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MAT 536 Problem Set 4

Homework Policy. Please read through all the problems. Please solve 5 of the problems. I will be happy to discuss the solutions during office hours.

Problems.

Problem 0.(The Cochain Functor of an Additive Functor) Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. Let

$$F : \mathcal{A} \to \mathcal{B}$$

be an additive functor. There is an induced additive functor,

$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$$

that associates to a cochain complex

$$A^\bullet = ((A^n)_{n\in\mathbb{Z}}, (d_A^n : A^n \to A^{n+1})_{n\in\mathbb{Z}}),$$

in $\mathcal{A}$ the cochain complex

$$F(A^\bullet) = ((F(A^n))_{n\in\mathbb{Z}}, (F(d_A^n) : F(A^n) \to F(A^{n+1}))_{n\in\mathbb{Z}}).$$

(a) Prove that F is half-exact, resp. left exact, right exact, exact, if and only if $\mathrm{Ch}(F)$ is half-exact, resp. left exact, right exact, exact.

(b) Prove that the functor $\mathrm{Ch}(F)$ induces natural transformations,

$$\theta^n_{B,F} : B^n \circ \mathrm{Ch}(F) \Rightarrow F \circ B^n,\ \theta^n_{F,Z} : F \circ Z^n \Rightarrow Z^n \circ \mathrm{Ch}(F).$$

Thus, for the functor $\overline{A}^n = A^n/B^n(A^\bullet)$, there is also an induced natural transformation,

$$\theta_{\overline{\cdot},F} : \overline{\cdot}^n \circ \mathrm{Ch}(F) \Rightarrow F \circ \overline{\cdot}^n.$$

(c) Assume now that F is right exact (half-exact and preserves epimorphisms). Denote by

$$p^n : Z^n \Rightarrow H^n,$$

the usual natural transformation of functors $\mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$. Conclude the existence of a unique natural transformation

$$\theta^n_{F,H} : F \circ H^n \Rightarrow H^n \circ \mathrm{Ch}(F),$$

such that for every $A^\bullet$ in $\mathrm{Ch}(\mathcal{A})$, the following diagram commutes,

$$\begin{tikzcd}
F(Z^n(A^\bullet)) \arrow[r, "F(p^n)"] \arrow[d, "\theta^n_{F,Z}(A^\bullet)"'] & F(H^n(A^\bullet)) \arrow[d, "\theta^n_{F,H}(A^\bullet)"] \\
Z^n(\mathrm{Ch}(F)(A^\bullet)) \arrow[r, "p^n"'] & H^n(\mathrm{Ch}(F)(A^\bullet))
\end{tikzcd}.$$

Finally, for every short exact sequence in $\mathrm{Ch}(\mathcal{A})$,

$$\Sigma : \ 0 \longrightarrow K^\bullet \xrightarrow{u^\bullet} A^\bullet \xrightarrow{v^\bullet} 0,$$

such that also $F(\Sigma)$ is a short exact sequence in $\mathrm{Ch}(\mathcal{B})$ (this holds, for instance, if Σ is term-by-term split), prove that the following diagram commutes,

$$\begin{tikzcd}
F(H^n(Q^\bullet)) \arrow[r, "F(\delta^n_\Sigma)"] \arrow[d, "\theta^n_{F,H}(Q^\bullet)"'] & F(H^{n+1}(K^\bullet)) \arrow[d, "\theta^{n+1}_{F,H}(K^\bullet)"] \\
H^n(F(Q^\bullet)) \arrow[r, "\delta^n_{F(\Sigma)}"'] & > H^{n+1}(F(K^\bullet))
\end{tikzcd}.$$

(d) Assume not that F is left exact (half-exact and preserves monomorphisms). Denote by

$$q^n : H^n(A^\bullet) \Rightarrow \overline{A}^n = A^n/B^n(A^\bullet),$$

the usual natural transformation of functors $\mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$. Conclude the existence of a unique natural transformation

$$\theta^n_{H,F} : H^n \circ \mathrm{Ch}(F) \Rightarrow F \circ H^n,$$

such that for every $A^\bullet$ in $\mathrm{Ch}(\mathcal{A})$, the following diagram commutes,

$$\begin{tikzcd}
H^n(\mathrm{Ch}(F)(A^\bullet)) \arrow[r, "q^n"] \arrow[d, "\theta^n_{H,F}(A^\bullet)"'] & \overline{\mathrm{Ch}(F)(A^\bullet)}^n \arrow[d, "\theta^n_{\cdot,F}(A^\bullet)"] \\
\overline{\mathrm{Ch}(F)(A^\bullet)}^n \arrow[r, "F(q^n)"'] & F(\overline{A}^n)
\end{tikzcd}.$$

