Jason Starr
Fall 2015

MAT 536 Problem Set 1

Homework Policy. Please read through all the problems. Please solve 5 of the problems. I will be happy to discuss the solutions during office hours.

Problems.

Problem 1. Let R and S be unital, associative rings. Let $\phi : F_1 \to F_0$ be a (left) R-module homomorphism of finite free R-modules. Let N be an $R-S$-bimodule. Denote by $\phi_N^\dagger : \mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)$ the adjoint morphism of (right) S-modules. Assuming that $\mathrm{Coker}(\phi)$ is finite free as a (left) R-module, prove that $\mathrm{Coker}(\phi_N^\dagger)$ is isomorphic as a (right) S-module to a direct sum of finitely many copies of N. In particular, if N is torsion-free as an S-module, then also $\mathrm{Coker}(\phi_N^\dagger)$ is torsion-free as an S-module.

Problem 2. Let $\mathcal{K}$ be a finite simplicial complex. Let

$$C_\bullet(\mathcal{K}) = ((C_k)_{k=0,1,\dots}, (d_k : C_k \to C_{k-1})_{k=1,2,\dots})$$

be the associated chain complex of simplicial chains with $\mathbb{Z}$-coefficients. For every Abelian group N, let

$$C^\bullet(\mathcal{K}; N) = ((C_N^k = \mathrm{Hom}(C_k, N))_{k=0,1,\dots}, (d_{k+1,N}^\dagger : C_N^k \to C_N^{k+1})_{k=0,1,\dots})$$

be the associated cochain complex of simplicial cochains with N-coefficients.

(i) For every connected component a of the underlying topological space $|\mathcal{K}|$, define $q_a : C_0 \to \mathbb{Z}\mathbf{e}_a$ to be the group homomorphism that sends every 0-simplex to $1\cdot\mathbf{e}_a$ if the 0-simplex is contained in the component a, and otherwise the 0-simplex maps to 0. Prove that $q_a \circ d_1$ is the zero homomorphism.

(ii) Since $\mathcal{K}$ is a finite simplicial complex, also the set $\pi_0(|\mathcal{K}|)$ of connected components a of $|\mathcal{K}|$ is a finite set. Prove that the induced homomorphism,

$$q : C_0 \to \prod_{a \in \pi_0(|\mathcal{K}|)} \mathbb{Z}\mathbf{e}_a,$$

is a cokernel of d_1.

(iii) Use **Problem 1** to conclude that the cokernel of the map d_N^0 is isomorphic to a direct sum of finitely many copies of N. Finally, let N be a (right) S-module for some unital, associative ring S, and assume that N is torsion-free as a (right) S-module. Conclude that the simplicial cohomology $H^1(\mathcal{K}, N) = \mathrm{Ker}(d_N^1)/\mathrm{Im}(d_N^0)$ is torsion-free as a (right) S-module.

Problem 3. For every integer $r \geq 1$, find an example of a finite simplicial complex $\mathcal{K}$ such that the simplicial homology $H_r(\mathcal{K}, \mathbb{Z})$ has nonzero torsion, yet $H^r(\mathcal{K}, \mathbb{Z})$ is torsion-free. What can you conclude about $H^{r+1}(\mathcal{K}, \mathbb{Z})$?

Problem 4. Let R be a unital, associative ring. Let M be a right R-module and let N be a left R-module. Let $i : M \times N \to F$ be the free Abelian group on the set $M \times N$. Let K be the subgroup of F generated by all elements of the form

$$i(m+m', n) - i(m, n) - i(m', n),\ i(m, n+n') - i(m, n) - i(m, n'),\ i(mr, n) - i(m, rn),$$

for all elements $m, m' \in M$, all elements $n, n' \in N$, and all elements $r \in R$. Denote by $q : F \to T$ the quotient group associated to the subgroup $K \leq F$. Denote by $\beta : M \times N \to T$ the composition $q \circ i$.

(i) Prove that $\beta : M \times N \to T$ is R-bilinear.

(ii) Let T' be an Abelian group, and let $\beta' : M \times N \to T'$ be an R-bilinear map. Prove that there exists a unique group homomorphism $u : T \to T'$ such that β' equals $u \circ \beta$. Conclude that $(T, \beta : M \times N \to T)$ is an R-tensor product of M and N.

