for every short exact sequence in $\mathcal{B}$,

$$\Sigma : \ 0 \longrightarrow B' \xrightarrow{q} B \xrightarrow{p} B'' \longrightarrow 0,$$

use exactness of $T_{P,-}$ to deduce that the following is a commutative diagram in $\mathcal{C}$ with exact rows,

$$\begin{tikzcd}
T_{K,-}(\Sigma): \arrow[d, "T_{v,-}(\Sigma)"'] & & T_{K,-}(B') \arrow[r, "T_{K,-}(q)"] \arrow[d, "T_{v,-}(B')"'] & T_{K,-}(B) \arrow[r, "T_{K,-}(p)"] \arrow[d, "T_{v,-}(B)"] & T_{K,-}(B'') \arrow[r] \arrow[d, "T_{v,-}(B'')"] & 0, \\
T_{P,-}(\Sigma): & 0 \arrow[r] & T_{P,-}(B') \arrow[r, "T_{P,-}(q)"] & T_{P,-}(B) \arrow[r, "T_{P,-}(p)"] & T_{P,-}(B'') \arrow[r] & 0,
\end{tikzcd}.$$

Apply the Snake Lemma to this commutative diagram to prove that $\widehat{L}_1 T_{T,-}$ is half-exact and to construct morphisms in $\mathcal{C}$

$$\delta_1^{\widehat{L}T_{T,-},\Sigma} : \widehat{L}_1 T_{T,-}(B'') \to T_{A,-}(B),$$

that are functorial in Σ and such that the following sequence in $\mathcal{C}$ is exact,

$$\widehat{L}_1 T_{T,-}(B) \xrightarrow{\widehat{L}_1 T_{T,-}(q)} \widehat{L}_1 T_{T,-}(B'') \xrightarrow{\delta_1^{\widehat{L}T_{T,-},\Sigma}} T_{A,-}(B') \xrightarrow{T_{A,-}(q)} T_{A,-}(B).$$

Moreover, from commutativity of the following square,

$$\begin{tikzcd}
L_1 T_{K,-}(B'') \arrow[r, "\delta_1^{LT_{K,-},\Sigma}"] \arrow[d, "L_1 T_{u,-}(B'')"'] & T_{K,-}(B') \arrow[d, "T_{u,-}(B')"] \\
L_1 T_{P,-}(B'') \arrow[r, "\delta_1^{LT_{P,-},\Sigma}"'] & T_{P,-}(B'),
\end{tikzcd}$$

and the fact that $L_1 T_{P,-}$ is the zero functor, conclude the existence of a unique factorization

$$\delta_2^{\widehat{L}T_{T,-},\Sigma} : L_1 T_{K,-}(B'') \to \widehat{L}T_{T,-}(B'),$$

of $\delta_1^{LT_{K,-},\Sigma}$, and conclude that the following sequence in $\mathcal{C}$ is exact,

$$L_1 T_{K,-}(B) \xrightarrow{L_1 T_{K,-}(v)} L_1 T_{K,-}(B'') \xrightarrow{\delta_2^{\widehat{L}T_{T,-},\Sigma}} \widehat{L}_1 T_{T,-}(B') \xrightarrow{\widehat{L}_1 T_{T,-}(u)} \widehat{L}_1 T_{T,-}(B).$$

For every $n \geq 2$, define

$$\widehat{L}_n T_{T,-} = L_{n-1} T_{K,-}, \quad \delta_{n+1}^{\widehat{L}T_{T,-},\Sigma} = \delta_n^{LT_{K,-},\Sigma}.$$

Prove that the induced sequence,

$$\widehat{L}T_{T,-} = ((\widehat{L}_n T_{T,-})_{n\geq 0}, (\delta_n^{\widehat{L}T_{T,-},\Sigma})_{n\geq 1}),$$

is a homological δ-functor extending $T_{A,-}$.

(e) Using (d) and the universality of $LT_{A,-}$, conclude that there exists a unique natural transformation of homological δ-functors,

$$(\theta_{T,n} : \widehat{L}_n T_{T,-} \Rightarrow L_n T_{A,-})_{n\geq 0}.$$

In particular, for $n = 1$, this gives morphisms in $\mathcal{C}$,

$$\theta_{T,n} : L_1 T_{-,B}(A) \to L_1 T_{A,-}(B).$$

(f) Finally, assume Hypotheses (H1) and (H2) also apply to $\mathcal{B}$ and the functors $T_{-,Q}$ for projective objects Q of $\mathcal{B}$. For every object B of $\mathcal{B}$ and for every projective resolution in $\mathcal{B}$,

$$\Pi : \quad 0 \longrightarrow H \longrightarrow Q \longrightarrow B \longrightarrow 0,$$

repeat the arguments above to conclude the existence of a homological δ-functor $\widehat{L}T_{-,\Pi}$ and a unique natural transformation of homological δ-functors

$$(\eta_{\Pi,n} : \widehat{L}_n T_{-,\Pi} \Rightarrow L_n T_{-,B})_{n\geq 0}.$$

In particular, for $n = 1$, this gives morphisms in $\mathcal{C}$,

$$\eta_{\Pi,1} : L_1 T_{A,-}(B) \to L_1 T_{-,B}(A).$$

Use the proof of uniqueness of natural transformations to left-derived functors to conclude that $\theta_{T,1}$ and $\eta_{\Pi,1}$ are inverse isomorphisms. In particular, use this to conclude that the morphisms $\theta_{T,1}$ and $\eta_{\Pi,1}$ are independent of the choices of projective resolutions T and Π. Finally, applying $\theta_{T,n}$ to the short exact sequence Π and use induction on n to conclude that every morphism,

$$\theta_{T,-}(B) : L_n T_{K,-}(B) \to L_{n+1} T_{A,-}(B),$$

is an isomorphism. From this conclude the existence of a binatural isomorphism,

$$L_2 T_{A,-}(B) \cong L_1 T_{K,-}(B) \cong L_1 T_{B,-}(K) \cong L_2 T_{B,-}(A).$$

Iterating this argument, prove by induction that for every integer $n \geq 1$ there is a binatural isomorphism,

$$L_n T_{A,-}(B) \cong L_n T_{-,B}(A).$$

(g) Let R be an associative, unital ring. Let $\mathcal{A}$ be the category or right R-modules. Let $\mathcal{B}$ be the category of left R-modules. Let $\mathcal{C}$ be the category of Abelian groups. Let $T(A, B)$ be the tensor product $A \otimes_R B$. Prove that $(\mathcal{A}, \mathcal{B}, T)$ satisfies the hypotheses above. Thus there are canonical isomorphisms,

$$L_n(A \otimes_R -)(B) \cong L_n(- \otimes_R B)(A).$$

These common Abelian groups are denote by $\mathrm{Tor}_n^R(A, B)$, the Tor groups.