Finally, for every short exact sequence in $\mathrm{Ch}(\mathcal{A})$,

$$\Sigma : \ 0 \longrightarrow K^\bullet \xrightarrow{u^\bullet} A^\bullet \xrightarrow{v^\bullet} 0,$$

such that also $F(\Sigma)$ is a short exact sequence in $\mathrm{Ch}(\mathcal{B})$ (this holds, for instance, if Σ is term-by-term split), prove that the following diagram commutes,

$$\begin{array}{ccc} H^n(F(Q^\bullet)) & \xrightarrow{\delta^n_{F(\Sigma)}} & H^{n+1}(F(K^\bullet)) \\ {\scriptstyle \theta^n_{H,F}(Q^\bullet)}\downarrow & & \downarrow{\scriptstyle \theta^{n+1}_{H,F}(K^\bullet)} \\ F(H^n(Q^\bullet)) & \xrightarrow[F(\delta^n_\Sigma)]{} & F(H^{n+1}(K^\bullet)) \end{array}$$

Problem 1.(Preservation of Direct Sums) Let $\mathcal{A}$ be an additive category. Let A_1 and A_2 be objects of $\mathcal{A}$. Let $(q_1 : A_1 \to A, q_2 : A_2 \to A)$ be a coproduct (direct sum) in $\mathcal{A}$. Define $p_1 : A \to A_1$ to be the unique morphism in $\mathcal{A}$ such that $p_1 \circ q_1$ equals Id_{A_1} and $p_1 \circ q_2$ is zero. Similarly define $p_2 : A \to A_2$ to be the unique morphism in $\mathcal{A}$ such that $p_2 \circ q_1$ is zero and $p_2 \circ q_2$ equals Id_{A_2}. Prove that $q_1 \circ p_1 + q_2 \circ p_2$ equals Id_A both compose with q_i to equal q_i, and thus both are equal. Conclude that $(p_1 : A \to A_1, p_2 : A \to A_2)$ is a product in $\mathcal{A}$.

Now let $\mathcal{B}$ be a second additive category, and let

$$F : \mathcal{A} \to \mathcal{B}$$

be an additive functor. Define $B_i = F(A_i)$ and $B = F(A)$. Prove that $F(p_i) \circ F(q_j)$ equals Id_{B_i} if $j = i$ and equals 0 otherwise. Also prove that Id_B equals $F(q_1)\circ F(p_1)+F(q_2)\circ F(p_2)$. Conclude that both $(F(q_1) : B_1 \to B, F(q_2) : B_2 \to B)$ is a coproduct in $\mathcal{B}$ and $(F(p_1) : B \to B_1, F(p_2) : B \to B_2)$ is a product in $\mathcal{B}$. Hence, additive functors preserve direct sums. In particular, additive functors send split exact sequences to split exact sequences.

Problem 2.(Homotopies) Let $\mathcal{A}$ be an Abelian category. Let $A^\bullet$ and $C^\bullet$ be cochain complexes in $\mathrm{Ch}(\mathcal{A})$. Let $f^\bullet : A^\bullet \to C^\bullet$ be a cochain morphism. A *homotopy* from $f^\bullet$ to 0 is a sequence $(s^n : A^n \to C^{n-1})_{n\in\mathbb{Z}}$ such that for every $n \in \mathbb{Z}$,

$$f^n = d_C^{n-1} \circ s^n + s^{n+1} \circ d_A^n.$$

In this case, $f^\bullet$ is called *homotopic* to 0 or *null homotopic*. Cochain morphisms $g^\bullet, h^\bullet : A^\bullet \to C^\bullet$ are *homotopic* if $f^\bullet = g^\bullet - h^\bullet$ is homotopic to 0.

(a) Prove that the null homotopic cochain morphisms form an Abelian subgroup of $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A^\bullet, C^\bullet)$. Moreover, prove that the precomposition or postcomposition of a null homotopic cochain morphism with an arbitrary cochain morphism is again null homotopic (the null homotopic cochain morphisms form a "left-right ideal" with respect to composition).