(iii) Let M_1, M_2 be left R-modules, let N_1, N_2 be right R-modules, let $\mu : M_1 \to M_2$ be a homomorphism of left R-modules, and let $\nu : N_1 \to N_2$ be a homomorphism of right R-modules. For $i = 1, 2$ and for $j = 1, 2$, let

$$(M_i \otimes_R N_j, \beta_{M_i,N_j} : M_i \times N_j \to M_i \otimes_R N_j)$$

be an R-tensor product. For $i = 1, 2$, prove that there exists a unique group homomorphism

$$\mathrm{Id}_{M_i} \otimes \nu : M_i \otimes_R N_1 \to M_i \otimes_R N_2$$

such that $(\mathrm{Id}_{M_i} \otimes \nu) \circ \beta_{M_i,N_1}$ equals $\beta_{M_i,N_2} \circ (\mathrm{Id}_{M_i} \times \nu)$. Similarly, for $j = 1, 2$, prove that there exists a unique group homomorphism

$$\mu \otimes \mathrm{Id}_{N_j} : M_1 \otimes_R N_j \to M_2 \otimes_R N_j$$

such that $\left(\mu \otimes textId_{N_j}\right) \circ \beta_{M_1,N_j}$ equals $\beta_{M_2,N_j} \circ (\mu \times \mathrm{Id}_{M_i})$.

(iv) For each left R-module M, prove that the rules above define a covariant additive functor

$$M \otimes_R - : \mathrm{mod} - R \to \mathbb{Z} - \mathrm{mod},\ N \mapsto M \otimes_R N,\ \nu \mapsto \mathrm{Id}_M \otimes \nu.$$

Similarly, for every right R-module N, prove that the rules above define a covariant additive functor

$$- \otimes_R N : R - \mathrm{mod} \to \mathbb{Z} - \mathrm{mod},\ M \mapsto M \otimes_R N,\ \mu \mapsto \mu \otimes \mathrm{Id}_N.$$

(v) For M_1, M_2, N_1, and N_2 as above, prove that the following two compositions are equals

$$M_1 \otimes_R N_1 \xrightarrow{\mu \otimes \mathrm{Id}_{N_2}} M_2 \otimes_R N_1 \xrightarrow{\mathrm{Id}_{M_2} \otimes \nu} M_2 \otimes_R N_2,$$

$$M_1 \otimes_R N_1 \xrightarrow{\mathrm{Id}_{M_1}\otimes\nu} M_1 \otimes_R N_2 \xrightarrow{\mu\otimes\mathrm{Id}_{N_2}} M_2 \otimes_R N_2.$$

These two common compositions are usually denote by $\mu \otimes \nu$.

(vi) Prove that $\mathrm{Id}_{M_i} \otimes \nu$ as defined above agrees with the morphism called $\mathrm{Id}_{M_i} \otimes \nu$ previously, and similarly for $\mu \otimes \mathrm{Id}_{N_j}$. In particular, check that $\mathrm{Id}_{M_i} \otimes \mathrm{Id}_{N_j}$ equals $\mathrm{Id}_{M_i \otimes_R N_j}$. For homomorphisms of right R-modules,

$$\mu : M_1 \to M_2,\ \mu' : M_2 \to M_3,$$

and for homomorphisms of left R-modules,

$$\nu : N_1 \to N_2,\ \nu' : N_2 \to N_3,$$

check that $(\mu \circ \mu') \otimes \nu$ equals $(\mu \otimes \nu) \circ (\mu' \otimes \nu)$, and $\mu \otimes (\nu \circ \nu')$ equals $(\mu \otimes \nu) \circ (\mu \otimes \nu')$.

All of the properties above together are what is meant by "bifunctoriality" of the tensor product.

(vii) Finally, for the zero homomorphism $0 : M_1 \to M_2$ and $0 : N_1 \to N_2$, prove that $0 \otimes \nu$ is the zero homomorphism $0 : M_1 \otimes N_1 \to M_2 \otimes N_2$ and $\mu \otimes 0$ is also the zero homomorphism. For homomorphisms of right R-modules,

$$\mu, \mu' : M_1 \to M_2,$$

and for homomorphisms of left R-modules,

$$\nu, \nu' : N_1 \to N_2,$$

prove that $(\mu + \mu') \otimes \nu$ equals $(\mu \otimes \nu) + (\mu' \otimes \nu)$, and $\mu \otimes (\nu + \nu')$ equals $(\mu \otimes \nu) + (\mu \otimes \nu')$. In this sense, tensor product is a biadditive functor.