(h) Similarly, let $\mathcal{A}$ be the category of left R-modules, let $\mathcal{B}$ be the opposite category of the category of left R-modules, and let $\mathcal{C}$ be the opposite category of the category of Abelian groups. Now let $T(A, B)$ be $\mathrm{Hom}_{R-\mathrm{mod}}(A, B)$ (it takes some time to unwind all the opposite categories!). Prove that this datum also satisfies the hypotheses above. The common Abelian groups are denoted by $\mathrm{Ext}_R^n(A, B)$, the Ext groups.

Problem 4. Let R be $\mathbb{Z}$. Let A and B be $\mathbb{Z}$-modules. Recall that there is a short exact sequence of Abelian groups,

$$T: \quad 0 \to A_{\mathrm{tor}} \to A \to A_{\mathrm{free}} \to 0,$$

where A_{tor} is the subgroup of all finite order elements of A, where the quotient group A_{free} is torsion-free.

(a) Review the theorem of finitely generated Abelian groups. In particular, note that the short exact sequence T is (non-canonically) split. For every integer $n \geq 0$, conclude the existence of short exact sequences that are (non-canonically) split,

$$\mathrm{Tor}_n^{\mathbb{Z}}(\Sigma, B): \quad 0 \to \mathrm{Tor}_n^{\mathbb{Z}}(A_{\mathrm{tor}}, B) \to \mathrm{Tor}_n^{\mathbb{Z}}(A, B) \to \mathrm{Tor}_n^{\mathbb{Z}}(A_{\mathrm{free}}, B) \to 0.$$

In particular, for $n = 0$, there is a short sequence,

$$T \otimes_{\mathbb{Z}} \mathrm{Id}_B : 0 \to A_{\mathrm{tor}} \otimes_{\mathbb{Z}} B \to A \otimes_{\mathbb{Z}} B \to A_{\mathrm{free}} \otimes_{\mathbb{Z}} B \to 0.$$

(b) Recall from the structure theorem that every finitely generated, torsion-free Abelian group is free of finite rank. Conclude that for all $n > 0$, $\mathrm{Tor}_n^{\mathbb{Z}}(A_{\mathrm{free}}, B)$ is a zero group (thus canonically isomorphic to "the" zero group). Thus, for every $n > 0$, there is a canonical isomorphism,

$$\mathrm{Tor}_n^{\mathbb{Z}}(A_{\mathrm{tor}}, B) \xrightarrow{\cong} \mathrm{Tor}_n^{\mathbb{Z}}(A, B).$$

Also, conclude that there is a canonical short exact sequence,

$$0 \to A_{\mathrm{tor}} \otimes_{\mathbb{Z}} B \to (A \otimes_{\mathbb{Z}} B)_{\mathrm{tor}} \to A_{\mathrm{free}} \otimes_{\mathbb{Z}} B_{\mathrm{tor}} \to 0,$$

as well as a canonical isomorphism,

$$A_{\mathrm{free}} \otimes_{\mathbb{Z}} B_{\mathrm{free}} \to (A \otimes_{\mathbb{Z}} B)_{\mathrm{free}}.$$

(c) Recall from the structure theorem that for every finitely generated, torsion Abelian group A_{tor} there is an increasing sequence of nonnegative integers $\underline{e} = (e_0, \ldots, e_m)$ with

$$1 < e_0, e_0|e_1, \ldots, e_k|e_{k+1}, \ldots, e_{m-1}|e_m$$

and a sequence of positive integers $(r_0, \ldots, r_m)$ such that there is an isomorphism,

$$A_{\mathrm{tor}} \cong (\mathbb{Z}/e_0\mathbb{Z})^{\oplus r_0} \oplus \cdots \oplus (\mathbb{Z}/e_m\mathbb{Z})^{\oplus r_m}.$$

The isomorphism is not canonical. However, this isomorphism does reduce the computation of the *isomorphism class* of $\mathrm{Tor}_n^{\mathbb{Z}}(A_{\mathrm{tor}}, B)$ as a $\mathbb{Z}$-module to the computation of every isomorphism class of the $\mathbb{Z}$-module,

$$\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/e\mathbb{Z}, B).$$

Use the free resolution,

$$0 \to \mathbb{Z} \xrightarrow{e} \mathbb{Z} \to \mathbb{Z}/e\mathbb{Z} \to 0,$$

to prove that there is an isomorphism of Abelian groups,

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/e\mathbb{Z}, B) \cong B_e := \{b \in B | e \cdot b = 0\},$$

and for every $n > 1$, $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/e\mathbb{Z}, B)$ is a zero group.

(d) For those who know about colimits, prove that every $\mathbb{Z}$-module A is a filtering colimit of finitely generated $\mathbb{Z}$-modules. Since tensor product commutes with filtering colimits, conclude that all of the functors Tor_n commute with filtering colimits. In particular, conclude that for every $\mathbb{Z}$-module A, finitely generated or not, for every integer $n \geq 2$, $\mathrm{Tor}_n^{\mathbb{Z}}(A, B)$ is a zero group.

Problem 5. Repeat the steps of the previous problem to compute the Ext groups $\mathrm{Ext}_{\mathbb{Z}}^n(A, B)$ when A is a finitely generated $\mathbb{Z}$-module. In particular, conclude that for every $n \geq 1$, $\mathrm{Ext}_{\mathbb{Z}}^n(A_{\mathrm{free}}, B)$ is zero, there is an isomorphism of $\mathbb{Z}$-modules,

$$\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/e\mathbb{Z}, B) \cong B/eB,$$

and for every $n \geq 2$, $\mathrm{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/e\mathbb{Z}, B)$ is a zero group. Again use commutation of the functor $\mathrm{Hom}_{\mathbb{Z}}(-, B)$ with filtering colimits to conclude that for every $\mathbb{Z}$-module A, for every $n \geq 2$, $\mathrm{Ext}_{\mathbb{Z}}^n(A, B)$ is a zero group. This property is denoted by saying that $\mathbb{Z}$ has *global homological dimension* 1. Note that this equals the Krull dimension of $\mathbb{Z}$.