(b) If $f^\bullet$ is homotopic to 0, prove that for every $n \in \mathbb{Z}$, the induced morphism,

$$H^n(f^\bullet) : H^n(A^\bullet) \to H^n(C^\bullet),$$

is the zero morphism. In particular, if $\mathrm{Id}_{A^\bullet}$ is homotopic to 0, conclude that every $H^n(A^\bullet)$ is a zero object.

(c) For a short exact sequence in $\mathcal{A}$

$$\Sigma: \ 0 \longrightarrow K \xrightarrow{q} A \xrightarrow{p} Q \longrightarrow 0,$$

considered as a cochain complex $A^\bullet$ in $\mathcal{A}$ concentrated in degrees -1, 0, 1, prove that a homotopy from $\mathrm{Id}_{A^\bullet}$ to 0 is the same thing as a splitting of the short exact sequence.

(d) Let $\mathcal{B}$ be an Abelian category. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. This induces an additive functor

$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B}).$$

If F is half-exact, resp. left exact, right exact, exact, prove that also $\mathrm{Ch}(F)$ is half-exact, resp. left exact, right exact, exact. Prove that $\mathrm{Ch}(F)$ preserves homotopies. In particular, if $g^\bullet$ and $h^\bullet$ are homotopic in $\mathrm{Ch}(\mathcal{A})$, then for every integer $n \in \mathbb{Z}$, $H^n(\mathrm{Ch}(F)(g^\bullet))$ equals $H^n(\mathrm{Ch}(F)(h^\bullet))$.

Problem 3.(Translation) Let $\mathcal{A}$ be an Abelian category. For every integer m, for every cochain complex $A^\bullet$ in $\mathrm{Ch}(\mathcal{A})$, define $T^m(A^\bullet) = A^\bullet[m]$ to be the cochain complex such that $T^m(A^\bullet)^n = A^{m+n}$, and with differential

$$d^n_{T^m(A^\bullet)} : T^m(A^\bullet)^n \to T^m(A^\bullet)^{n+1}$$

equal to $(-1)^m d^{m+n}_{A^\bullet}$. For every cochain morphism

$$f^\bullet : A^\bullet \to C^\bullet,$$

define

$$T^m(f^\bullet)^n : T^m(A^\bullet)^n \to T^m(C^\bullet)^n$$

to be f^{m+n}. Finally, for every homotopy $s^\bullet$ from $g^\bullet - h^\bullet$ to 0, define

$$T^m(s^\bullet)^n = (-1)^m s^{m+n}.$$

(a) Prove that $T^m : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ is an additive functor that is exact. Prove that T^0 is the identity functor. Also prove that $T^m \circ T^\ell$ equals $T^{m+\ell}$. Prove that not only are T^m and T^{-m} inverse functors, but also (T^m, T^{-m}) is an adjoint pair of functors (which implies that also (T^{-m}, T^m) is an adjoint pair). Finally, if $s^\bullet$ is a homotopy from $g^\bullet - h^\bullet$ to 0, prove that $T^m(s^\bullet)$ is a homotopy from $T^m(g^\bullet) - T^m(h^\bullet)$ to 0.

(b) Via the identification $T^m(A^\bullet)^n = A^{m+n}$, prove that the subfunctor $Z^n(T^m(A^\bullet))$ is naturally identified with $Z^{m+n}(A^\bullet)$. Similarly, prove that the subfunctor $B^n(T^m(A^\bullet))$ is naturally identified with $B^{m+n}(A^\bullet)$. Thus, show that the epimorphism $(T^m(A^\bullet))^n \to \overline{T^m(A^\bullet)}^n$ is identified with the epimorphism $A^{m+n} \to \overline{A}^{m+n}$. Finally, use these natural equivalences to deduce a natural equivalence of half-exact, additive functors $\mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$,

$$\iota^{m,n} : H^{m+n} \Rightarrow H^n \circ T^m.$$

(c) For a short exact sequence in $\mathrm{Ch}(\mathcal{A})$,

$$\Sigma: \ K^\bullet \xrightarrow{q^\bullet} A^\bullet \xrightarrow{p^\bullet} Q^\bullet \longrightarrow 0,$$

for the associated short exact sequence,

$$\Sigma[+1] = T(\Sigma):\ T(K^\bullet) \xrightarrow{T(q^\bullet)} T(A^\bullet) \xrightarrow{T(p^\bullet)} T(Q^\bullet) \longrightarrow 0,$$

prove that the following diagram commutes,

$$\begin{tikzcd}
H^{n+1}(Q^\bullet) \arrow[r, "-\delta^{n+1}_{\Sigma}"] \arrow[d, "\iota^n(Q^\bullet)"'] & H^{n+1}(K^\bullet) \arrow[d, "\iota^{n+1}(K^\bullet)"] \\
H^n(T(Q^\bullet)) \arrow[r, "\delta^n_{T(\Sigma)}"'] & H^{n+1}(T(K^\bullet))
\end{tikzcd}$$

Iterate this to prove that for every $m \in \mathbb{Z}$, $\delta^n_{\Sigma[m]}$ is identified with $(-1)^m \delta^{n+m}_{\Sigma}$.

