MAT 536 Problem Set 10

Homework Policy. Please read through all the problems. Please solve 5 of the problems. I will be happy to discuss the solutions during office hours.

Problems.

Problem 1.(The Mapping Cone is a Homotopy Limit and a Homotopy Colimit) Let $\mathcal{A}$ be an Abelian category. Let $A^\bullet$ and $B^\bullet$ be objects in $\mathrm{Ch}^\bullet(\mathcal{A})$. Let $f^\bullet : A^\bullet \to B^\bullet$ be a morphism in $\mathrm{Ch}^\bullet(\mathcal{A})$. For every object $T^\bullet$ of $\mathrm{Ch}^\bullet(\mathcal{A})$, a *left homotopy annihilator* to $T^\bullet$ is a pair $(Q^\bullet, \sigma^\bullet)$ of a morphism $Q^\bullet : B^\bullet \to T^\bullet$ in $\mathrm{Ch}^\bullet(\mathcal{A})$ and a nullhomotopy $(\sigma^n : A^n \to T^{n-1})_{n\in\mathbb{Z}}$ of $Q^\bullet \circ f^\bullet$, i.e., for every $n \in \mathbb{Z}$,

$$Q^n \circ f^n = d_T^{n-1} \circ \sigma^n + \sigma^{n+1} \circ d_A^n.$$

For every object $S^\bullet$ of $\mathrm{Ch}^\bullet(\mathcal{A})$, a *right homotopy annihilator* from $S^\bullet$ is a pair $(\Delta^\bullet, \tau^\bullet)$ of a morphism $\Delta^\bullet : S^\bullet \to A^\bullet[+1]$ and a nullhomotopy $(\tau^n : S^n \to B[+1]^n = B^n)_{n\in\mathbb{Z}}$ of $f^\bullet[+1] \circ \Delta^\bullet$, i.e., for every $n \in \mathbb{Z}$,

$$f^{n+1} \circ \Delta^n = d_{B[+1]}^{n-1} \circ \tau^n + \tau^{n+1} \circ d_S^n = -d_B^n \circ \tau^n + \tau^{n+1} \circ d_S^n.$$

Denote by $LHA_{f^\bullet}(T^\bullet)$ the set of left homotopy annihilators to $T^\bullet$, and denote by $RHA_{f^\bullet}(S^\bullet)$ the set of right homotopy annihilators from $S^\bullet$.

(a)(Homotopy Annihilators are Additive Functors) **Prove** that $(0,0)$ is a left homotopy annihilator to $T^\bullet$. For left homotopy annihilators $(Q^\bullet, \sigma^\bullet)$ and $(\widetilde{Q}^\bullet, \widetilde{\sigma}^\bullet)$ to $T^\bullet$, **prove** that $(Q^\bullet - \widetilde{Q}^\bullet, \sigma^\bullet - \widetilde{\sigma}^\bullet)$ is also a left homotopy annihilator to $T^\bullet$. Conclude that $LHA_{f^\bullet}(T^\bullet)$ is an Abelian group, and the set map,

$$\eta_{T^\bullet} : LHA_{f^\bullet}(T^\bullet) \to \mathrm{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(B^\bullet, T^\bullet),\ (Q^\bullet, s^\bullet) \mapsto Q^\bullet,$$

is a homomorphism of Abelian groups. For every morphism $g^\bullet : T^\bullet \to \widetilde{T}^\bullet$ in $\mathrm{Ch}^\bullet(\mathcal{A})$, for every left homotopy annihilator $(Q^\bullet, \sigma^\bullet)$ to $T^\bullet$, **prove** that $(g^\bullet \circ Q^\bullet, g^\bullet \circ \sigma^\bullet)$ is a left homotopy annihilator to $\widetilde{T}^\bullet$. Denote the associated set map by

$$LHA_{f^\bullet}(g^\bullet) : LHA_{f^\bullet}(T^\bullet) \to LHA_{f^\bullet}(\widetilde{T}^\bullet),\ (Q^\bullet, \sigma^\bullet) \to (g^\bullet \circ Q^\bullet, g^\bullet \circ \sigma^\bullet).$$

Prove that $LHA_{f^\bullet}(g^\bullet)$ is a morphism of Abelian groups, and the following diagram commutes,

$$\begin{tikzcd} LHA_{f^\bullet}(T^\bullet) \arrow[r, "LHA_{f^\bullet}(g^\bullet)"] \arrow[d, "\eta_{T^\bullet}"'] & LHA_{f^\bullet}(\widetilde{T}^\bullet) \arrow[d, "\eta_{\widetilde{T}^\bullet}"] \\ h^{B^\bullet}(T^\bullet) \arrow[r, "h^{B^\bullet}(g^\bullet)"'] & h^{B^\bullet}(\widetilde{T}^\bullet) \end{tikzcd}.$$

Prove that $LHA_{f^\bullet}(\mathrm{Id}_{T^\bullet})$ is the identity map on $LHA_{f^\bullet}(T^\bullet)$. Also **prove** that $LHA_{f^\bullet}(\widehat{g}^\bullet \circ g^\bullet)$ equals $LHA_{f^\bullet}(\widehat{g}^\bullet) \circ LHA_{f^\bullet}(g^\bullet)$. Conclude that these rules define a functor,

$$LHA_{f^\bullet} : \mathrm{Ch}^\bullet(\mathcal{A}) \to \mathbb{Z} - \mathbf{mod},$$

together with a natural transformation of functors,

$$\eta : LHA_{f^\bullet} \Rightarrow h^{B^\bullet}.$$

In a similar way, extend $RHA_{f^\bullet}$ to a contravariant functor

$$RHA_{f^\bullet} : \mathrm{Ch}^\bullet(\mathcal{A})^{\mathrm{opp}} to \mathbb{Z} - \mathbf{mod},$$

together with homomorphisms of Abelian groups,

$$\eta^{S^\bullet} : RHA_{f^\bullet}(S^\bullet) \to h_{A^\bullet[+1]}(S^\bullet),$$

that forms a natural transformation of (contravariant) functors.

(b)(The Universal Homotopy Annihilators) As usual, define the mapping cone $C(f^\bullet)$ to be

$$C(f^\bullet)^n = B^n \oplus A^{n-1},$$

with differential

$$d^n_{C(f)} : C(f^\bullet)^n \to C(f^\bullet)^{n+1}, \begin{bmatrix} b_n \\ a_{n+1} \end{bmatrix} \mapsto \begin{bmatrix} d^n_B(b_n) & + & u^{n+1}(a_{n+1}) \\ & & -d^{n+1}_A(a_{n+1}) \end{bmatrix}.$$

Prove that this defines an object of $\mathrm{Ch}^\bullet(\mathcal{A})$. For every $n \in \mathbb{Z}$, define

$$q^n_{f^\bullet} : B^n \to C(f^\bullet)^n, \quad q^n_{f^\bullet}(b_n) = \begin{bmatrix} b_n \\ 0 \end{bmatrix}.$$

Prove that this is a morphism in $\mathrm{Ch}^\bullet(\mathcal{A})$. For every $n \in \mathbb{Z}$, define

$$s^{n+1}_{f^\bullet} : A^{n+1} \to C(f^\bullet)^n, \quad s^{n+1}_{f^\bullet}(a_{n+1}) = \begin{bmatrix} 0 \\ a_{n+1} \end{bmatrix}.$$

Prove that $(s^n_{f^\bullet})_{n \in \mathbb{Z}}$ is a nullhomotopy of $q^\bullet_{f^\bullet} \circ f^\bullet$. Thus, $(q^\bullet_{f^\bullet}, s^\bullet_{f^\bullet})$ is an element of $LHA_{f^\bullet}(C(f^\bullet))$.

In a similar way, for every $n \in \mathbb{Z}$, define

$$\delta^n_{f^\bullet} : C(f^\bullet)^n \to A[+1]^n = A^{n+1}, \quad \begin{bmatrix} b_n \\ a_{n+1} \end{bmatrix} \mapsto a_{n+1}.$$

Prove that this is a morphism $\delta^\bullet_{f^\bullet} : C(f^\bullet) \to A[+1]$ in $\mathrm{Ch}^\bullet(\mathcal{A})$. For every $n \in \mathbb{Z}$, define

$$t^n_{f^\bullet} : C(f^\bullet)^n \to B[+1]^{n+1} = B^n, \quad \begin{bmatrix} b_n \\ a_{n+1} \end{bmatrix} \mapsto b_n.$$

Prove that $(t^n_{f^\bullet})_{n\in\mathbb{Z}}$ is a nullhomotopy of $f^\bullet[+1]\circ\delta^\bullet_{f^\bullet}$. Thus $(\delta^\bullet_{f^\bullet}, t^\bullet_{f^\bullet})$ is an element of $RHA_{f^\bullet}(C(f^\bullet))$.

(c)(Mapping Cones Represent the Homotopy Annihilator Functors) **Prove** that the element $(q^\bullet_{f^\bullet}, s^\bullet_{f^\bullet}) \in LHA_{f^\bullet}(C(f^\bullet))$ represents the functor $LHA_{f^\bullet}(C(f^\bullet))$, i.e., for every object $T^\bullet$ of $\mathrm{Ch}^\bullet(\mathcal{A})$ and for every element $(Q^\bullet, \sigma^\bullet) \in LHA_{f^\bullet}(T^\bullet)$, there exists a unique morphism $g^\bullet : C(f^\bullet) \to T^\bullet$ such that $LHA_{f^\bullet}(g^\bullet)$ maps $(q_{f^\bullet}, s_{f^\bullet})$ to (Q, σ).