Problem 6. Let k be a field and let R be $k[x]$, the polynomial ring in one variable x with coefficients in k. Use the structure theorem for finitely generated $k[x]$-modules to prove all of the following.

(a) For every $k[x]$-module A, defining A_{tor} to be the submodule of all elements that are annihilated by some nonzero element of $k[x]$, there is a short exact sequence of $k[x]$-modules,

$$T : 0 \to A_{\mathrm{tor}} \to A \to A_{\mathrm{free}} \to 0,$$

such that A_{free} is torsion-free. If A is finitely generated, then this short exact sequence is (non-canonically) split. In that case, for every integer $n \geq 0$ there are short exact sequences that are (non-canonically) split,

$$0 \to \mathrm{Tor}_n^{k[x]}(A_{\mathrm{tor}}, B) \to \mathrm{Tor}_n^{k[x]}(A, B) \to \mathrm{Tor}_n^{k[x]}(A_{\mathrm{free}}, B) \to 0,$$

$$0 \to \mathrm{Ext}_{k[x]}^n(A_{\mathrm{free}}, B) \to \mathrm{Ext}_{k[x]}^n(A, B) \to \mathrm{Ext}_{k[x]}^n(A_{\mathrm{tor}}, B) \to 0.$$

(b) Every finitely generated, torsion-free $k[x]$-module A_{free} is a finitely generated, free $k[x]$-module. In this case, conclude for every $n \geq 1$ that both $\mathrm{Tor}_n^{k[x]}(A_{\mathrm{free}}, B)$ and $\mathrm{Ext}_{k[x]}^n(A_{\mathrm{free}}, B)$ are zero groups.

(c) Every finitely generated, torsion $k[x]$-module A_{tor} is (non-canonically) isomorphic to a direct sum of $k[x]$-modules of the form $k[x]/e(x)k[x]$ for noninvertible, nonzerodivisors $e(x) \in k[x]$. If we choose each $e(x)$ to be monic, then the sequence of distinct elementary divisors $(e_0(x), \dots, e_m(x))$ with $e_0|e_1, \dots, e_{m-1}|e_m$, and the sequence of multiplicities $(r_0, \dots, r_m)$ as in the previous problem is unique. Reduce the computation of the isomorphism type of $\text{Tor}_n^{k[x]}(A_{\text{tor}}, B)$ and $\text{Ext}^n_{k[x]}(A_{\text{tor}}, B)$ to the computation of the groups $\text{Tor}_n^{k[x]}(k[x]/e(x)k[x], B)$ and $\text{Ext}^n_{k[x]}(k[x]/ek[x], B)$.

(d) Prove that there are isomorphisms,

$$\text{Tor}_1^{k[x]}(k[x]/e(x)k[x], B) \cong B_{e(x)} := \{b \in B | e(x) \cdot b = 0\},$$

$$\text{Ext}^1_{k[x]}(k[x]/e(x)k[x], B) \cong B/e(x)B.$$

Also prove that for every $n \geq 2$, both $\text{Tor}_n^{k[x]}(k[x]/e(x)k[x], B)$ and $\text{Ext}^n_{k[x]}(k[x]/e(x)k[x], B)$ are zero groups.

(e) Again use filtering colimits to conclude that for every $k[x]$-module A, for every $n \geq 2$, both $\text{Tor}_n^{k[x]}(A, B)$ and $\text{Ext}^n_{k[x]}(A, B)$ are zero groups. Thus $k[x]$ has global homological dimension 1, which also equals the Krull dimension of $k[x]$.

Problem 7. Read the definition of Noetherian ring R, and in particular the proof that every finitely generated R-module A is finitely presented, and every R-submodule A' of a finitely generated R-module A is again a finitely generated R-module. Use this to prove that every finitely generated R-module A has a projective resolution,

$$P_\bullet \to A[0],$$

such that for every $n \geq 0$, P_n is a finitely generated, free R-module. Conclude that for every finitely generated R-module B, for every integer $n \geq 0$, both $\text{Tor}_n^R(A, B)$ and $\text{Ext}^n_R(A, B)$ is a finitely generated R-module.

Problem 8. Let p be a prime integer, and let R be the ring $\mathbb{Z}/p^2\mathbb{Z}$. Let A be the R-module $A = R/pR \cong \mathbb{Z}/p\mathbb{Z}$. For every integer $n \geq 0$, define $P_n = R = \mathbb{Z}/p^2\mathbb{Z}$. For every integer $n \geq 1$, define

$$d_n : P_n \to P_{n-1}, \quad \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z}.$$

Define $\epsilon : P_0 \to A$ to be the quotient map $\mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. Prove that $P_\bullet \to A[0]$ is a projective resolution. Compute that for every $n \geq 0$, $\text{Tor}_n^R(A, A)$ and $\text{Ext}^n_R(A, A)$ are both isomorphic to A. Conclude that $R = \mathbb{Z}/p^2\mathbb{Z}$ does not have finite global homological dimension.

Problem 9. Repeat Problem 8 for the ring $R = k[x]/x^2k[x]$.

Problem 10. Let R be a commutative, unital ring. An *associative, unital, graded commutative R-algebra* (with homological indexing) is a triple

$$A_\bullet = ((A_n)_{n\in\mathbb{Z}}, (m_{p,q} : A_p \times A_q \to A_{p+q})_{p,q\in\mathbb{Z}}, (\epsilon : R \to A_0))$$

of a sequence $(A_n)_{n\in\mathbb{Z}}$ of R-modules, of a sequence $(m_{p,q})_{p,q\in\mathbb{Z}}$ of R-bilinear maps, and an R-module morphism ϵ such that the following hold.