(d) For every integer m, define

$$e_{\geq m} : \mathrm{Ch}^{\geq m}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$$

to be the full additive subcategory whose objects are complexes $A^\bullet$ such that for every $n < m$, A^n is a zero object. (From here on, writing $A = 0$ for an object A means "A is a zero object".) Check that $\mathrm{Ch}^{\geq m}(\mathcal{A})$ is an Abelian category, and $e_{\geq m}$ is an exact functor. For every integer m, define the "brutal truncation"

$$\sigma_{\geq m} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}^{\geq m}(\mathcal{A}),$$

to be the additive functor such that for every object $A^\bullet$

$$(\sigma_{\geq m} A^\bullet)^n = \begin{cases} A^n, & n \geq m \\ 0, & n < m \end{cases}$$

and for every morphism $u^\bullet : A^\bullet \to C^\bullet$,

$$(\sigma_{\geq m} f^\bullet)^n = \begin{cases} f^n, & n \geq m, \\ 0, & n < m \end{cases}$$

Check that $\sigma_{\geq m}$ is exact and is right adjoint to $e_{\geq m}$. For the natural transformation,

$$\eta_{\geq m} : e_{\geq m} \circ \sigma_{\geq m} \Rightarrow \mathrm{Id}_{\mathrm{Ch}(\mathcal{A})},$$

check that the induced natural transformation,

$$\overline{\eta_{\geq m}(A^\bullet)}^n : \overline{(\sigma_{\geq m}(A))^n}\overline{A^n},$$

is zero for $n < m$, is the identity for $n > m$, and for $n = m$ it is the epimorphism,

$$A^m \twoheadrightarrow \overline{A^m}.$$

Check that the induced natural transformation

$$Z^n(\eta_{\geq m}(A^\bullet)) : Z^n(\sigma_{\geq m}(A^\bullet)) \to Z^n(A^\bullet),$$

is zero for $n < m$, and it is the identity for $n \geq m$. Check that the induced natural transformation,

$$B^n(\eta_{\geq m}(A^\bullet)) : B^n(\sigma_{\geq m}(A^\bullet)) \to B^n(A^\bullet),$$

is zero for $n \leq m$, and it is the identity for $n > m$. Check that the induced natural transformation,

$$H^n(\eta_{\geq m}(A^\bullet)) : H^n(\sigma_{\geq m}(A^\bullet)) \to H^n(A^\bullet),$$

is zero for $n < m$, is the identity for $n > m$, and for $n = m$ it is the epimorphism,

$$Z^m(A^\bullet) \twoheadrightarrow H^n(A^\bullet).$$

Check that for every integer ℓ, there is a unique (exact) equivalence of categories,

$$T^\ell_m : \mathrm{Ch}^{\geq m}(\mathcal{A}) \to \mathrm{Ch}^{\geq \ell+m}(\mathcal{A}),$$

such that $T^\ell_m \circ \sigma_{\geq m}$ equals $\sigma_{\geq \ell+m} \circ T^\ell$, and T^ℓ_m. Check that $(T^\ell_m, T^{-\ell}_{\ell+m})$ is an adjoint pair of functors, so that also $(T^{-\ell}_{\ell+m}, T^\ell_m)$ is an adjoint pair of functors.