Use this to reprove the following result (actually, this is essentially the same as the original proof). For the short exact sequence in $\mathrm{Ch}^\bullet(\mathcal{A})$,

$$C_f : 0 \longrightarrow B \xrightarrow{q_f} C(f) \xrightarrow{\delta_f} A[+1] \longrightarrow 0,$$

splittings of the exact sequence (if any exist) are equivalent to nullhomotopies of f. In particular, conclude that there exist splittings of the following two short exact sequences,

$$C_{q_f} : 0 \longrightarrow C(f) \xrightarrow{q_{q_f}} C(q_f) \xrightarrow{\delta_{q_f}} B[+1] \longrightarrow 0,$$

$$C_{\delta_f[-1]} : 0 \longrightarrow A \xrightarrow{q_{\delta_f[-1]}} C(\delta_f[-1]) \xrightarrow{\delta_{\delta_f[-1]}} C(f) \longrightarrow 0.$$

Finally, for every object $D^\bullet$ of $\mathrm{Ch}^\bullet(\mathcal{A})$, for the zero morphism $0_D : 0 \to D^\bullet$, conclude that $q_{0_D} : D^\bullet \to C(0_D)^\bullet$ is a natural isomorphism. Similarly, for the zero morphism $0^D : D^\bullet \to 0$, conclude that $\delta_{0_D} : C(0^D)^\bullet \to D[+1]^\bullet$ is a natural isomorphism.

(d)(Compatibility with Homotopy Commutative Diagrams) A *homotopy commutative diagram* e in $\mathrm{Ch}^\bullet(\mathcal{A})$ is a pair of a diagram in $\mathrm{Ch}^\bullet(\mathcal{A})$ (**not** strictly commutative),

$$\begin{tikzcd} f^\bullet : \arrow[d, "e^\bullet"'] & A^\bullet \arrow[r, "f^\bullet"] \arrow[d, "e^\bullet_A"] & B^\bullet \arrow[d, "e^\bullet_B"] \\ \widetilde{f}^\bullet : & \widetilde{A}^\bullet \arrow[r, "\widetilde{f}^\bullet"'] & \widetilde{B}^\bullet \end{tikzcd}$$

and a nullhomotopy $e^\bullet_s = (e^n_s : A^n \to \widetilde{B}^{n-1})_n$ of $\widetilde{f}^\bullet \circ e^\bullet_A - e^\bullet_B \circ f^\bullet$. For every homotopy commutative diagram e, for every object $T^\bullet$ and for every left homotopy annihilator $(\widetilde{Q}^\bullet, \widetilde{\sigma}^\bullet)$ to $T^\bullet$ relative to $\widetilde{f}^\bullet$, **prove** that $(\widetilde{Q}^\bullet \circ e^\bullet_B, \widetilde{\sigma}^\bullet \circ e^\bullet_A - \widetilde{Q}^\bullet \circ e^\bullet_s)$ is a left homotopy annihilator relative to $\widetilde{f}^\bullet$. Denote the associated set map by

$$LHA_{e^\bullet}(T^\bullet) : LHA_{\widetilde{f}^\bullet}(T^\bullet) \to LHA_{f^\bullet}(T^\bullet), \quad (\widetilde{Q}^\bullet, \widetilde{\sigma}^\bullet) \mapsto (\widetilde{Q}^\bullet \circ e^\bullet_B, \widetilde{\sigma}^\bullet \circ e^\bullet_A - \widetilde{Q}^\bullet \circ e^\bullet_s).$$

Prove that $LHA_{e^\bullet}$ is a natural transformation of functors. In particular, associated to the left homotopy annihilator $(q_{\widetilde{f}^\bullet} \circ e_B, s_{\widetilde{f}^\bullet} \circ e_A - q_{\widetilde{f}^\bullet} \circ e_s)$, conclude that there is a unique morphism in $\mathrm{Ch}^\bullet(\mathcal{A})$,

$$C(e^\bullet) : C(f^\bullet) \to C(\widetilde{f}^\bullet),$$

such that

$$(q_{\widetilde{f}^\bullet} \circ e_B, s_{\widetilde{f}^\bullet} \circ e_A - q_{\widetilde{f}^\bullet} \circ e_s) = (C(e^\bullet) \circ q_{f^\bullet}, C(e^\bullet) \circ s_{f^\bullet}).$$

On the level of elements, this is

$$C(e^\bullet) : \begin{bmatrix} b_n \\ a_{n+1} \end{bmatrix} \mapsto \begin{bmatrix} e_B^n(b_n) - e_s^{n+1}(a_{n+1}) \\ e_A^{n+1}(a_{n+1}) \end{bmatrix}$$

In particular, **prove** that this induces a morphism of short exact sequences in $\mathrm{Ch}^\bullet(\mathcal{A})$,

$$\begin{tikzcd} C_f : 0 \arrow[r] \arrow[d, "C_e"'] & B \arrow[r, "q_f"] \arrow[d, "e_B"'] & C(f) \arrow[r, "\delta_f"] \arrow[d, "C(e)"] & A[+1] \arrow[r] \arrow[d, "e_A"] & 0 \\ C_{\widetilde{f}} : 0 \arrow[r] & \widetilde{B} \arrow[r, "q_{\widetilde{f}}"] & C(\widetilde{f}) \arrow[r, "\delta_{\widetilde{f}}"] & \widetilde{A}[+1] \arrow[r] & 0 \end{tikzcd}$$

For every object $S^\bullet$ and for every right homotopy annihilator $(\Delta^\bullet, \tau^\bullet)$ from $S^\bullet$ relative to $f^\bullet$, **prove** that $(e_A[+1]^\bullet \circ \Delta^\bullet, e_B[+1]^\bullet \circ \tau^\bullet + e_s[+1]^\bullet \circ \Delta^\bullet)$ is a right homotopy annihilator relative to $\widetilde{f}^\bullet$. (Please recall, for a nullhomotopy $s^\bullet = (s^n : C^n \to D^{n-1})_n$ of a morphism $g^\bullet : C^\bullet \to D^\bullet$, the sequence $s[+1]^\bullet = (-s^{n+1} : C^{n+1} \to D^n)_n$ is a nullhomotopy of $g[+1]^\bullet : C[+1]^\bullet \to D[+1]^\bullet$.) Denote the associated set map by

$$RHA_{e^\bullet}(S^\bullet) : RHA_{f^\bullet}(S^\bullet) \to RHA_{\widetilde{f}^\bullet}(S^\bullet).$$

Prove that $RHA_{e^\bullet}$ is a natural transformation of functors. In particular, associated to the right homotopy annihilator $(e_A[+1] \circ \delta_{f^\bullet}, e_B[+1] \circ t_{f^\bullet} + e_s[+1] \circ \delta_f))$, conclude that there is a unique morphism in $\mathrm{Ch}^\bullet(\mathcal{A})$,

$$C(e^\bullet) : C(f^\bullet) \to C(\widetilde{f}^\bullet),$$

such that

$$(e_A[+1]^\bullet \circ \delta_{f^\bullet}, e_B[+1]^\bullet \circ t_{f^\bullet} + e_s[+1] \circ \delta_f) = (\delta_{\widetilde{f}^\bullet} \circ C(e^\bullet), t_{\widetilde{f}^\bullet} \circ C(e^\bullet)).$$

Prove that this is the same morphism as above.

(e)(Decomposition of the Diagonal of the Mapping Cone) In particular, associated to the homotopy commutative diagram,

$$\begin{tikzcd} f : \arrow[d, "q_f"'] & A \arrow[r, "f"] \arrow[d, "0^A"] & B \arrow[d, "q_f"] \\ 0_{C(f)} : & 0 \arrow[r, "0_{C(f)}"'] & C(f) \end{tikzcd}$$

with nullhomotopy $-s_f$, **prove** that the associated morphism of cones is

$$s_f^{n+1} \circ \delta_f^n : C(f)^n \to C(f)^n.$$

Similarly, associated to the homotopy commutative diagram,

$$\begin{tikzcd} 0^{C(f)} : \arrow[d, "\delta_f"'] & C(f) \arrow[r, "0^{C(f)}"] \arrow[d, "\delta_f"] & 0 \arrow[d, "0_{B[+1]}"] \\ f[+1] : & A[+1] \arrow[r, "f[+1]"'] & B[+1] \end{tikzcd}$$

with nullhomotopy t_f, **prove** that the associated morphism of cones is

$$q_f^n \circ t_f^n : C(f)^n \to C(f)^n.$$

Prove that these morphisms give morphisms of short exact sequences,

$$\begin{tikzcd}
C_f: 0 \arrow[r] \arrow[d, "s_f\circ\delta_f"'] & B \arrow[r, "q_f"] \arrow[d, "0"'] & C(f) \arrow[r, "\delta_f"] \arrow[d, "s_f\circ\delta_f"] & A[+1] \arrow[r] \arrow[d, "\mathrm{Id}_{A[+1]}"] & 0 \\
C_f: 0 \arrow[r] & B \arrow[r, "q_f"] & C(f) \arrow[r, "\delta_f"] & A[+1] \arrow[r] & 0
\end{tikzcd},$$

$$\begin{tikzcd}
C_f: 0 \arrow[r] \arrow[d, "q_f\circ t_f"'] & B \arrow[r, "q_f"] \arrow[d, "\mathrm{Id}_B"'] & C(f) \arrow[r, "\delta_f"] \arrow[d, "q_f\circ t_f"] & A[+1] \arrow[r] \arrow[d, "0"] & 0 \\
C_f: 0 \arrow[r] & B \arrow[r, "q_f"] & C(f) \arrow[r, "\delta_f"] & A[+1] \arrow[r] & 0
\end{tikzcd}.$$

Using Baer sum, **prove** that $s_f \circ \delta_f + q_f \circ t_f$ equals $\mathrm{Id}_{C(f)}$.