(i) For the associated R-module $A = \bigoplus_{n\in\mathbb{Z}} A_n$ and the induced morphism $m : A \times A \to A$ whose restriction to each $A_p \times A_q$ equals $m_{p,q}$, $(A, m, \epsilon(1))$ is an associative, unital, R-algebra.

(ii) For every $p, q \in \mathbb{Z}$, for every $a_p \in A_p$ and for every $a_q \in A_q$, $m_{q,p}(a_q, a_p)$ equals $(-1)^{pq} m_{p,q}(a_p, a_q)$.

(a) Prove that the R-submodules of A,

$$A_{\geq 0} = \bigoplus_{n\geq 0} A_n, \quad A_{\leq 0} = \bigoplus_{n\leq 0} A_n,$$

are both associative, unital R-subalgebras. Moreover, prove that the R-submodule,

$$A_{>0} = \bigoplus_{n>0} A_n, \text{ resp. } A_{<0} = \bigoplus_{n<0} A_n,$$

is a left-right ideal in $A_{\geq 0}$, resp. in $A_{\leq 0}$.

(b) For associative, unital, graded commutative R-algebras $A_\bullet$ and $B_\bullet$, a graded homomorphism of R-algebras is a collection

$$f_\bullet = (f_n : A_n \to B_n)_{n\geq 0}$$

such that for the unique R-module homomorphism $f : A \to B$ whose restriction to every A_n equals f_n, f is an R-algebra homomorphism. Prove that such $f_\bullet$ is uniquely reconstructed from the homomorphism f. Prove that Id_A comes from a unique graded homomorphism $\mathrm{Id}_{A_\bullet}$. Prove that for a graded homomorphism of R-algebras, $g_\bullet : B_\bullet \to C_\bullet$, the composition $g \circ f$ arises from a unique graded homomorphism of R-algebras, $A_\bullet \to C_\bullet$. Using this to define composition of homomorphisms of graded R-algebras, prove that composition is associative and the identity morphisms abe are left-right identities for composition. Conclude that these notions form a category $R-\mathrm{GrComm}$ of associative, unital, graded commutative R-algebras. Prove that the rule $A_\bullet \mapsto A$, $f_\bullet \mapsto f$ defines a faithful functor

$$R - \mathrm{GrComm} \to R - \mathrm{Algebra}.$$

Give an example showing that this functor is not typically full.

(c) Let $A_\bullet$ be an associative, unital, graded commutative R-algebra. Prove that R is commutative (in the usual sense) if and only if A_n is a zero module for every even integer n. Denote by $R-\mathrm{Comm}$ the category of associative, unital R-algebras S that are commutative. Denote by $\mathbb{Z} - R - \mathrm{Comm}$ the faithful (but not full) subcategory whose objects are triples,

$$S_\bullet = ((S_n)_{n\in\mathbb{Z}}, (m_{p,q} : S_p \times S_q \to S_{p+q})_{p,q\in\mathbb{Z}}, (\epsilon : R \to S_0))$$

as above, but such that the multiplication is commutative rather than graded commutative, i.e., $m_{q,p}(s_q, s_p) = m_{p,q}(s_p, s_q)$. Prove that there is a functor,

$$v_{\mathrm{even}} : R - \mathrm{GrComm} \to \mathbb{Z} - R - \mathrm{Comm},$$

$((A_n)_{n\in\mathbb{Z}}, (m_{p,q}: A_p \times A_q \to A_{p+q})_{p,q\in\mathbb{Z}}, (\epsilon: R \to A_0)) \mapsto ((A_{2n})_{n\in\mathbb{Z}}, (m_{2p,2q}: A_{2p}\times A_{2q} \to A_{2(p+q)})_{p,q\in\mathbb{Z}}, (\epsilon: R \to$

and $f_\bullet : A_\bullet \to B_\bullet$ maps to $v_{\text{ev}}(f) = (f_{2n})_{n\in\mathbb{Z}}$. Also prove that there is a left adjoint to v_{even},

$$w_{\text{even}} : \mathbb{Z} - R - \text{Comm} \to R - \text{GrComm},$$

where $w_{\text{even}}(S_\bullet)_{2n}$ equals S_n, where $w_{\text{even}}(S_\bullet)_p$ is the zero module for every odd p, where

$$A_{2p} \times A_{2q} \to A_{2(p+q)}$$

is $m_{p,q}$ for $S_\bullet$, and where $R \to A_0$ is $\epsilon : R \to S_0$. For a morphism $f_\bullet : S_\bullet \to T_\bullet$ in $\mathbb{Z} - R - \text{Comm}$, $w_{\text{even}}(f_\bullet)$ is the unique morphism whose component in degree $2n$ equals f_n for every $n \in \mathbb{Z}$.

(d) Let e be an odd integer. For every associative, unital, graded commutative R-algebra $A_\bullet$ define $v_e(A_\bullet)$ to be the collection

$$((A_{ne})_{n\in\mathbb{Z}}, (m_{pe,qe} : A_{pe} \times A_{qe} \to A_{(p+q)e})_{p,q\in\mathbb{Z}}, \epsilon : R \to A_0 = A_{0e}).$$

Prove that $v_e(A_\bullet)$ is again an associative, unital, graded commutative R-algebra. For every morphism of associative, unital, graded commutative R-algebras, $f_\bullet : A_\bullet \to B_\bullet$, the collection $v_e(f_\bullet) = (f_{ne})_{n\in\mathbb{Z}}$ is a morphism of associative, unival, graded commutative R-algebras, $v_e(A_\bullet) \to v_e(B_\bullet)$. Prove that this defines a functor,

$$v_e : R - \text{GrComm} \to R - \text{GrComm}.$$

This is sometimes called the *Veronese functor* (it is closely related to the Veronese morphism of projective spaces). If e is positive, prove that the induced morphism $v_e(A_{\geq 0}) \to v_e(A_\bullet)$, resp. $v_e(A_{\leq 0}) \to v_e(A_\bullet)$, is an isomorphism to $(v_e(A_\bullet))_{\geq 0}$, resp. to $(v_e(A_\bullet))_{\leq 0}$. Similarly, if e is negative (e.g., if e equals -1), this defines an isomorphism to $(v_e(A_\bullet))_{\leq 0}$, resp. to $(v_e(A_\bullet))_{\geq 0}$. Prove that v_1 is the identity functor. For odd integers d and e, construct a natural isomorphism of functors,

$$v_{d,e} : v_d \circ v_e \Rightarrow v_{de},$$

prove that $v_{d,1}$ and $v_{1,e}$ are identity natural transformations, and prove that these natural isomorphisms are associative: $v_{de,f} \circ (v_{d,e} \circ v_f)$ equals $v_{d,ef} \circ (v_d \circ v_{e,f})$ for all odd integers d, e and f.