(d)bis Similarly, define the "good truncation"

$$\tau_{\geq m} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}^{\geq m}(\mathcal{A}),$$

to be the additive functor such that for every object $A^\bullet$

$$(\tau_{\geq m}A^\bullet)^n = \begin{cases} A^n, & n > m, \\ \overline{A^m}, & n = m, \\ 0, & n < m \end{cases}$$

and for every morphism $u^\bullet : A^\bullet \to C^\bullet$,

$$(\tau_{\geq m}f^\bullet)^n = \begin{cases} f^n, & n > m, \\ \overline{f^m}, & n = m, \\ 0, & n < m \end{cases}$$

Check that τ_m is right exact and is left adjoint to $e_{\geq m}$. For the natural transformation

$$\theta_m : \mathrm{Id}_{\mathrm{Ch}(\mathcal{A})} \Rightarrow e_m \circ \tau_{\geq m},$$

check that the induced morphism,

$$Z^n(\theta_{A^\bullet}) : Z^n(A^\bullet) \to Z^n(\tau_{\geq m}(A^\bullet)),$$

is zero for $n < m$, is the identity for $n > m$, and for $n = m$ it is the epimorphism,

$$Z^n(A^\bullet) \to H^n(A^\bullet).$$

Check that the induced natural transformation,

$$B^n(\theta_{A^\bullet}) : B^n(A^\bullet) \to B^n(\tau_{\geq m}(A^\bullet)),$$

is zero for $n \leq m$, and it is the identity for $n > m$. Check that the induced natural transformation,

$$\overline{\theta_{A^\bullet}}^n : \overline{A}^n \to \overline{\tau_{\geq m}(A^\bullet)}^n$$

is zero for $n < m$, and it is the identity for $n \geq m$. Check that the induced natural transformation,

$$H^n(\theta_{A^\bullet}) : H^n(A^\bullet) \to H^n(\tau_{\geq m}(A^\bullet)),$$

is zero for $n < m$, and it is the identity for $n \geq m$.

Finally, e.g., using the opposite category, formulate and prove the corresponding results for the full embedding,

$$e_{\leq m} : \mathrm{Ch}^{\leq m}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A}),$$

whose objects are complexes $A^\bullet$ such that A^n is a zero object for all $n > m$. In particular, note that although the sequence of brutal truncations,

$$0 \longrightarrow \sigma_{\geq m}(A^\bullet) \xrightarrow{\eta_{\geq m}(A^\bullet)} A^\bullet \xrightarrow{\theta_{\leq m-1}(A^\bullet)} \sigma_{\leq m-1}(A^\bullet) \longrightarrow 0$$

is exact, the corresponding morphisms of good truncations,

$$\mathrm{Ker}(\theta_{\geq m}(A^\bullet)) \hookrightarrow \tau_{\leq m}(A^\bullet), \quad \tau_{\geq m}(A^\bullet) \twoheadrightarrow \mathrm{Coker}(\eta_{\leq m}(A^\bullet)),$$

are not isomorphisms; in the first case the cokernel is $H^m(A^\bullet)[m]$, and in the second case the kernel is $H^m(A^\bullet)[m]$. However, check that the natural morphisms,

$$\tau_{\leq m-1}(A^\bullet) \xrightarrow{\eta_{\leq m-1}} \mathrm{Ker}(\theta_{\geq m}(A^\bullet),$$

$$\mathrm{Coker}(\eta_{\leq m-1}(A^\bullet)) \xrightarrow{\theta_{\geq m}} \tau_{\geq m}(A^\bullet),$$

are quasi-isomorphisms. (One reference slightly misstates this, claiming that the morphisms are isomorphisms, which is "morally" correct after passing to the derived category.)

(e) Beginning with the cohomological δ-functor (in all degrees) $\mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$,

$$H^\bullet = ((H^n)_{n\in\mathbb{Z}}, (\delta^n)_{n\in\mathbb{Z}}),$$

the associated cohomological δ-functor,

$$H^\bullet \circ T^\ell = ((H^n \circ T^\ell)_{n\in}, (\delta^n \circ T^\ell)_{n\in\mathbb{Z}}),$$

the cohomological δ-functor

$$H^{\bullet+\ell} = ((H^{n+\ell})_{n\in\mathbb{Z}}, (\delta^{n+\ell})_{n\in\mathbb{Z}}),$$

and the equivalence,
$$\iota^{\ell,0} : H^\ell \Rightarrow H^0 \circ T^\ell,$$
prove that there exists a unique natural transformation of cohomological δ-functors,
$$\theta_\ell : H^{\bullet+\ell} \Rightarrow H^\bullet \circ T^\ell, \ \ (\theta_\ell^n : H^{n+\ell} \Rightarrow H^n \circ T^\ell)_{n\in\mathbb{Z}},$$
and that $\theta_\ell^n = (-1)^{n\ell}\iota^{\ell,n}$.