Problem 5. Let Q, R and S be unital, associative rings. Let M be a $Q - R$-bimodule, and let N be an $R - S$-bimodule. For every $q \in Q$, left multiplication by q on M,

$$\mu_q : M \to M,\ m \mapsto q \cdot m,$$

is a homomorphism of right R-modules. For every $s \in S$, right multiplication by s on N,

$$\nu_s : N \to N,\ n \mapsto n \cdot s,$$

is a homomorphism of left R-modules. Thus, there are associated group homomorphisms,

$$\mu_q \otimes \mathrm{Id}_N : M \otimes_R N \to M \otimes_R N,$$

$$\mathrm{Id}_M \otimes \nu_s : M \otimes_R N \to M \otimes_R N.$$

(i) Prove that these maps define a structure of $Q - S$-bimodule on $M \otimes_R N$.

(ii) Let T' be a $Q-S$-bimodule. Let $\beta' : M \times N \to T'$ be an R-bilinear that is also Q-linear and S-linear, i.e., $\alpha(q \cdot m, n) = q \cdot \alpha(m, n)$ and $\alpha(m, n \cdot s) = \alpha(m, n) \cdot s$. For the unique group homomorphism,

$$u : M \otimes_R N \to T',$$

such that $u \circ \beta$ equals β', prove that u is a homomorphism of $Q-S$-bimodules.

(iii) Finally, assume that R is a unital, associative ring that is also **commutative**. Thus, every left or right R-module has a unique extension to a left-right R-bimodule. For every pair of R-modules, M and N, denote by

$$\sigma_{M,N} : M \times N \to N \times M, \ (m, n) \mapsto (n, m),$$

the evident bijection whose inverse is $\sigma_{N,M}$. For R-tensor products,

$$(M \otimes_R N, \beta_{M,N} : M \times N \to M \otimes_R N), \ (N \otimes_R M, \beta_{N,M} : N \times M \to N \otimes_R M),$$

prove that both $\beta_{N,M} \circ \sigma_{M,N}$ and $\beta_{M,N} \circ \sigma_{N,M}$ are R-bilinear. Hence there are unique left-right R-module homomorphisms,

$$u_{M,N} : M \otimes_R N \to N \otimes_R M, \quad u_{N,M} : N \otimes_R M \to M \otimes_R N,$$

such that $u_{M,N} \circ \beta_{M,N}$ equals $\beta_{N,M} \circ \sigma_{M,N}$, resp. $u_{N,M} \circ \beta_{N,M}$ equals $\beta_{M,N} \circ \sigma_{M,N}^{-1}$. Prove that $u_{M,N}$ and $u_{N,M}$ are inverse isomorphisms of left-right R-modules. Moreover, prove that $u_{M,N}$ is "binatural" in M and N.

(iv) As above, assume that R is commutative. For every natural number n, denote by $M^{\otimes n}$ the iterated tensor product of n copies of M as an R-module, i.e., $M^{\otimes n}$ is defined inductively by,

$$M^{\otimes 0} = R, M^{\otimes 1} = M, M^{\otimes (n+1)} = \left(M^{\otimes n}\right) \otimes_R M.$$

Extend the previous part to define, for every element permutation $\phi \in \mathfrak{S}_n$ a left-right R-module isomorphism,

$$\widetilde{\phi}_M : M^{\otimes n} \to M^{\otimes n}, \ m_1 \otimes \cdots \otimes m_i \otimes \cdots \otimes m_n \mapsto m_{\phi^{-1}(1)} \otimes \cdots \otimes m_{\phi^{-1}(i)} \otimes \cdots \otimes m_{\phi^{-1}(n)}.$$

Prove that this defines an action of the group $\mathfrak{S}_n$ on $M^{\otimes n}$. In particular, for every n, define

$$s : M^{\otimes n} \to \mathrm{Sym}^n_R(M),$$

to be the initial $\mathfrak{S}_n$-invariant left-right R-module homomorphism. Similarly, define

$$a : M^{\otimes n} \to \bigwedge\nolimits_R^n M,$$

to be the initial quotient such that for every $\phi \in \mathfrak{S}_n$, $a(\widetilde{\phi}_M(m \otimes m \otimes m_3 \otimes \cdots \otimes m_n))$ equals 0 for every $m, m_3, \ldots, m_n$ in M. Prove that all of these operations are functorial / natural in M.