(e) For every associative, unital, graded commutative R-algebra $A_\bullet$, for every odd integer e, define

$$w_e : R - \text{GrComm} \to R - \text{GrComm},$$

where $w_e(A_\bullet)_{ne}$ equals A_n for every integer n, and where $w_e(A_\bullet)_m$ is a zero module if e does not m. For every morphism $f_\bullet : A_\bullet \to B_\bullet$, define $w_e(f_\bullet)$ to the be the unique morphism whose component in degree en equals f_n for every $n \in \mathbb{Z}$. Prove that w_e is a functor. For the natural isomorphism,

$$\theta_e(A_\bullet) : A_\bullet \to v_e(w_e(A_\bullet)), (A_n \xrightarrow{=} A_n)_{n\in\mathbb{Z}}$$

and the natural monomorphisms

$$\eta_e(B_\bullet) : w_e(v_e(B_\bullet)) \to B_\bullet, (B_{ne} \xrightarrow{=} B_{ne})_{n\in\mathbb{Z}},$$

prove that $(w_e, v_e, \theta_e, \eta_e)$ is an adjoint pair.

(f) For every integer $n \geq 0$, recall from Problem 5(iv) of Problem Set 1, that there is a functor,

$$\bigwedge_R^n : R-\text{mod} \to R-\text{mod},\ M \mapsto \bigwedge_R^n(M).$$

In particular, there is a natural isomorphism

$$\epsilon(M) : R \to \bigwedge_R^0(M),$$

and there is a natural isomorphism,

$$\theta(M) : M \to \bigwedge_R^1(M).$$

By convention, for every integer $n < 0$, define $\bigwedge_R^n(M)$ to be the zero module. For every pair of integers $q, r \geq 0$, prove that the natural R-bilinear map

$$\otimes : M^{\otimes q} \times M^{\otimes r} \to M^{\otimes(q+r)},\ ((m_1 \otimes \cdots \otimes m_q), (m_1' \otimes \cdots \otimes m_r')) \mapsto m_1 \otimes \ldots m_q \otimes m_1' \otimes \cdots \otimes m_r',$$

factors uniquely through an R-bilinear map,

$$\wedge : \bigwedge_R^q(M) \times \bigwedge_R^r(M) \to \bigwedge_R^{q+r}(M).$$

Prove that $\bigwedge_R^\bullet(M)$ is an associative, unital, graded commutative R-algebra. For every R-module homomorphism $\phi : M \to N$, prove that the associated R-module homomorphisms,

$$\bigwedge_R^n(\phi) : \bigwedge_R^n(M) \to \bigwedge_R^n(N),$$

define a morpism of associative, unital, graded commutative R-algebras,

$$\bigwedge_R^\bullet(\phi) : \bigwedge_R^\bullet(M) \to \bigwedge_R^\bullet(N).$$

Prove that for every R-module homomorphism $\psi : N \to P$, $\bigwedge_R^\bullet(\psi \circ \phi)$ equals $\bigwedge_R^\bullet(\psi) \circ \bigwedge_R^\bullet(\phi)$. Also prove that $\bigwedge_R^\bullet(\text{Id}_M)$ is the identity morphism of $\bigwedge_R^\bullet(M)$.

(g) An associative, unital, graded commutative R-algebra $A_\bullet$ is (strictly) 0-*connected*, resp. *weakly* 0-*connected*, if the inclusion $A_{\geq 0} \to A$ is an isomorphism and the R-module homorphism ϵ is an isomorphism, resp. an epimorphism. If R is a field, prove that every weakly 0-connected algebra is strictly 0-connected. Denote by

$$R - \mathrm{GrComm}_{\geq 0}, \text{ resp. } R - \mathrm{GrComm}'_{\geq 0}$$

the full subcategory of $R - \mathrm{GrComm}$ whose objects are the 0-connected algebras, resp. the weakly 0-connected algebras. Prove that v_{even} restricts to a functor,

$$R - \mathrm{GrComm}_{\geq 0} \to \mathbb{Z}_+ - R - \mathrm{Comm},$$

where $\mathbb{Z}_+ - R - \mathrm{Comm}$ is the full subcategory of $\mathbb{Z} - R - \mathrm{Comm}$ of algebras graded in nonnegative degrees such that $R \to S_0$ is an isomorphism. For e an odd positive integer, prove that v_e and w_e restrict to an adjoint pair of functors,

$$v_e : R - \mathrm{GrComm}_{\geq 0} \to R - \mathrm{GrComm}_{\geq 0},$$

$$w_e : R - \mathrm{GrComm}_{\geq 0} \to R - \mathrm{GrComm}_{\geq 0}.$$

For every odd positive integer e, define a functor

$$\Phi_e : R - \mathrm{GrComm}_{\geq 0} \to R - \mathrm{mod},$$

that sends $A_\bullet$ to A_e and sends $f_\bullet$ to f_e. Of course, for every odd positive integer d, $\Phi_e \circ v_d$ is naturally isomorphic to Φ_{de} and $\Phi_{de} \circ w_d$ is Φ_e. By the previous part, there is a functor

$$\bigwedge_R^\bullet : R - \mathrm{mod} \to R - \mathrm{GrComm}_{\geq 0}$$

that sends every module M to the 0-connected, associative, unital, graded commutative R-algebra $(\bigwedge_R^n(M))_{n\geq 0}$. Moreover, there is a natural transformation,

$$\theta : \mathrm{Id}_{R-\mathrm{mod}} \Rightarrow \Phi_1 \circ \bigwedge_R^\bullet.$$