(e)bis The truncation $\tau_{\geq m}H^\bullet$ in degrees $\geq m$ is obtained by replacing H^m by the subfunctor Z^m. Check that θ_ℓ restricts to a natural transformation $\tau_{\geq \ell+m}H^{\bullet+\ell} \to \tau_{\geq m}H^\bullet \circ T^\ell$. Assuming that $\tau_{\geq m}H^\bullet$ is a universal cohomological δ-functor in degrees $\geq m$, conclude that also $\tau_{\geq \ell+m}H^\bullet$ is a universal cohomological δ-functor in degrees $\geq \ell+m$. Also, formulate and prove the corresponding result for the universal δ-functors $\tau_{\leq 0}H^\bullet$ and $\tau_{\leq m}H^\bullet$.

(f) Let $\mathcal{B}$ be an Abelian category. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. This induces an additive functor
$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B}).$$
Prove that $\mathrm{Ch}(F) \circ T_{\mathcal{A}}$ equals $T_{\mathcal{B}} \circ \mathrm{Ch}(F)$.

Problem 4.(Compatibility with automorphisms.) Let $\mathcal{A}$ be an Abelian variety. Let
$$\Sigma : \ 0 \longrightarrow K^\bullet \xrightarrow{q^\bullet} A^\bullet \xrightarrow{p^\bullet} Q^\bullet \longrightarrow 0$$
be a short exact sequence in $\mathrm{Ch}(\mathcal{A})$. Let
$$u^\bullet : K^\bullet \to K^\bullet, \ \ v^\bullet : Q^\bullet \to Q^\bullet$$
be isomorphisms in $\mathrm{Ch}(\mathcal{A})$.

(a) Prove that the following sequence is a short exact sequence,
$$\Sigma_{u^\bullet,v^\bullet} : \ 0 \longrightarrow K^\bullet \xrightarrow{q^\bullet\circ u^\bullet} A^\bullet \xrightarrow{v^\bullet\circ p^\bullet} Q^\bullet \longrightarrow 0.$$

(b) Prove that the following diagrams are commutative diagrams.
$$\begin{array}{ccccccccccc}
\Sigma_{u^\bullet,\mathrm{Id}_Q} : & 0 & \longrightarrow & K^\bullet & \xrightarrow{q^\bullet\circ u^\bullet} & A^\bullet & \xrightarrow{p^\bullet} & Q^\bullet & \longrightarrow & 0 \\
\widetilde{u}\downarrow & & & u^\bullet\downarrow & & \downarrow\mathrm{Id}_A & & \downarrow\mathrm{Id}_Q & & \\
\Sigma_{\mathrm{Id}_K,\mathrm{Id}_Q} : & 0 & \longrightarrow & K^\bullet & \xrightarrow{q^\bullet} & A^\bullet & \xrightarrow{p^\bullet} & Q^\bullet & \longrightarrow & 0
\end{array},$$
$$\begin{array}{ccccccccccc}
\Sigma_{u^\bullet,\mathrm{Id}_Q} : & 0 & \longrightarrow & K^\bullet & \xrightarrow{q^\bullet\circ u^\bullet} & A^\bullet & \xrightarrow{p^\bullet} & Q^\bullet & \longrightarrow & 0 \\
\widetilde{v}\downarrow & & & \mathrm{Id}_K\downarrow & & \downarrow\mathrm{Id}_A & & \downarrow v^\bullet & & \\
\Sigma_{u^\bullet,v^\bullet} : & 0 & \longrightarrow & K^\bullet & \xrightarrow{q^\bullet\circ u^\bullet} & A^\bullet & \xrightarrow{v^\bullet\circ p^\bullet} & Q^\bullet & \longrightarrow & 0
\end{array}.$$

(c) Use the commutative diagram of long exact sequences associated to a commutative diagrams of short exact sequences to prove that

$$\delta^n_\Sigma = H^{n+1}(u^\bullet) \circ \delta^n_{\Sigma_{u^\bullet,v^\bullet}} \circ H^n(v^\bullet),$$

for every integer n.