(f)(Mapping Cone and the Total Complex of a Double Complex) Interpret every morphism $f^\bullet : A^\bullet \to B^\bullet$ in $\mathrm{Ch}^\bullet(\mathcal{A})$ as an object of $\mathrm{Ch}^{[-1,0]}(\mathrm{Ch}^\bullet(\mathcal{A}))$, or, equivalently, as a double complex. Prove that the mapping cone is an additive functor,

$$C : \mathrm{Ch}^{[-1,0]}(\mathrm{Ch}^\bullet(\mathcal{A})) \to \mathrm{Ch}^\bullet(\mathcal{A}),$$

that is the same as the functor,

$$\mathrm{Tot} : \mathrm{Ch}^{\bullet,*}(\mathcal{A}) \to \mathrm{Ch}^\bullet(\mathcal{A}),$$

sending a double complex to the associated total complex.

For every homotopy commutative diagram e in $\mathrm{Ch}^\bullet(\mathcal{A})$,

$$\begin{tikzcd}
f^\bullet : \arrow[d, "e^\bullet"'] & A^\bullet \arrow[r, "f^\bullet"] \arrow[d, "e_A^\bullet"] & B^\bullet \arrow[d, "e_B^\bullet"] \\
\widetilde{f}^\bullet : & \widetilde{A}^\bullet \arrow[r, "\widetilde{f}^\bullet"'] & \widetilde{B}^\bullet
\end{tikzcd}$$

and a nullhomotopy $e_s^\bullet = (e_s^n : A^n \to \widetilde{B}^{n-1})_n$ of $\widetilde{f}^\bullet \circ e_A^\bullet - e_B^\bullet \circ f^\bullet$, prove that there is a corresponding morphism

$$C(e) : C(f) \to C(\widetilde{f}),$$

and prove that this is functorial in homotopy commutative diagrams in a suitable sense. This suggests an enrichment of the categories of cochain complexes to properly capture this functoriality.

(g)(Cones Commute with Limits and Colimits) Use the universal property of C with respect to right homotopy annihilators to **prove** that C commutes with the limit for every functor from a small category to $\mathrm{Ch}^{[-1,0]}(\mathrm{Ch}^\bullet(\mathcal{A}))$ for which the limit of the functor exists in $\mathrm{Ch}^{[-1,0]}(\mathrm{Ch}^\bullet(\mathcal{A}))$. Formulate and **prove** a version when the "functor" only preserves composition up to (specified) homotopies, as in the previous part. Similarly, use the universal property of C with respect to left homotopy annihilators to **prove** that C commutes with colimits of functors, resp. "functors up to specified homotopies", such that the colimit exists in $\mathrm{Ch}^{[-1,0]}(\mathrm{Ch}^\bullet(\mathcal{A}))$.

Problem 2.(The DG Category of Complexes) Let $\mathcal{A}$ be an Abelian category. Let $A^\bullet$ and $B^\bullet$ be objects in $\mathrm{Ch}^\bullet(\mathcal{A})$. For every integer n, a *degree n graded morphism*, $f^\bullet : A^\bullet \to B^\bullet$ is a sequence $(f^m : A^m \to B^{m+n})_{m\in\mathbb{Z}}$ of morphisms f^m in $\mathcal{A}$. Define $d^n_{\mathrm{Hom}}(f^\bullet)$ to be the degree $n+1$ graded morphism

$$(d_B^{m+n} \circ f^m + (-1)^{n+1} f^{m+1} \circ d_A^m)_{m\in\mathbb{Z}}.$$

(a)(The Hom Complex) For every integer n, **prove** that the zero morphisms $0 : A^m \to B^{m+n}$ define a degree m graded morphism from $A^\bullet$ to $B^\bullet$. **Prove** that the identity morphisms $\mathrm{Id}_{A^m} : A^m \to A^m$ define a degree 0 graded morphism $\mathrm{Id}_{A^\bullet} : A^\bullet \to A^\bullet$. For two degree m graded morphisms from $A^\bullet$ to $B^\bullet$, $f^\bullet = (f^m)_{m\in\mathbb{Z}}$ and $(g^m)_{m\in\mathbb{Z}}$, define $f^\bullet - g^\bullet$ to be $(f^m - g^m)_{m\in\mathbb{Z}}$. **Prove** that $f^\bullet - g^\bullet$ is a degree m graded morphism. With these operations, **prove** that the set of degree m graded morphisms from $A^\bullet$ to $B^\bullet$ forms an Abelian group. This group is denoted

$$\mathrm{Hom}^n_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet).$$

Prove that $d^n_{\mathrm{Hom}}(A^\bullet, B^\bullet)$ is a degree $n+1$ graded morphism from $A^\bullet$ to $B^\bullet$. **Prove** that the induced set map,

$$d^n_{\mathrm{Hom}} : \mathrm{Hom}^n_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet) \to \mathrm{Hom}^{n+1}_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet),$$

is a homomorphism of Abelian groups. **Prove** that $d^{n+1}_{\mathrm{Hom}} \circ d^n_{\mathrm{Hom}}$ is the zero homomorphism. Use this to interpret the data

$$((\mathrm{Hom}^n_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet))_n, (d^n_{\mathrm{Hom}})_n)$$

as an element of $\mathrm{Ch}^\bullet(\mathbb{Z}-\mathbf{mod})$. This datum is often called a *differential graded $\mathbb{Z}$-module*, or a *dg module* for short.

(b)(The Symmetric Monoidal Category of DG Modules) For associative, unital rings R, S, and T, for every object $K^\bullet = ((K^n)_n, (d_K^n)_n)$ in $\mathrm{Ch}^\bullet(R-S-\mathbf{Bimod})$, and for every object $L^\bullet = ((L^n)_n, (d_L^n)_n)$ in $\mathrm{Ch}^\bullet(S-T-\mathbf{Bimod})$, define $K^\bullet \otimes_S L^\bullet$ to be the direct sum total complex of the double complex

$$K^\bullet \otimes_S L^* = ((K^m \otimes_S L^n)_{(m,n)}, (d_K^m \otimes \mathrm{Id}_{L^n})_{(m,n)}, (\mathrm{Id}_{K^m} \otimes d_L^n)_{(m,n)}).$$

In other words,

$$(K \otimes_S L)^\ell = \oplus_{(m,n),m+n=\ell} K^m \otimes_S L^n,$$

with the family of summands $\iota_{m,n} : K^m \otimes_S L^n \to (K \otimes_S L)^\ell$. Also, the differential is

$$d^\ell \circ \iota_{m,n} = \iota_{m+1,n} \circ (d_K^m \otimes \mathrm{Id}_{L^n}) + (-1)^m \iota_{m,n+1} \circ (\mathrm{Id}_{K^m} \otimes d_L^n).$$

Since the double complex $K^\bullet \otimes_S L^*$ is bifunctorial in $K^\bullet$ and L^*, **prove** that also $K^\bullet \otimes_S L^\bullet$ is bifunctorial in $K^\bullet$ and $L^\bullet$. Since the double complex is biadditive, **prove** that also $K^\bullet \otimes_S L^\bullet$ is biadditive. For the complex $S[0]$ of $S-S$-bimodules with $S[0]^0 = S$ and all other $S[0]^n$ zero, **prove** that the usual natural isomorphisms $K \otimes_S S[0] \cong K$ and $S[0] \otimes_S L \cong L$ induce natural isomorphisms in $\mathrm{Ch}^\bullet(R - S - \mathbf{Bimod})$, resp. in $\mathrm{Ch}^\bullet(S - T - \mathbf{Bimod})$,

$$\rho_{K^\bullet} : K^\bullet \otimes_S S[0] \xrightarrow{\cong} K^\bullet,$$

and

$$\lambda_{K^\bullet} : S[0] \otimes_S K^\bullet \xrightarrow{\cong} K^\bullet.$$

For an associative, unital ring U, for an object $M^\bullet$ of $\mathrm{Ch}^\bullet(T - U - \mathbf{Bimod})$, prove that the tri-natural isomorphisms $(K^m \otimes_S L^n) \otimes_T N^p \cong K^m \otimes_S (L^n \otimes_T N^p)$ induce a tri-natural isomorphism of dg $R - T$-bimodules,

$$\alpha_{K^\bullet, L^\bullet, M^\bullet} : (K^\bullet \otimes_S L^\bullet) \otimes_T N^\bullet \cong K^\bullet \otimes_S (L^\bullet \otimes_T N^\bullet).$$

Read the definition of monoidal category. For every commutative, unital ring k, **prove** that these operations make $\mathrm{Ch}^\bullet(k - \mathbf{mod})$ into a monoidal category. For objects $K^\bullet$ and $L^\bullet$, prove that the binatural isomorphisms $K^m \otimes_k L^n \cong L^n \otimes_k K^m$ induce a binatural isomorphism of dg k-modules,

$$s_{K^\bullet, L^\bullet} : K^\bullet \otimes_k L^\bullet \to L^\bullet \otimes_k K^\bullet.$$

Read the definition of symmetric monoidal category. **Prove** that the binatural isomorphism $s_{K^\bullet, L^\bullet}$ makes the monoidal category $\mathrm{Ch}^\bullet(k - \mathbf{mod})$ into a symmetric monoidal category. In particular, for $k = \mathbb{Z}$, prove that the Hom dg modules in the previous part are objects in the symmetric monoidal category of dg $\mathbb{Z}$-modules.