Prove that this extends uniquely to an adjoint pair of functors

$$(\bigwedge_R^\bullet, \Phi_1, \theta, \eta).$$

Using the natural isomorphisms $\Phi_e \circ v_d = \Phi_{de}$ and $\Phi_{de} \circ w_d = \Phi_e$, prove that there is also an adjoint pair of functors

$$(w_e \circ \bigwedge_R^\bullet, \Phi_e, \theta, \eta_e).$$

Problem 11. Let R be a commutative, unital ring. A (homological, unital, associative, graded commutative) *differential graded R-algebra* is a pair

$$((C_n)_{n\in\mathbb{Z}}, (\wedge : C_p \times C_q \to C_{p+q})_{p,q\in\mathbb{Z}}, (\epsilon : R \to C_0), (d_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}),$$

of an associative, unital, graded commutative R-algebra $C_\bullet$ together with R-linear morphisms $(d_n)_{n\in\mathbb{Z}}$ such that $d_{n-1} \circ d_n$ equals 0 for every $n \in \mathbb{Z}$, and that satisfies the graded Leibniz identity,

$$d_{p+q}(c_p \wedge c_q) = d_p(c_p) \wedge c_q + (-1)^p c_p \wedge d_q(c_q),$$

for every $p, q \in \mathbb{Z}$, for every $c_p \in C_p$, and for every $c_q \in C_q$. A *morphism* of differential graded R-algebras,

$$\phi_\bullet : C_\bullet \to A_\bullet,$$

is a morphism $\phi_\bullet = (\phi_n)_{n\in\mathbb{Z}}$ that is simultaneously a morphism of chain complexes of R-modules and a morphism of associative, unital, graded commutative R-algebras.

(a) For morphisms of differential graded R-algebras, $\phi_\bullet : C_\bullet \to A_\bullet$, $\psi_\bullet : D_\bullet \to C_\bullet$, prove that the composition of $\psi_\bullet \circ \phi_\bullet$ of graded R-modules is both a morphism of chain complexes of R-modules and a morphism of associative, unital, graded commutative R-algebras. Thus, it is a composition of morphisms of differential graded R-algebras. With this composition, prove that this defines a category $R - \mathrm{CDGA}$ of differential graded R-algebras.

(b) For every associative, unital, graded commutative R-algebra $A_\bullet$, for every integer n, define $d_{E(A),n} : A_n \to A_{n-1}$ to be the zero morphism. Prove that this gives a differential graded R-algebra, denoted $E(A_\bullet)$. For every morphism $f_\bullet : A_\bullet \to B_\bullet$ of associative, unital, graded commutative R-algebras, prove that $f_\bullet : E(A_\bullet) \to E(B_\bullet)$ is a morphism of differential graded R-algebras, denoted $E(f_\bullet)$. Prove that this defines a functor

$$E : R - \mathrm{GrComm} \to R - \mathrm{CDGA}.$$

For every differential graded R-algebra $C_\bullet$, prove that the subcomplex $Z_\bullet(C_\bullet)$ is a differential graded R-subalgebra with zero differential, and the inclusion,

$$\eta(C_\bullet) : E(Z_\bullet(C_\bullet)) \to C_\bullet,$$

is a morphism of differential graded R-algebras. Also, for every morphism $\phi_\bullet : C_\bullet \to D_\bullet$ of differential graded R-algebras, prove that the induced morphism $Z_\bullet(f_\bullet) : Z_\bullet(C_\bullet) \to Z_\bullet(D_\bullet)$ is a morphism of associative, unital, graded commutative R-algebras. Prove that this defines a functor

$$Z_\bullet : R - \mathrm{CDGA} \to R - \mathrm{GrComm}.$$

For every associative, unital, graded commutative R-algebra $A_\bullet$, the inclusion $Z_\bullet(E(A_\bullet)) \to E(A_\bullet)$ is just the identity map, whose inverse,

$$\theta(A_\bullet) : A_\bullet \to Z_\bullet(E(A_\bullet)),$$

is an isomorphism. Prove that $(E, Z_\bullet, \theta, \eta)$ is an adjoint pair of functors. Finally, prove that the subcomplex $B_\bullet(C_\bullet) \subset Z_\bullet(C_\bullet)$ is a left-right ideal in the associative, unital, graded commutative R-algebra $Z_\bullet(C_\bullet)$. Conclude that there is a unique structure of associative, unital, graded commutative R-algebra on the cokernel $H_\bullet(C_\bullet)$ such that the quotient morphism $Z_\bullet(C_\bullet) \to H_\bullet(C_\bullet)$ is a morphism of differential graded R-algebras. Prove that altogether this defines a functor,

$$H : R - \mathrm{CDGA} \to R - \mathrm{GrComm}.$$

(c) A differential graded R-algebra $C_\bullet$ is (strictly) 0-*connected*, resp. *weakly* 0-*connected*, if the underlying associative, unital, graded commutative R-algebra is 0-connected, resp. weakly 0-connected. Denote by $R - \mathrm{CDGA}_{\geq 0}$, resp. $R - \mathrm{CDGA}'_{\geq 0}$, the full subcategory of $R - \mathrm{CDGA}$ whose objects are the 0-connected differential graded R-algebras, resp. those that are weakly 0-connected. Prove that the functors above restrict to functors,

$$E : R - \mathrm{GrComm}_{\geq 0} \to R - \mathrm{CDGA}_{\geq 0},$$

$$Z_\bullet : R - \mathrm{CDGA}_{\geq 0} \to R - \mathrm{GrComm}_{\geq 0},$$

such that (E, Z, θ, η) is still an adjoint pair. Similarly, show that H restricts to a functor

$$H : R - \mathrm{CDGA}_{\geq 0} \to R - \mathrm{GrComm}'_{\geq 0}.$$

(d) Denote by $R - \mathrm{CDGA}_{[0,1]}$ the full subcategory of $R - \mathrm{CDGA}_{\geq 0}$ whose objects are 0-connected differential graded R-algebras $C_\bullet$ such that C_n is a zero object for $n > 1$. Prove that every such object is uniquely determined by the data of an R-module C_1 and an R-module homomorphism $d_{C,1} : C_1 \to C_0 = R$, and conversely such data uniquely determine an object of $R - \mathrm{CDGA}_{[0,1]}$. Prove that for such algebras $C_\bullet$ and $D_\bullet$, every morphism $\phi_\bullet : C_\bullet \to D_\bullet$ of differential graded R-algebras is uniquely determined by an R-module homomorphism $\phi_1 : C_1 \to D_1$ such that $d_{D,1} \circ \phi_1$ equals $d_{C,1}$, and conversely, such an R-module homomorphism uniquely determines a morphism of differential graded R-algebras. Conclude that there is a functor