Problem 5.(Exactness and adjoint pairs) Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. Let (L, R, θ, η) be an adjoint pair of additive functors

$$L : \mathcal{A} \to \mathcal{B},\ \ R : \mathcal{B} \to \mathcal{A}.$$

(a) For every short exact sequence in $\mathcal{A}$,

$$\Sigma :\ \ 0 \longrightarrow A' \xrightarrow{q_A} A \xrightarrow{p_A} A'' \longrightarrow 0,$$

for every object B in $\mathcal{B}$, prove that the induced morphism of Abelian groups,

$$\mathrm{Hom}_{\mathcal{A}}(p_A, R(B)) : \mathrm{Hom}_{\mathcal{A}}(A'', R(B)) \to \mathrm{Hom}_{\mathcal{A}}(A, R(B)),$$

is a monomorphism. Conclude that also the associated morphism of Abelian groups,

$$\mathrm{Hom}_{\mathcal{B}}(L(p_A), B) : \mathrm{Hom}_{\mathcal{B}}(L(A''), B) \to \mathrm{Hom}_{\mathcal{B}}(L(A), B),$$

is a monomorphism. In the special case that B equals $\mathrm{Coker}(L(p_A))$, use this to conclude that B must be a zero object. Conclude that R preserves epimorphisms.

(b) Prove that the following induced diagram of Abelian groups is exact,

$$\mathrm{Hom}_{\mathcal{A}}(A'', R(B)) \xrightarrow{p_A^*} \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \xrightarrow{q_A^*} \mathrm{Hom}_{\mathcal{A}}(A', R(B))\cdot$$

Conclude that also the following associated diagram of Abelian groups is exact,

$$\mathrm{Hom}_{\mathcal{B}}(L(A''), B) \xrightarrow{p_A^*} \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{q_A^*} \mathrm{Hom}_{\mathcal{B}}(L(A'), B)\cdot$$

In the special case that B equals $\mathrm{Coker}(L(q_A))$, conclude that the induced epimorphism $B \to L(A'')$ is split. Conclude that L is half-exact, hence right exact.

(c) Use similar arguments, or opposite categories, to conclude that also R is left exact.

(d) In case R is exact (not just left exact), prove that for every projective object P of $\mathcal{A}$, also $L(P)$ is a projective object of $\mathcal{B}$. Similarly, if L is exact (not just right exact), prove that for every injective object I of $\mathcal{A}$, also $R(I)$ is an injective object of $\mathcal{A}$.

Problem 6.(Complexes concentrated in one degree) Let $\mathcal{A}$ be an Abelian category. For every integer n, define the functor

$$E_n : \mathcal{A} \to \mathrm{Ch}(\mathcal{A}),\ \ \ A \mapsto A[n],$$

where $A[n]$ is the complex whose only nonzero term is $(A[n])^n = A$. For every morphism $f : A \to C$, the cochain morphism $E_n(f) : A[n] \to C[n]$ is defined to be the unique cochain morphism such that $(f[n])^n$ equals f. Although $A[n]$ is the standard notation, in what follows, also denote the functor by $E_n(A)$ to avoid confusion.

(a) Prove that $T \circ E_{n+1}$ equals E_n.

(b) Prove that E_n is an additive functor that is exact.

(c) Prove that E_n is left adjoint to the additive, left-exact functor,

$$Z^n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}.$$

(d) Prove that E_n is right adjoint to the additive, right-exact functor,

$$\overline{(-)}^n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}, \quad C^\bullet \mapsto \overline{C}^n = C^n/B^n(C^\bullet).$$

Problem 7.(A first mapping cone) Let $\mathcal{A}$ be an Abelian category. Let $u : K \to A$ be a monomorphism in $\mathcal{A}$. Define $\mathrm{Cone}(u)$ to be the cochain complex whose only nonzero terms are

$$d^{-1} : \mathrm{Cone}(u)^{-1} \to \mathrm{Cone}(u)^0,$$

which equals,

$$u : K \to A.$$

Define $q(u) : A[0] \to \mathrm{Cone}(u)$ to be the unique cochain morphism such that $q(u)^0$ equals $\mathrm{Id}_A : A \to A$. Define $p(u) : \mathrm{Cone}(u) \to K[+1]$ to be the unique cochain morphism such that $p(u)^{-1}$ equals $\mathrm{Id}_K : K \to K$.

(a) Prove that the following is a short exact sequence in $\mathrm{Ch}(\mathcal{A})$,

$$\Sigma : \ 0 \longrightarrow A[0] \xrightarrow{q(u)} \mathrm{Cone}(u) \xrightarrow{p(u)} K[+1] \longrightarrow 0.$$

(b) Prove that the associated long exact sequence of cohomology has only three nonzero terms,

$$H^{-1}(K[+1]) \xrightarrow{\delta_\Sigma^{-1}} H^0(A[0]) \xrightarrow{H^0(q(u))} H^0(\mathrm{Cone}(u))$$

which are canonically identified with

$$K \xrightarrow{u} A \longrightarrow \mathrm{Coker}(u).$$

Be careful to check that it is u and not $-u$.