(c)(The DG Category of Complexes) For objects $A^\bullet$, $B^\bullet$ and $C^\bullet$ of $\mathrm{Ch}^\bullet(\mathcal{A})$, for a degree m graded morphism $f^\bullet : A^\bullet \to B^\bullet$ and for a degree n graded morphism $g^\bullet : B^\bullet \to C^\bullet$, **prove** that $(g^{\ell+m} \circ f^\ell : A^\ell \to C^{\ell+m+n})_{\ell \in \mathbb{Z}}$ is a degree $m + n$ graded morphism $A^\bullet \to C^\bullet$. Denote the corresponding set map by

$$\circ^{n,m}_{A,B,C} : \mathrm{Hom}^n_{\mathrm{Ch}^\bullet(\mathcal{A})}(B^\bullet, C^\bullet) \times \mathrm{Hom}^m_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet) \to \mathrm{Hom}^{m+n}_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, C^\bullet).$$

Prove that this is biadditive, and hence defines a homomorphism

$$\circ^{n,m}_{A,B,C} : \mathrm{Hom}^n_{\mathrm{Ch}^\bullet(\mathcal{A})}(B^\bullet, C^\bullet) \otimes_{\mathbb{Z}} \mathrm{Hom}^m_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet) \to \mathrm{Hom}^{m+n}_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, C^\bullet).$$

Prove that these homomorphisms define a morphism of dg modules,

$$\circ^\bullet_{A,B,C} : \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(B^\bullet, C^\bullet) \otimes_{\mathbb{Z}} \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet) \to \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, C^\bullet).$$

Read the definition of a category enriched over a monoidal category. **Prove** that these compositions, together with the identity morphisms $\mathrm{Id}_{A^\bullet}$, make $\mathrm{Ch}^\bullet(\mathcal{A})$ into a category enriched over dg modules. Such a category is a *dg category*.

(d)(Cycles and Homology of the Hom Complex) Prove that a degree 0 graded morphism $f^\bullet \in \mathrm{Hom}^0_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet)$ is a morphism of cochain complexes if and only if $d^0_{\mathrm{Hom}}(f^\bullet)$ is the zero morphism, i.e.,

$$\mathrm{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet) = Z^0(\mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet)).$$

Similarly, **prove** that every $s^\bullet \in \mathrm{Hom}^{-1}_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet)$ with $d^{-1}_{\mathrm{Hom}}(s^\bullet) = f^\bullet$ is a nullhomotopy of $f^\bullet$. Conclude that there is a binatural isomorphism,

$$H^0(\mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet)) \cong \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet),$$

where, as usual, $K(\mathcal{A})$ denotes the homotopy category of $\mathrm{Ch}^\bullet(\mathcal{A})$.

(e)(Composition in the Homotopy Category) Use the abutment morphisms of the spectral sequence of a double complex to **prove** the existence of a binatural, biadditive homomorphism,

$$H^0(\mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(B^\bullet, C^\bullet)) \otimes_{\mathbb{Z}} H^0(\mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet)) \to H^0(\mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, C^\bullet)),$$

that is associative and that respects identities on both the left and the right. Use this to give a second proof that the composition in $K(\mathcal{A})$ is well-defined, associative, biadditive, and respects identities, i.e., $K(\mathcal{A})$ is an additive category.

(f)(Compatibility with Translation; Yoneda Functors) **Prove** that the dg category structure is compatible with translation in the sense that for every integer n, there are binatural isomorphisms of dg modules,

$$\mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B[+n]^\bullet) \cong \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet)[+n] \cong \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A[-n]^\bullet, B^\bullet).$$

In particular, **prove** that there are binatural, biadditive isomorphisms compatible with identities,

$$\mathrm{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B[+n]^\bullet) \cong Z^n(\mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet)) \cong \mathrm{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(A[-n]^\bullet, B^\bullet),$$

and similar isomorphisms

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B[+n]^\bullet) \cong H^n(\mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet)) \cong \mathrm{Hom}_{K(\mathcal{A})}(A[-n]^\bullet, B^\bullet).$$

In particular, for objects $A^\bullet$ and $B^\bullet$ of $\mathrm{Ch}^\bullet(\mathcal{A})$, the Yoneda functors are enriched to functors,

$$h^B(-)^\bullet : \mathrm{Ch}^\bullet(\mathcal{A}) \to \mathrm{Ch}^\bullet(\mathbb{Z}-\mathbf{mod}),\ T^\bullet \mapsto \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(B^\bullet, T^\bullet),$$

$$h_A(-)^\bullet : \mathrm{Ch}^\bullet(\mathcal{A})^{\mathrm{opp}} \to \mathrm{Ch}^\bullet(\mathbb{Z}-\mathbf{mod}),\ S^\bullet \mapsto \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(S^\bullet, A^\bullet).$$

Prove that these are additive functors that are compatible with translation functors. Moreover, prove that the associativity of the composition dg module homomorphisms enrich these to functors of dg categories, i.e.,

$$h^B(-) : \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(T^\bullet, \widetilde{T}^\bullet) \to \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathbb{Z}-\mathbf{mod})}(h^B(T^\bullet), h^B(\widetilde{T}^\bullet))$$

$$h_A(-) : \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(S^\bullet, \widetilde{S}^\bullet) \to \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathbb{Z}-\mathbf{mod})}(h_A(\widetilde{S}^\bullet), h_A(S^\bullet)).$$

Such an enriched functor h^B, resp. h_A, is a *left dg module*, resp. *right dg module*, on $\mathrm{Ch}^\bullet(\mathcal{A})$.

(g)(Cones of Yoneda Functors are Yoneda Functors of Cones) For every $f^\bullet \in \mathrm{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet)$, for every objects $S^\bullet$ of $\mathrm{Ch}^\bullet(\mathcal{A})$, **prove** that the composition morphism,

$$h^f : h^A(-) \to h^B(-),$$

respectively

$$h_f : h_B(-) \to h_A(-),$$

is a morphism of left dg modules, resp. right dg modules, on $\mathrm{Ch}^\bullet(\mathcal{A})$. As above, interpret the morphism h^f of left dg modules as a functor from $\mathrm{Ch}^\bullet(\mathcal{A})$ to $\mathrm{Ch}^{[-1,0]}(\mathrm{Ch}^\bullet(\mathbb{Z}-\mathbf{mod}))$ considered as a double complex of Abelian groups. **Prove** that the total complex of this double complex defines a left dg module on $\mathrm{Ch}^\bullet(\mathcal{A})$,

$$C(h^f) : \mathrm{Ch}^\bullet(\mathcal{A}) \to \mathrm{Ch}^\bullet(\mathbb{Z}-\mathbf{mod}).$$

Similarly, interpret the morphism h_f of right dg modules as a contravariant functor from $\mathrm{Ch}^\bullet(\mathcal{A})$ to $\mathrm{Ch}^{[-1,0]}(\mathrm{Ch}^\bullet(\mathbb{Z}-\mathbf{mod}))$ considered as a double complex of Abelian groups. **Prove** that the total complex of this double complex defines a right dg module on $\mathrm{Ch}^\bullet(\mathcal{A})$,

$$C(h_f) : \mathrm{Ch}^\bullet(\mathcal{A})^{\mathrm{opp}} \to \mathrm{Ch}^\bullet(\mathbb{Z}-\mathbf{mod}).$$

Rework Problem 1 to **prove** that both of these dg modules over $\mathrm{Ch}^\bullet(\mathcal{A})$ are representable by the mapping cone,

$$C(h^f) = h^{C(f)}, \quad C(h_f) = h_{C(f)}.$$

(h)(Adjointness of Tensor and Hom) Let R, S and T be unital, associative rings. For every element $K^\bullet$ of $\mathrm{Ch}^\bullet(R-S-\mathbf{Bimod})$, for every element $L^\bullet$ of $\mathrm{Ch}^\bullet(S-T-\mathbf{Bimod})$, and for every element $M^\bullet$ of $\mathrm{Ch}^\bullet(R-T-\mathbf{Bimod})$, formulate and **prove** a dg enhanced version of adjointness of tensor product and Hom,

$$\mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(R-T-\mathbf{Bimod})}(K^\bullet \otimes_S L^\bullet, M^\bullet) \cong \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(R-S-\mathbf{Bimod})}(K^\bullet, \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathbf{Mod}-T)}(L^\bullet, M^\bullet)).$$

In particular, setting R, S and T all equal to $\mathbb{Z}$, setting $K^\bullet$ equal to $\mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(B^\bullet, C^\bullet)$, setting $L^\bullet$ equal to $\mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, B^\bullet)$, and setting $M^\bullet$ equal to $\mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet, C^\bullet)$, associated to the composition,

$$\circ_{A,B,C} : K^\bullet \otimes_{\mathbb{Z}} L^\bullet \to M^\bullet,$$

there is an adjoint morphism of dg modules,

$$h^A(B, C) : \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathcal{A})}(B^\bullet, C^\bullet) \to \mathrm{Hom}^\bullet_{\mathrm{Ch}^\bullet(\mathbb{Z}-\mathbf{mod})}(h^A(B), h^A(C)).$$

Prove that this gives another approach (perhaps the same approach, actually) to the dg enhancement of the covariant Yoneda functor h^A. Formulate and **prove** an analogue for the dg enhancement of the contravariant Yoneda functor h_A.

(i) Read more about dg categories, e.g., in the lecture notes of Bernhard Keller or Bertrand Töen. Read about the homotopy category of a dg category, and then read about the "derived category", i.e., the homotopy category of the associated dg category of dg modules over the original dg category. Read the definition of a triangulated dg category, i.e., a dg category $\mathcal{T}$ such that for the derived category of the opposite category of $\mathcal{T}$, every compact object is quasi-representable; an object F is *compact* if the covariant Yoneda functor $h^F(-)$ commutes with small direct sums, and a dg module F on a dg category $\mathcal{T}$ is *quasi-representable* if F is isomorphic in the derived category (homotopy category of the dg category of dg modules on $\mathcal{T}$) to the covariant Yoneda functor h^A of an object A in the homotopy category of $\mathcal{T}$. Read about the functorial construction of cones in every triangulated dge category. (This is one of many reasons that many modern authors choose to work with triangulated dg categories rather than arbitrary triangulated categories.)