$$\sigma_{[0,1]} : R - \mathrm{CDGA}_{\geq 0} \to R - \mathrm{CDGA}_{[0,1]},$$

that associates to every 0-connected differential graded R-algebra $C_\bullet$ the algebra $\sigma_{[0,1]}(C_\bullet)$ uniquely determined by the R-module homomorphism $d_{C,1} : C_1 \to C_0 = R$ and that associates to every morphism $\phi_\bullet : C_\bullet \to D_\bullet$ of 0-connected differential graded R-algebras the morphism,

$$\sigma_{[0,1]}(\phi_\bullet) : \sigma_{[0,1]}(C_\bullet) \to \sigma_{[0,1]}(D_\bullet),$$

uniquely determined by the morphism $\phi_1 : C_1 \to D_1$.

(e) For every R-module M and for every R-module homomorphism $\phi : M \to R$, prove that there exists a unique sequence of R-module homomorphisms,

$$\left(d_{M,\phi,n} : \bigwedge_R^n(M) \to \bigwedge_R^{n-1}(M)\right)_{n>0},$$

such that d_1 equals ϕ and such that $(\bigwedge_R^\bullet(M), d_{M,\phi})$ is a 0-connected differential graded R-algebra. It may be convenient to begin with the case of a free R-module P and a morphism $\psi : P \to R$, in which case every $\bigwedge_R^n(P)$ is also free and the R-module homomorphisms d_n is uniquely determined by its restriction to a convenient basis. Given a presentation $M = P/K$ such that ψ factors uniquely through $\phi : M \to R$, prove that the associative, unital, graded commutative R-algebra $\bigwedge_R^\bullet(M)$ is the quotient of $\bigwedge_R^\bullet(P)$ by the left-right ideal generated by $K \subset P = \bigwedge_R^1(P)$. Also prove that $d_{P,\psi}$ maps this ideal to itself, i.e., the ideal is differentially-closed. Conclude that there is a unique structure of differential graded algebra on the quotient $\bigwedge_R^\bullet(M)$ such that the quotient map is a morphism of differential graded R-algebras.

(f) Prove that the construction of the previous part defines a functor,

$$\bigwedge_R^\bullet : R - \mathrm{CDGA}_{[0,1]} \to R - \mathrm{CDGA}_{\geq 0}.$$

Prove that for every object $(\phi : M \to R)$ of $R - \mathrm{CDGA}_{[0,1]}$, the morphism

$$\theta(M,\phi) : M \xrightarrow{=} \bigwedge_R^1(M)$$

is a natural isomorphism

$$\theta : \mathrm{Id}_{R-\mathrm{CDGA}_{[0,1]}} \Rightarrow \sigma_{[0,1]} \circ \bigwedge_R^\bullet.$$

Similarly, for every object 0-connected differential graded R-algebra $C_\bullet$, prove that the natural transformation from Problem 10(g),

$$\eta(C_\bullet) : \bigwedge_R^\bullet(C_1) \to C_\bullet,$$

is compatible with the differential on $\bigwedge_R^\bullet(C_1)$ induced by $d_{C,1} : C_1 \to C_0 = R$, i.e., $\eta(C_\bullet)$ is a natural transformation,

$$\eta : \bigwedge_R^\bullet \circ \sigma_{[0,1]} \to \mathrm{Id}_{R-\mathrm{CDGA}_{\geq 0}}.$$

Conclude that $(\bigwedge_R^\bullet, \sigma_{[0,1]}, \theta, \eta)$ is an adjoint pair of functors. For every $\phi : M \to R$ in $R-\mathrm{CDGA}_{[0,1]}$, the associated 0-complete differential graded R-algebra structure on $\bigwedge_R^\bullet(M)$ is called the *Koszul algebra* of $\phi : M \to R$ and denoted $K_\bullet(M,\phi)$.

(g) For every R-module M, and for every R-submodule M' of M, denote by $F^1 \subset \bigwedge_R^\bullet(M)$ the left-right ideal generated by $M' \subset M = \bigwedge_R^1(M)$. For every integer $n \leq 0$, denote by $F^n \subset \bigwedge_R^\bullet(M)$ the entire algebra. For every integer $n \geq 1$, denote by F^n the left-right ideal of $\bigwedge_R^\bullet(M)$ generated by the n-fold self-product $F^1 \cdot \dots \cdot F^1$. For every pair of nonnegative integers p, q, prove that the ideal $F^p \cdot F^q$ equals F^{p+q}. In particular, prove that there is a natural epimorphism,

$$\bigwedge_R^p (F_1^1) \otimes_R \bigwedge_R^q (M) \to F_{p+q}^p.$$

Denote the quotient M/M' by M'', and denote by Σ the short exact sequence,

$$\Sigma: \ 0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0.$$

For every nonnegative integer q, prove that the R-module morphism,

$$\bigwedge_R^q (v) : \bigwedge_R^q (M) \to \bigwedge_R^q (M''),$$

is an epimorphism whose kernel equals F_q^1. Conclude that the composite epimorphism

$$\bigwedge_R^p (M') \otimes_R \bigwedge_R^q (M) \to F_{p+q}^p \to F_{p+q}^p / F_{p+q}^{p+1}$$

factors uniquely through an R-module epimorphism

$$c_{\Sigma,p,q} : \bigwedge_R^p (M') \otimes_R \bigwedge_R^q (M'') \to F_{p+q}^p / F_{p+q}^{p+1}.$$

In case there exists a splitting of Σ, prove that every epimorphism $c_{\Sigma,p,q}$ is an isomorphism. On the other hand, find an example where Σ is not split and some morphism $c_{\Sigma,p,q}$ is not a monomorphism (there exist such examples for $R = \mathbb{C}[x, y]$).