Problem 8.(Mapping cones) Let $\mathcal{A}$ be an Abelian category. Let $u^\bullet : D^\bullet \to C^\bullet$ be a morphism in $\mathrm{Ch}(\mathcal{A})$. Define $\mathrm{Cone}(u^\bullet)$ to be the cochain complex such that for every $n \in \mathbb{Z}$, $\mathrm{Cone}(u^\bullet)^n$ is

$C^n \oplus D^{n+1}$ with the canonical morphisms $(q_1 : C^n \to C^n \oplus D^{n+1}, q_2 : D^{n+1} \to C^n \oplus D^{n+1})$ and $(p_1 : C^n \oplus D^{n+1} \to C_n, p_2 : C^n \oplus D^{n+1} \to D^{n+1})$. For every integer n, define

$$d^n_{\text{Cone}(u)} : \text{Cone}(u^\bullet)^n \to \text{Cone}(u^\bullet)^{n+1},$$

to be the unique morphism,

$$C^n \oplus D^{n+1} \to C^{n+1} \oplus D^{n+2},$$

such that $p_1 \circ d \circ q_1$ equals d^n_C, $p_2 \circ d \circ q_1$ equals 0, $p_1 \circ d \circ q_2$ equals u^{n+1}, and $p_2 \circ d \circ q_2$ equals $-d^{n+1}_D = d^n_{D[+1]}$.

(a) Check that $\text{Cone}(u^\bullet)$ is a cochain complex, i.e., $d^{n+1}_{\text{Cone}(u)} \circ d^n_{\text{Cone}(u)}$ equals 0 for every integer n.

(b) For every integer n, define

$$q(u^\bullet)^n : C^n \to \text{Cone}(u^\bullet)^n$$

to be $q_1 : C^n \to C^n \oplus D^{n+1}$. Similarly, define

$$p(u^\bullet)^n : \text{Cone}(u^\bullet)^n \to D[+1]^n$$

to be $p_2 : C^n \oplus D^{n+1} \to D^{n+1}$. Prove that both of these morphisms are cochain morphisms, i.e., they commute with the cochain differentials.

(c) Prove that the following is a short exact sequence in $\text{Ch}(\mathcal{A})$,

$$\Gamma(u) : \ 0 \longrightarrow C^\bullet \xrightarrow{q(u^\bullet)} \text{Cone}(u^\bullet) \xrightarrow{p(u^\bullet)} D^\bullet[+1] \longrightarrow 0.$$

In fact, for every n, prove that the corresponding short exact sequence in $\mathcal{A}$,

$$\Gamma(u)^n : \ 0 \longrightarrow C^n \xrightarrow{q(u^\bullet)^n} \text{Cone}(u^\bullet)^n \xrightarrow{p(u^\bullet)^n} D[+1]^n \longrightarrow 0,$$

is split by q_2 and p_1. However, the morphisms q_2 and p_1 do not (typically) commute with the differentials, hence they are not cochain morphisms. Prove that there is a natural equivalence between the splittings of $\Gamma(u)$ in $\text{Ch}(\mathcal{A})$ and homotopies of $u^\bullet$ to 0 (tautological if neither exists).

(d) Check carefully that for every integer n,

$$\delta^{n-1}_{\Gamma(u)} : H^{n-1}(D^\bullet[+1]) \to H^n(C^\bullet),$$

equals

$$H^n(u^\bullet) : H^n(D^\bullet) \to H^n(C^\bullet).$$

In particular, check carefully the sign.

(e) Prove that $\text{Cone}(u^\bullet)$ and $\Gamma(u^\bullet)$ are additive functors on the additive category whose objects are morphisms in $\text{Ch}(\mathcal{A})$ and whose morphisms are commutative diagrams. Prove that $q(u^\bullet)$ and $p(u^\bullet)$ are natural transformations of additive functors.

(f) Let $\mathcal{B}$ be an Abelian category. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. This induces an additive functor

$$\text{Ch}(F) : \text{Ch}(\mathcal{A}) \to \text{Ch}(\mathcal{B}).$$

Prove that $\text{Ch}(F)$ sends $\text{Cone}(u^\bullet)$, resp. $\Gamma(u^\bullet)$, to $\text{Cone}(\text{Ch}(F)(u^\bullet))$, resp. $\Gamma(\text{Ch}(F)(u^\bullet))$.