Problem 3.($K(\mathcal{A})$ is a Triangulated Category)

(a)(TR1) For every object $A^\bullet$ of $\mathrm{Ch}^\bullet(\mathcal{A})$, since the identity morphism on $C(\mathrm{Id}_A)$ equals $q_{\mathrm{Id}_A} \circ t_{\mathrm{Id}_A} + s_{\mathrm{Id}_A} \circ \delta_{\mathrm{Id}_A}$, **prove** that $(s_f^n \circ t_f^n : C(\mathrm{Id}_A)^n \to C(\mathrm{Id}_A)^{n-1}$ is a nullhomotopy of the identity morphism. Conclude that $C(\mathrm{Id}_A)$ is homotopy equivalent to the zero object. Also, **prove** that this nullhomotopy and homotopy equivalence with zero is functorial in $A^\bullet$.

(b)(TR2) Let $f^\bullet : A^\bullet \to B^\bullet$ be a morphism in $\mathrm{Ch}^\bullet(\mathcal{A})$. Since $\delta_f \circ q_f$ is the zero morphism, in particular δ_f is a left homotopy annihilator of q_f via the zero homotopy. Therefore there exists a unique morphism

$$\lambda_f^\bullet : C^\bullet(q_f) \to A[+1]^\bullet,$$

such that $\lambda_f \circ q_{q_f}$ equals δ_f and such that $\lambda_f \circ s_{q_f}$ equals 0. Similarly, the morphism $-f[+1]^\bullet : A[+1]^\bullet \to B[+1]^\bullet$ together with the homotopy $-s_f[+1]^\bullet$ is a right homotopy annihilator of q_f (recall that $s_f[+1]^n = -s_f^{n+1}$, which is the source of the negative sign). Therefore there exists a unique morphism

$$\rho_f^\bullet : A[+1]^\bullet \to C^\bullet(q_f)$$

such that $\delta_{q_f} \circ \rho_f$ equals $-f[+1]$ and such that $t_{q_f} \circ \rho_f$ equals $-s_f[+1]$. **Prove** that $\lambda_f \circ \rho_f$ equals the identity morphism on $A[+1]$. (It is probably easiest to check this via the embedding theorem and chasing elements; this also explains the sign on $-f[+1]$ above.) For every integer n, define $\widetilde{s}_f^n : C(q_f)^n \to C(q_f)^{n-1}$ as the composition

$$C(q_f)^n \xrightarrow{t_{q_f}^n} C(f)^n \xrightarrow{t_f^n} B^n \xrightarrow{s_{q_f}^n} C(q_f)^{n-1}.$$

Prove that $(\widetilde{s}_f^n)_n$ is a homotopy from $\rho_f \circ \lambda_f$ to $\mathrm{Id}_{C(q_f)}$. Conclude that ρ_f and λ_f are homotopy inverses. Thus, conclude that the translate of the strict triangle C_f is homotopy equivalent to the strict triangle C_{q_f}. Combined with the fact that $C_{f[+1]}$ equals $C_f[+1]$, conclude that $K(\mathcal{A})$ satisfies Axiom (TR2).

(c)(TR3) Use Problem 1(d) to **prove** that $K(\mathcal{A})$ satisfies Axiom (TR3). Axiom (TR4) was verified in lecture. Conclude that $K(\mathcal{A})$ is a triangulated category.

Problem 4.(Distinguished Triangles are Unique up to Non-unique Isomorphism) Let $\mathcal{T}$ be an additive category, let $-[+1] : \mathcal{T} \to \mathcal{T}$ be an additive equivalence of $\mathcal{T}$, and let Δ be a collection of triangles.

(a)(Translation Invariance of Distinguished Triangles) Assuming Axioms (TR1) and (TR2), **prove** that for every triangle Σ,

$$\Sigma : \ A \xrightarrow{f} B \xrightarrow{q} C \xrightarrow{\delta} A[+1],$$

Σ is a distinguished triangle if and only if the triangle $\Sigma[+1]$ is distinguished,

$$\Sigma[+1] : \ A[+1] \xrightarrow{f[+1]} B[+1] \xrightarrow{q[+1]} C[+1] \xrightarrow{\delta[+1]} A[+2].$$

Use induction to **prove** that for every integer n, Σ is a distinguished triangle if and only if the triangle $\Sigma[+n]$ is distinguished,

$$\Sigma[+n] : \ A[+n] \xrightarrow{f[+n]} B[+n] \xrightarrow{q[+n]} C[+n] \xrightarrow{\delta[+n]} A[+n+1].$$

(b)(Distinguished Triangles are "Complexes") Assuming Axioms (TR1) and (TR3), for every distinguished triangle Σ, consider the commutative diagram,

$$\begin{tikzcd}
\mathrm{Id}_A : \arrow[d, "f"'] & A \arrow[r, "\mathrm{Id}_A"] \arrow[d, "\mathrm{Id}_A"] & A \arrow[r, "0^A"] \arrow[d, "f"] & 0 \arrow[r, "0_{A[+1]}"] & A[+1] \arrow[d, "\mathrm{Id}_{A[+1]}"] \\
\Sigma : & A \arrow[r, "f"'] & B \arrow[r, "q"'] & C \arrow[r, "\delta"'] & A[+1]
\end{tikzcd}$$

Prove that there exists a morphism $r : 0 \to C$ such that $q \circ f = r \circ 0^A$. Conclude that $q \circ f$ equals 0. Assuming Axiom (TR2) as well, also **prove** that $\delta \circ q$ and $f[+1] \circ \delta$ equal 0.

(c)(Yoneda Functors are Cohomological Functors) Assuming Axioms (TR1), (TR2) and (TR3), for every distinguished triangle Σ,

$$\Sigma : \ A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta} A[+1],$$

for every object S of $\mathcal{T}$, for the additive Yoneda functor $h_S : \mathcal{T} \to \mathbb{Z}-\mathbf{mod}$, **prove** that the sequence

$$\cdots \to h_S(A[+n]) \xrightarrow{h_S(f[+n])} h_S(B[+n]) \xrightarrow{h_S(q[+n])} h_S(C[+n]) \xrightarrow{h_S(\delta[+n])} h_S(A[+n+1]) \to \dots$$

forms a complex of Abelian groups. Let $s : S \to A[+n+1]$ be a morphism such that $f[+n+1] \circ s$ equals 0. Consider the following commutative diagram,

$$\begin{tikzcd}
\mathrm{Id}_S : \arrow[d, "s"'] & S[-1] \arrow[r, "0^{S[-1]}"] \arrow[d, "s[-1]"] & 0 \arrow[r, "0_S"] \arrow[d, "0_B"] & S \arrow[r, "\mathrm{Id}_S"] & S \arrow[d, "s"] \\
\Sigma[+n] : & A[+n] \arrow[r, "f[+n]"'] & B[+n] \arrow[r, "q[+n]"'] & C[+n] \arrow[r, "\delta[+n]"'] & A[+n+1]
\end{tikzcd}.$$

Prove that there exists a morphism $s' : S \to C[+n]$ such that s equals $\delta[+n] \circ s'$. Conclude that the complex above is everywhere acyclic, i.e., it is a long exact sequence. Prove that every commutative diagram of distinguished triangles gives rise to a commutative diagram of long exact sequences. Conclude that h_S is a *cohomological functor*.

For every object T of $\mathcal{T}$, repeat this argument (or use opposite categories) to conclude that the following sequence is a long exact sequence,

$$\dots \to h^T(A[+n+1]) \xrightarrow{h^T(\delta[+n])} h^T(C[+n]) \xrightarrow{h^T(q[+n])} h^T(B[+n]) \xrightarrow{h^T(f[+n])} h^T(A[+n]) \to \dots$$

In particular, for every morphism $t : B[+n] \to T$ such that $t \circ f[+n]$ equals 0, **prove** that there exists a morphism $t' : C[+n] \to T$ such that t equals $t' \circ q[+n]$.

(d)(Automorphisms of a Distinguished Triangle) Assuming Axioms (TR1), (TR2), and (TR3), for every commutative diagram of distinguished triangle,

$$\begin{tikzcd}
\Sigma: \arrow[d, "u"'] & A \arrow[r, "f"] \arrow[d, "\mathrm{Id}_A"] & B \arrow[r, "q"] \arrow[d, "\mathrm{Id}_B"] & C \arrow[r, "\delta"] \arrow[d, "u"] & A[+1] \arrow[d, "\mathrm{Id}_{A[+1]}"] \\
\Sigma: & A \arrow[r, "f"] & B \arrow[r, "q"] & C \arrow[r, "\delta"] & A[+1]
\end{tikzcd}$$

use the associated commutative diagram of long exact sequences of the Yoneda functor h^C to **prove** that there exists $s : C \to B$ such that $\mathrm{Id}_C = u + q \circ s$. Defining $v = \mathrm{Id}_C + q \circ s = 2\mathrm{Id}_C - u$, **prove** that

$$u \circ v = u + u \circ (q \circ s) = u + (u \circ q) \circ s = u + q \circ s = \mathrm{Id}_C.$$

Thus v is a right inverse of u. Moreover, **prove** that $\delta \circ v = \delta$ and

$$v \circ q = (2\mathrm{Id}_C - u) \circ q = 2q - (u \circ q) = 2q - q = q.$$

Thus, by the same argument with v in place of u, **prove** that there exists $w : C \to C$ such that $v \circ w$ equals Id_C. **Prove** that w equals u. Conclude that v is an inverse of u. Thus u is an isomorphism.