(h) Continuing the previous problem, assume that M'' is isomorphic to R as an R-module (or, more generally, projective of constant rank 1), so that Σ is split. For every nonnegative integer p, conclude that there exists a short exact sequence,

$$\Sigma_{p,1} : \ 0 \longrightarrow \textstyle\bigwedge_R^{p+1}(M') \xrightarrow{\bigwedge_R^{p+1}(u)} \bigwedge_R^{p+1}(M) \xrightarrow{c_{\Sigma,p,1}^{-1}} \bigwedge_R^{p}(M') \otimes_R M'' \longrightarrow 0,$$

that is split. Check that this is compatible with the product structure and, thus, defines a short exact sequence of graded (left) $\bigwedge_R^\bullet(M)$-modules,

$$\textstyle\bigwedge_R^\bullet(\Sigma) : \ 0 \longrightarrow \bigwedge_R^\bullet(M') \xrightarrow{\bigwedge_R^\bullet(u)} \bigwedge_R^\bullet(M) \xrightarrow{c_\Sigma^{-1}} \bigwedge_R^\bullet(M') \otimes_R M''[+1] \longrightarrow 0.$$

(i) Now, let $\phi : M \to R$ be an R-module homomorphism. Denote by $\phi' : M' \to R$ the restriction $\phi \circ u$. These morphisms define structures of differential graded R-algebra, $K_\bullet(M, \phi)$ on $\bigwedge_R^\bullet(M)$, and $K_\bullet(M',\phi')$ on $\bigwedge_R^\bullet(M')$. Moreover, the morphism $\bigwedge_R^\bullet(u)$ above is a morphism of differential graded R-modules,

$$K(u) : K_\bullet(M',\phi') \to K_\bullet(M,\phi).$$

Prove that the induced morphism

$$c_\Sigma^{-1} : K_\bullet(M,\phi) \to K_\bullet(M',\phi') \otimes_R M''[+1]$$

is a morphism of cochain complexes. Moreover, for a choice of splitting $s : M'' \to M$, for the induced morphism $\phi'' : M'' \to R$, $\phi'' = \phi \circ s$, for the induced morphism of cochain complexes,

$$\mathrm{Id}_{K_\bullet(M',\phi')} \otimes \phi'' : K_\bullet(M',\phi') \otimes_R M'' \to K_\bullet(M',\phi'),$$

prove that there is a unique commutative diagram of short exact sequences,

$$\begin{array}{ccccccccc}
T_{\mathrm{Id}\otimes\phi''}: & 0 & \longrightarrow & K_\bullet(M',\phi') & \xrightarrow{q_{\mathrm{Id}\otimes\phi''}} & \mathrm{Cone}(\mathrm{Id}\otimes\phi'') & \xrightarrow{p_{\mathrm{Id}\otimes\phi''}} & K_\bullet(M',\phi')\otimes_R M''[+1] & \longrightarrow & 0 \\
\widetilde{s}\downarrow & & & \downarrow \mathrm{Id} & & \downarrow \widetilde{s} & & \downarrow \mathrm{Id} & & \\
K(\Sigma) & 0 & \longrightarrow & K_\bullet(M',\phi') & \xrightarrow{K_\bullet(u)} & K_\bullet(M,\phi) & \xrightarrow{c_\Sigma^{-1}} & K_\bullet(M',\phi')\otimes_R M'' & \longrightarrow & 0.
\end{array}$$

(j) With the same hypotheses as above, conclude that there is an exact sequence of homology (remember the shift $[+1]$ above is cohomological),

$$H_0(K_\bullet(M',\phi')) \otimes_R M'' \xrightarrow{\mathrm{Id}\otimes\phi''} H_0(K_\bullet(M',\phi')) \xrightarrow{K_0(u)} H_0(K_\bullet(M,\phi)) \to 0,$$

i.e., $H_0(K_\bullet(M,\phi)) \cong H_0(K_\bullet(M,\phi))/\phi(M'') \cdot H_0(K_\bullet(M,\phi))$ as a quotient algebra of R. Also, for every $n > 0$, conclude the existence of a short exact sequence of Koszul homologies,

$$0 \to K_n(M',\phi') \otimes_R R/\mathrm{Im}(\phi'') \xrightarrow{\psi''} K_n(M,\phi) \to K_{n-1}(M',\phi';M'')_{\mathrm{Im}(\phi'')} \to 0,$$

where for every R-module N, $N_{\mathrm{Im}(\phi'')}$ denotes the submodule of elements that are annihilated by the ideal $\mathrm{Im}(\phi'') \subset R$. As graded modules over the associative, unital, graded commutative R-algebra $K_*(M',\phi') = H_*(K_\bullet(M',\phi'))$, this is a short exact sequence,

$$0 \to K_*(M',\phi') \otimes_R R/\mathrm{Im}(\phi'') \xrightarrow{\psi''} K_*(M,\phi) \to K_{*-1}(M',\phi';M'')_{\mathrm{Im}(\phi'')} \to 0,$$

As a special case, if $K_\bullet(M',\phi')$ is acyclic, and if the morphism

$$H_0(K_\bullet(M',\phi')) \otimes_R M'' \xrightarrow{\mathrm{Id}\otimes\phi''} H_0(K_\bullet(M',\phi'))$$

is injective, conclude that also $K_\bullet(M,\phi)$ is acyclic.

(k) Repeat this exercise for the cohomological Koszul complexes $K^\bullet(M,\phi)$.

Problem 12. Let R be the ring $k[x_1,\dots,x_m]$. For the sequence of elements $\underline{x} = (x_1,\dots,x_m)$, check that this is a regular sequence for the R-module R, itself. For the quotient R-module $A = R/\langle x_1, dots, x_m\rangle$, use the Koszul resolution in the previous problem to prove that for every R-module B, for every integer $n > m$, both $\mathrm{Tor}_n^R(A,B)$ and $\mathrm{Ext}_R^n(A,B)$ are zero. Moreover, check that there is an isomorphism $\mathrm{Ext}_R^n(A,B) \cong \mathrm{Tor}_{m-n}^R(A,B)$ for every $n = 0,\dots,m$. When B is A, compute the associative, unital graded R-algebras (homological, resp. cohomological) given by $\mathrm{Tor}_\bullet^R(A,A)$, resp. $\mathrm{Ext}_R^\bullet(A,A)$.