Prove that for every morphism $s : A[+1] \to B$, the morphism $u = \mathrm{Id}_C + q \circ s \circ \delta$ makes the diagram above commute, and it has inverse $v = \mathrm{Id}_C - q \circ s \circ \delta$. For $K(\mathcal{A})$ with its standard triangulated structure, prove that every morphism $u : C(f) \to C(f)$ as above is of the form $\mathrm{Id}_C + q \circ s \circ \delta$ for a morphism $s : A[+1] \to B$.

(e)(Characterization of Direct Sums via Distinguished Triangles) Assuming Axioms (TR1), (TR2) and (TR3), for every distinguished triangle with one term 0,

$$\Sigma : \ A \xrightarrow{f} B \xrightarrow{0^B} 0 \xrightarrow{0_{A[+1]}} A[+1],$$

prove that there exists a morphism $s : B \to A$ such that $f \circ s$ equals Id_B, and **prove** that there exists $t : B \to A$ such that $t \circ f$ equals Id_A. Finally, **prove** that t equals s. Conclude that f is an isomorphism.

(f)(Characterization of Isomorphisms via Distinguished Triangles) Assuming Axioms (TR1), (TR2) and (TR3), for every distinguished triangle extending a zero morphism,

$$\Sigma: \ A \xrightarrow{0} B \xrightarrow{q} C \xrightarrow{\delta} A[+1],$$

prove that there exists a morphism $s : A[+1] \to C$ such that $\delta \circ s$ equals $\mathrm{Id}_{A[+1]}$, and **prove** that there exists $t : C \to B$ such that $t \circ q$ equals Id_B. Use the long exact sequence to prove that the pair $(q : B \to C, s : A[+1] \to C)$ satisfies the universal property of a direct sum. Thus the distinguished triangle above is isomorphic to the following triangle (which, therefore, is a distinguished triangle),

$$A > 0 >> B \xrightarrow{q_B} B \oplus A[+1] \xrightarrow{\pi_{A[+1]}} A[+1].$$

(g)(Split Distinguished Triangles are Direct Sums) Assuming Axioms (TR1), (TR2) and (TR3), for every distinguished triangle,

$$\Sigma: \ A \xrightarrow{f} B \xrightarrow{q} C \xrightarrow{\delta} A[+1],$$

if there exists a morphism $s : A[+1] \to C$ such that $\delta \circ s$ equals $\mathrm{Id}_{A[+1]}$, then **prove** that $f[+1]$ is 0. Conclude that f is 0, and C is a direct sum $B \oplus A[+1]$ as above. Similarly, if there exists a morphism $r : C \to B$ such that $r \circ q$ equals Id_B, again **prove** that f is 0.

(h)(Distinguished Triangles are Unique up to Isomorphism) Now, in addition to (TR1), (TR2), and (TR3), also assume the Octahedral Axiom (TR4): for every triple of distinguished triangles (X, Y, Z', f, q, β), (Y, Z, X', g, r, γ), and $(X, Z, Y', g \circ f, s, \delta)$, there exists a distinguished triangle $(Z', Y', Z', h, t, q[+1] \circ \gamma)$ such that $t \circ s$ equals r, such that $\delta \circ h$ equals β, such that $f \circ \delta$ equals $\gamma \circ t$, and such that $h \circ q$ equals $s \circ g$. For every pair of distinguished triangles,

$$\begin{array}{cccccccc}
\Sigma: & A & \xrightarrow{f} & B & \xrightarrow{q} & C & \xrightarrow{\delta} & A[+1] \\
& \downarrow{\scriptstyle \mathrm{Id}_A} & & \downarrow{\scriptstyle \mathrm{Id}_B} & & & & \\
\Sigma': & A & \xrightarrow{f} & B & \xrightarrow{q'} & C' & \xrightarrow{\delta'} & A[+1]
\end{array},$$

associated to the triple of distinguished triangles (A, B, C, f, q, δ), $(B, B, 0, \mathrm{Id}_B, 0, 0)$, $(A, B, C', f, q', \delta')$, **prove** that there exists a distinguished triangle $(C, C', 0, u, 0, 0)$ such that q' equals $u \circ q$, and such that $\delta' \circ u$ equals δ. Conclude that u is an isomorphism such that there is an isomorphism of distinguished triangles,

$$\begin{array}{cccccccc}
\Sigma: & A & \xrightarrow{f} & B & \xrightarrow{q} & C & \xrightarrow{\delta} & A[+1] \\
{\scriptstyle u}\downarrow & \downarrow{\scriptstyle \mathrm{Id}_A} & & \downarrow{\scriptstyle \mathrm{Id}_B} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \mathrm{Id}_{A[+1]}} \\
\Sigma': & A & \xrightarrow{f} & B & \xrightarrow{q'} & C' & \xrightarrow{\delta'} & A[+1]
\end{array}$$

(i)(Nonuniqueness of the Isomorphism) As above, assume (TR1), (TR2), (TR3) and (TR4). By combining (d) and (i), for every commutative diagram of distinguished triangles,

$$\begin{array}{lccccccc} \Sigma: & A & \xrightarrow{f} & B & \xrightarrow{q} & C & \xrightarrow{\delta} & A[+1] \\ & \downarrow{\scriptstyle \mathrm{Id}_A} & & \downarrow{\scriptstyle \mathrm{Id}_B} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \mathrm{Id}_{A[+1]}} \\ \Sigma': & A & \xrightarrow{f} & B & \xrightarrow{q'} & C' & \xrightarrow{\delta'} & A[+1] \end{array}$$

prove that u is an isomorphism. As in (d), conclude that the isomorphism u may be nonunique.

Although the morphism may be nonunique, there are some constraints. For instance, for every commutative diagram of distinguished triangles,

$$\begin{array}{lccccccc} \Sigma: & A & \xrightarrow{f} & B & \xrightarrow{q} & C & \xrightarrow{\delta} & A[+1] \\ & \downarrow{\scriptstyle u_A} & & \downarrow{\scriptstyle u_B} & & \downarrow{\scriptstyle \mathrm{Id}_C} & & \downarrow{\scriptstyle u_A[+1]} \\ \Sigma': & A' & \xrightarrow{f'} & B' & \xrightarrow{q'} & C & \xrightarrow{\delta'} & A'[+1] \end{array}$$

if $u_C : C \to C$ is any other morphism that makes the diagram commute, then **prove** that $\delta' \circ u_C = u_A[+1] \circ \delta = \delta'$ and $u_C \circ q = q' \circ u_B = q$. Conclude that there is a commutative diagram of distinguished triangles,

$$\begin{array}{lccccccc} \Sigma: & A & \xrightarrow{f} & B & \xrightarrow{q} & C & \xrightarrow{\delta} & A[+1] \\ & \downarrow{\scriptstyle \mathrm{Id}_A} & & \downarrow{\scriptstyle \mathrm{Id}_B} & & \downarrow{\scriptstyle u_C} & & \downarrow{\scriptstyle \mathrm{Id}_{A[+1]}} \\ \Sigma: & A & \xrightarrow{f} & B & \xrightarrow{q} & C & \xrightarrow{\delta} & A[+1] \end{array}$$

Now use (d) to **prove** that u_C is an isomorphism.

Problem 5.(Verdier's Nine Diagram) Let $(\mathcal{T}, -[+1], \Delta)$ be a triangulated category. Let

$$\begin{array}{ccccccc} A'' & \xrightarrow{f''} & B'' & \xrightarrow{q''} & C'' & \xrightarrow{\delta''} & A''[+1] \\ {\scriptstyle u_A}\downarrow & & \downarrow{\scriptstyle u_B} & & & & \downarrow{\scriptstyle u_A[+1]} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{q'} & C' & \xrightarrow{\delta'} & A'[+1] \\ {\scriptstyle r_A}\downarrow & & \downarrow{\scriptstyle r_B} & & & & \downarrow{\scriptstyle r_A[+1]} \\ A & & B & & & & A[+1] \\ {\scriptstyle \gamma_A}\downarrow & & \downarrow{\scriptstyle \gamma_B} & & & & \downarrow{\scriptstyle \gamma_A[+1]} \\ A''[+1] & \xrightarrow{f''[+1]} & B''[+1] & \xrightarrow{q''[+1]} & C''[+1] & \xrightarrow{\delta''[+1]} & A''[+2] \end{array}$$

be a commutative diagram of morphisms in $\mathcal{T}$ whose rows and columns are distinguished. Denote by $\widetilde{f} : A'' \to B'$ the common composition, $u_B \circ f'' = f' \circ u_A$. Denote by $\widehat{q} : B \to C''[+1]$ the composition $q''[+1] \circ \gamma_B$, and denote by $\overline{\delta} : C' \to A[+1]$ the composition $r_A[+1] \circ \delta'$.

(a) Use Axiom (TR2) to **prove** that there exists a distinguished triangle,

$$A'' \xrightarrow{\widetilde{f}} B' \xrightarrow{\widetilde{q}} D \xrightarrow{\widetilde{\delta}} A''[+1].$$

Associated to the triple of distinguished triangles, $(A'', B'', C'', f'', q'', \delta'')$, $(B'', B', B, u_B, r_B, \gamma_B)$ and $(A'', B', D, \widetilde{f}, \widetilde{q}, \widetilde{\delta})$, by Axiom (TR4), **prove** that there exists a distinguished triangle $(C'', D, B, \widehat{\delta}, \widehat{f}, \widehat{q})$ such that $\widetilde{\delta} \circ \widehat{\delta}$ equals δ'', such that $\gamma_B \circ \widehat{f}$ equals $f''[+1] \circ \widetilde{\delta}$, such that $\widehat{\delta} \circ q''$ equals $\widetilde{q} \circ u_B$, and such that $\widehat{f} \circ \widetilde{q}$ equals r_B.

(b) Associated to the triple of distinguished triangles, $(A'', A', A, u_A, r_A, \gamma_A)$, $(A', B', C', f', q', \delta')$ and $(A'', B', D, \widetilde{f}, \widetilde{q}, \widetilde{\delta})$, by Axiom (TR4), **prove** that there exists a distinguished triangle $(A, D, C', \overline{f}, \overline{q}, \overline{\delta})$ such that $\overline{q} \circ \widetilde{q}$ equals q', such that $\widetilde{q} \circ f'$ equals $\overline{f} \circ r_A$, such that $\delta' \circ \overline{q}$ equals $u_A[+1] \circ \widetilde{\delta}$, and such that $\gamma_A \circ \overline{f}$ equals $\widetilde{\delta}$.

(c) Define $f : A \to B$ to be $\widehat{f} \circ f$, and define $u_C : C'' \to C'$ to be $\overline{q} \circ \widehat{\delta}$. **Prove** that $\gamma_B \circ f$ equals $f''[+1] \circ \gamma_A$. **Prove** that $f \circ r_A$ equals $r_B \circ f'$. **Prove** that $\delta' \circ u_C$ equals $u_A[+1] \circ \delta''$. **Prove** that $u_C \circ q''$ equals $q' \circ u_B$.

(d) Use Axiom (TR2) to **prove** that there exists a distinguished triangle,

$$A \xrightarrow{f} B \xrightarrow{q} C \xrightarrow{\delta} A[+1].$$

Associated to the triple of distinguished triangles, $(A, D, C', \overline{f}, \overline{q}, \overline{\delta})$, $(D, B, C''[+1], \widehat{f}, \widehat{q}, -\widehat{\delta}[+1])$ and (A, B, C, f, q, δ), by Axiom (TR4), **prove** that there exists a distinguished triangle

$$(C', C, C''[+1], r_C, \gamma_C, -u_C[+1]),$$

such that $\gamma_C \circ q$ equals $\widehat{q}$, i.e., $q''[+1] \circ \gamma_B$, such that $\delta \circ r_C$ equals $\overline{\delta}$, i.e., $r_A[+1] \circ \delta'$, and such that $\overline{f}[+1] \circ \delta$ equals $-\widehat{\delta}[+1] \circ \gamma_C$. Conclude that

$$\gamma_A[+1] \circ \delta = \widetilde{\delta}[+1] \circ \overline{f}[+1] \circ \delta = -\widetilde{\delta}[+1] \circ \widehat{\delta}[+1] \circ \gamma_C = -\delta''[+1] \circ \gamma_C.$$

Conclude that the commutative diagram extends to a diagram

$$\begin{tikzcd}
A'' \arrow[r, "f''"] \arrow[d, "u_A"'] & B'' \arrow[r, "q''"] \arrow[d, "u_B"] & C'' \arrow[r, "\delta''"] \arrow[d, "u_C"] & A''[+1] \arrow[d, "u_A[+1]"] \\
A' \arrow[r, "f'"] \arrow[d, "r_A"'] & B' \arrow[r, "q'"] \arrow[d, "r_B"] & C' \arrow[r, "\delta'"] \arrow[d, "r_C"] & A'[+1] \arrow[d, "r_A[+1]"] \\
A \arrow[r, "f"] \arrow[d, "\gamma_A"'] & B \arrow[r, "q"] \arrow[d, "\gamma_B"] & C \arrow[r, "\delta"] \arrow[d, "\gamma_C"] & A[+1] \arrow[d, "\gamma_A[+1]"] \\
A''[+1] \arrow[r, "f''[+1]"] & B''[+1] \arrow[r, "q''[+1]"] & C''[+1] \arrow[r, "\delta''[+1]"] & A''[+2]
\end{tikzcd}$$

such that every row and every column is a distinguished triangle, and such that every small square commutes except in the bottom right, which anticommutes.

(e) As a special case, let

$$\begin{tikzcd}
A'' \arrow[r, "f''"] \arrow[d, "u_A"'] & B'' \arrow[r, "q''"] \arrow[d, "u_B"] & C'' \arrow[r, "\delta''"] & A''[+1] \arrow[d, "u_A[+1]"] \\
A' \arrow[r, "f'"] \arrow[d, "r_A"'] & B' \arrow[r, "q'"] \arrow[d, "r_B"] & C' \arrow[r, "\delta'"] & A'[+1] \arrow[d, "r_A[+1]"] \\
A \arrow[d, "\gamma_A"'] & B \arrow[d, "\gamma_B"] & & A[+1] \arrow[d, "\gamma_A[+1]"] \\
A''[+1] \arrow[r, "f''[+1]"] & B''[+1] \arrow[r, "q''[+1]"] & C''[+1] \arrow[r, "\delta''[+1]"] & A''[+2]
\end{tikzcd}$$

be a commutative diagram of morphisms in $\mathcal{T}$ whose rows and columns are distinguished. Assume, moreover, that there exists an *isomorphism* $v : C'' \to C'$ completing the first two rows to a commutative diagram, i.e., $v \circ q''$ equals $q' \circ u_B$ and $u_A[+1] \circ \delta''$ equals $\delta' \circ v$. Use Problem 4(i) to **prove** that in the nine diagram above,

$$\begin{tikzcd}
A'' \arrow[r, "f''"] \arrow[d, "u_A"'] & B'' \arrow[r, "q''"] \arrow[d, "u_B"] & C'' \arrow[r, "\delta''"] \arrow[d, "u_C"] & A''[+1] \arrow[d, "u_A[+1]"] \\
A' \arrow[r, "f'"] \arrow[d, "r_A"'] & B' \arrow[r, "q'"] \arrow[d, "r_B"] & C' \arrow[r, "\delta'"] \arrow[d, "r_C"] & A'[+1] \arrow[d, "r_A[+1]"] \\
A \arrow[r, "f"] \arrow[d, "\gamma_A"'] & B \arrow[r, "q"] \arrow[d, "\gamma_B"] & C \arrow[r, "\delta"] \arrow[d, "\gamma_C"] & A[+1] \arrow[d, "\gamma_A[+1]"] \\
A''[+1] \arrow[r, "f''[+1]"] & B''[+1] \arrow[r, "q''[+1]"] & C''[+1] \arrow[r, "\delta''[+1]"] & A''[+2]
\end{tikzcd}$$

the morphism u_C is an isomorphism. Use this to **prove** that also f is an isomorphism.

Problem 6.(Cohomological Functors, Full Triangulated Subcategories, and Multiplicative Systems) Let $(\mathcal{T}, -[+1], \Delta)$ be a triangulated category. A *full triangulated subcategory* is a full subcategory $\mathcal{T}'$ of $\mathcal{T}$ that is an additive subcategory, that is mapped to itself by $-[+n]$ for every integer n, and such that for every distinguished triangle of $\mathcal{T}$, if two of the objects are in $\mathcal{T}'$, then also the third object is in $\mathcal{T}'$. Let $\mathcal{A}$ be an Abelian category. Let $H : \mathcal{T} \to \mathcal{A}$ be a *cohomological functor*, i.e., an additive functor such that for every distinguished triangle Σ,

$$\Sigma : \ A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta} A[+1],$$

the following sequence is a long exact sequence,

$$\dots \to H(A[+n]) \xrightarrow{H(f[+n])} H(B[+n]) \xrightarrow{H(q[+n])} H(C[+n]) \xrightarrow{H(\delta[+n])} H(A[+n+1]) \to \dots$$

Define $\mathcal{T}_H$ to be the full subcategory of $\mathcal{T}$ consisting of those objects A such that for every integer n, $H(A[+n])$ is a zero object. A collection Σ of morphisms of a category $\mathcal{C}$ is a (left-right) *multiplicative system* if all of the following hold.

(F1) Every identity morphism is in Σ.

(F2) The composition of every pair of (composable) morphisms in Σ is a morphism in Σ.

(F3L) For every morphism $q : B \to C$ in Σ and for every morphism $\gamma_B : B \to B''$, there exists a morphism $q'' : B'' \to C''$ in Σ and a morphism $\gamma_C : C \to C''$ such that $\gamma_C \circ q$ equals $q'' \circ \gamma_B$.

(F3R) For every morphism $q : B \to C$ in Σ and for every morphism $r_C : C' \to C$, there exists a morphism $q' : B' \to C'$ in Σ and a morphism $r_B : B' \to B$ such that $q \circ r_B$ equals $r_C \circ q'$.

(F4L) For every pair of morphisms $f, g : S \to B$ and for every morphism $q : B \to C$ in Σ with $q \circ f$ equal to $q \circ g$, there exists a morphism $r : R \to S$ in Σ such that $f \circ r$ equals $g \circ r$,

(F4R) For every pair of morphisms $f, g : S \to B$ and for every morphism $r : R \to S$ in Σ with $f \circ r$ equal to $g \circ r$, there exists a morphism $q : B \to C$ in Σ such that $q \circ f$ equals $q \circ g$.

(a)(The Kernel of a Cohomological Functor is a Full Triangulated Subcategory) **Prove** that every object of $\mathcal{T}$ that is isomorphic to an object of $\mathcal{T}_H$ is an object of $\mathcal{T}_H$. **Prove** that $\mathcal{T}_H$ is mapped to itself by $-[+n]$ for every integer n. For every distinguished triangle Σ, if two of (A, B, C) are objects of $\mathcal{T}_H$, **prove** that the remaining object is an object of $\mathcal{T}_H$. Conclude that $\mathcal{T}_H$ is a full triangulated subcategory of $\mathcal{T}$.

(b)(Full Triangulated Subcategories are Triangulated Subcategories) For every full triangulated subcategory $\mathcal{T}'$ of $\mathcal{T}$, **prove** that an object of $\mathcal{T}$ is an object of $\mathcal{T}'$ if and only if it is isomorphic to an object of $\mathcal{T}'$. Also **prove** that there is a unique functor $-[+n] : \mathcal{T}' \to \mathcal{T}'$ and a unique family of triangles Δ' on $\mathcal{T}'$ such that the full embedding $\mathcal{T}' \to \mathcal{T}$ commutes with $-[+n]$ and sends Δ' to Δ. In particular, conclude that $(\mathcal{T}', -[+1], \Delta')$ is a triangulated category.

(d)(Examples of Full Triangulated Subcategories) For every Abelian category $\mathcal{A}$, **prove** that $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ are full triangulated subcategories of $K(\mathcal{A})$. For every Abelian category $\mathcal{A}$ and for every full Abelian subcategory $\mathcal{B} \subset \mathcal{A}$ such that the full embedding is exact, prove that $K(\mathcal{B})$ is a full triangulated subcategory of $K(\mathcal{A})$.

(e)(Sorites of Full Triangulated Subcategories) If $\mathcal{T}'$ is a full triangulated subcategory of $\mathcal{T}$, and if $\mathcal{T}''$ is a full triangulated subcategory of $\mathcal{T}$, then $\mathcal{T}''$ is a full triangulated subcategory of $\mathcal{T}$. For every collection $(\mathcal{T}_i')_{i \in I}$ of full triangulated subcategories of $\mathcal{T}$, **prove** that the full subcategory $\mathcal{T}'$ of $\mathcal{T}$ of those objects contained in every $\mathcal{T}_i'$ is a full triangulated subcategory of $\mathcal{T}$.

(f)(Orthogonals) For a class $\mathcal{C}$ of objects of $\mathcal{T}$, the *left orthogonal* to $\mathcal{C}$ is the full subcategory of $\mathcal{T}$ of all objects X such that for every object Y of $\mathcal{C}$ and for every integer n, $h^Y(X[+n])$ is a zero object. **Prove** that the left orthogonal is a full triangulated subcategory of $\mathcal{T}$. Similarly, the *right orthogonal* to $\mathcal{C}$ is the full subcategory of $\mathcal{T}$ of all objects X such that for every object Y of $\mathcal{C}$ and for every integer n, $h_Y(X[+n])$ is a zero object. **Prove** that the right orthogonal is a full triangulated subcategory of $\mathcal{T}$. The *left-right orthogonal* is the full subcategory of objects X such that both $h^Y(X[+n])$ and $h_Y(X[+n])$ are zero objects for every object X of $\mathcal{C}$ and for every integer n. **Prove** that the left-right orthogonal is a full triangulated subcategory of $\mathcal{T}$.

(g)(The Multiplicative System of a Full Triangulated Subcategory) For every full triangulated subcategory $\mathcal{T}'$ of $\mathcal{T}$, define $\Sigma = \Sigma_{\mathcal{T}'}$ to be the collection of all morphisms of $\mathcal{T}$, $q : B \to C$, such that there exists a distinguished triangle,

$$A \xrightarrow{f} B \xrightarrow{q} C \xrightarrow{\delta} A[+1]$$

with A and object of $\mathcal{T}'$. Using Axioms (TR1) and (TR2), **prove** that identity morphism is in Σ. Using Axiom (TR4) particularly, **prove** that if $q : B \to C$ and $r : C \to D$ are in Σ, then also $r \circ q : B \to D$ is in Σ. For every morphism $q : B \to C$ in Σ and for every morphism $\gamma_B : B \to B''$, use Axioms (TR1) and (TR3) to **prove** that there exists a morphism $q'' : B'' \to C''$ in Σ and a morphism $\gamma_C : C \to C''$ such that $q'' \circ \gamma_B$ equals $\gamma_C \circ q$. Similarly, for every morphism $r_C : C' \to C$, **prove** that there exists a morphism $q' : B' \to C'$ in Σ and a morphism $r_B : B' \to B$ such that $q \circ r_B$ equals $r_C \circ q'$. Finally, for every morphism $q : B \to C$ and for every morphism $h : S \to B$ such that $q \circ h$ equals 0, **prove** that there exists a morphism $\eta : S \to A$ such that h equals $f \circ \eta$. Use Axioms (TR1) and (TR2) to **prove** that there exists a distinguished triangle

$$A[-1] \xrightarrow{\phi} R \xrightarrow{r} S \xrightarrow{\gamma} A.$$

Conclude that r is a morphism in Σ such that $h \circ r$ equals 0. Conversely, for every morphism $r : R \to S$ in Σ and for every morphism $h : S \to B$ such that $h \circ r$ equals 0, **prove** that there exists a morphism $q : B \to C$ in Σ such that $q \circ h$ equals 0. Finally, conclude that Σ is a (left-right) multiplicative system of morphisms in $\mathcal{T}$.

Moreover, prove the following additional properties. Using Axioms (TR1), (TR2) and (TR3) and Problem 4(f), **prove** that Σ is *isomorphism closed*, i.e., all isomorphisms in $\mathcal{T}$ are in Σ. Use Axiom (TR2) to **prove** that Σ is *translation invariant*, i.e., a morphism $q : B \to C$ is in Σ if and only if $q[+1] : B[+1] \to C[+1]$ is in Σ. Finally, use Axiom (TR4) to **prove** that Σ is *left cancellative*, resp. *right cancellative*, i.e., for morphisms $q : B \to C$ and $r : C \to D$ if q and $r \circ q$ are in Σ, resp. if r and $r \circ q$ are in Σ, then also r is in Σ, resp. also q is in Σ. Finally, use Axiom (TR4) and Problem 5(e) to **prove** the following *distinguished* property. For every commutative diagram with distinguished rows,

$$\begin{array}{ccccccc}
B'' & \xrightarrow{u_B} & B & \xrightarrow{r_B} & D & \xrightarrow{\gamma_B} & B''[+1] \\
\downarrow{\scriptstyle q''} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \mathrm{Id}_D} & & \downarrow{\scriptstyle q''[+1]} \\
C'' & \xrightarrow{u_C} & C & \xrightarrow{r_C} & D & \xrightarrow{\gamma_C} & C''[+1]
\end{array}$$

q'' is in Σ if and only if q is in Σ.

(h)(The Full Triangulated Subcategory of a Multiplicative System) Let Σ be a (left-right) multiplicative system that is invertible saturated, that is translation invariant, and that is left-right cancellative. Define $\mathcal{T}' = \mathcal{T}'_\Sigma$ to be the full subcategory of all objects A of $\mathcal{T}$ such that there exists a distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{q} C \xrightarrow{\delta} A[+1]$$

with q in Σ. Prove that every zero object is in $\mathcal{T}'$. Prove that every object isomorphic to an object of $\mathcal{T}'$ is in $\mathcal{T}'$. Prove that an object is in $\mathcal{T}'$ if and only if its translates are in $\mathcal{T}'$. Finally, use Axiom (TR4) together with (F2) and left-right cancellativity to prove that $\mathcal{T}'$ is a full triangulated subcategory.

(i) For every full triangulated subcategory $\mathcal{T}'$, for the associated multiplicative system $\Sigma_{\mathcal{T}'}$, **prove** that $\mathcal{T}'_\Sigma$ equals $\mathcal{T}'$. Conversely, let Σ be a multiplicative system that satisfies the distinguished property. Let $\mathcal{T}'_\Sigma$ be the associated full triangulated subcategory. For every distinguished triangle

$$A \xrightarrow{f'} B' \xrightarrow{q'} C' \xrightarrow{\delta'} A[+1]$$

with q in Σ, for every distinguished triangle,

$$A \xrightarrow{f''} B'' \xrightarrow{q''} C'' \xrightarrow{\delta''} A[+1],$$

use Axiom (TR1) to find a distinguished diagram,

$$A \xrightarrow{(f'',f')} B'' \oplus B' \xrightarrow{q} C \xrightarrow{\delta} A[+1],.$$

Then use Axiom (TR3) to find a commutative diagram of distinguished triangles,

$$\begin{tikzcd}
A \arrow[r, "f''"] \arrow[d, "\mathrm{Id}_A"'] & B'' \arrow[r, "q''"] \arrow[d, "q_{B''}"] & C'' \arrow[r, "\delta''"] \arrow[d, "r_C"] & A[+1] \arrow[d, "\mathrm{Id}_{A[+1]}"] \\
A \arrow[r, "(f'',f')"] & B'' \oplus B' \arrow[r, "q"] & C \arrow[r, "\delta"] & A[+1],
\end{tikzcd}$$

Use the distinguished property to **prove** that q is in Σ. Next, use Axiom (TR3) once more to find a commutative diagram of distinguished triangles,

$$\begin{tikzcd}
A \arrow[r, "f'"] \arrow[d, "\mathrm{Id}_A"'] & B' \arrow[r, "q'"] \arrow[d, "q_{B'}"] & C' \arrow[r, "\delta'"] \arrow[d, "u_C"] & A[+1] \arrow[d, "\mathrm{Id}_{A[+1]}"] \\
A \arrow[r, "(f'',f')"] & B'' \oplus B' \arrow[r, "q"] & C \arrow[r, "\delta"] & A[+1],
\end{tikzcd}$$

Use the distinguished property once more to **prove** that q' is in Σ. Conclude that the multiplicative system $\Sigma_{\mathcal{T}'}$ equals Σ.