Problem 6 Let R be a unital, associative ring. Let

$$N' \xrightarrow{q} N \xrightarrow{p} N'' \to 0,$$

be an exact sequence of left R-modules. Let M be a right R-module. Prove that the associated sequence of Abelian groups,

$$M \otimes_R N' \xrightarrow{\mathrm{Id}_M \otimes q} M \otimes_R N \xrightarrow{\mathrm{Id}_M \otimes p} M \otimes_R N'' \to 0,$$

is also exact. Prove the symmetric result in the second argument. Finally, give an example proving that even for a monomorphism q of left R-modules, the associated morphism $\mathrm{Id}_M \otimes q$ may not be a monomorphism.

Problem 7 Let R be a unital, associative ring. Let Γ and Δ be sets. Let $(i_\Gamma : \Gamma \to F_\Gamma)$, resp. $(i_\Delta : \Delta \to F_\Delta)$, be a free (left-right) R-module on the set Γ, resp. Δ. Let $(i_{\Gamma\times\Delta} : \Gamma \times \Delta \to F_{\Gamma\times\Delta})$ be a free (left-right) R-module on the product set $\Gamma \times \Delta$.

(i) Prove that there exists a unique R-bilinear map

$$\iota_{\Gamma,\Delta} : F_\Gamma \times F_\Delta \to F_{\Gamma\times\Delta}$$

such that $\iota_{\Gamma,\Delta} \circ (i_\Gamma \times i_\Delta)$ equals $i_{\Gamma\times\Delta}$.

(ii) Prove that $(F_{\Gamma\times\Delta}, \iota_{\Gamma,\Delta})$ is an R-tensor product. Thus, for every R-tensor product $(F_\Gamma \otimes_R F_\Delta, \beta_{F_\Gamma,F_\Delta})$ of F_Γ and F_Δ, there exists a unique group homomorphism $\theta_{\Gamma,\Delta} : F_\Gamma \otimes_R F_\Delta \to F_{\Gamma\times\Delta}$ such that $\theta_{\Gamma,\Delta} \circ \beta_{F_\Gamma,F_\Delta}$ equals $\iota_{\Gamma,\Delta}$.

(iii) Let there be given an exact sequence of left R-modules,

$$F_{\Sigma'} \xrightarrow{q_N} F_\Sigma \xrightarrow{p_N} N \to 0,$$

and an exact sequence of right R-modules

$$F_{\Xi'} \xrightarrow{q_M} F_\Xi \xrightarrow{p_M} M \to 0.$$

Use the previous parts to prove that the group homomorphism

$$p_M \otimes p_N : F_{\Sigma\times\Xi} \to M \otimes_R N$$

is an epimorphism whose kernel is the image of the group homomorphism,

$$(q_M \otimes \mathrm{Id}_{F_\Xi}) \oplus (\mathrm{Id}_{F_\Sigma} \otimes q_N) : F_{\Sigma'\times\Xi} \oplus F_{\Sigma\times\Xi'} \to F_{\Sigma\times\Xi}.$$

Thus, given presentations of M and N, this defines a presentation of $M \otimes_R N$.

(iv) Let k be a commutative, unital ring, let V be a finite free k-module, let $V^\vee = \mathrm{Hom}_k(V, k)$ be the dual finite free k-module, and let R be $\mathrm{Hom}_k(V, V)$ with its usual structure of associative, unital k-algebra. Let N be V with its usual left R-module structure. Let M be $V^\vee$ with its usual right R-module structure. Check that the usual pairing

$$\beta : V^\vee \times V \to k,\ \beta(\chi, \vec{v}) = \chi(\vec{v}),$$

is R-bilinear. Let $\vec{v}_0 \in V$ and $\chi_0 \in V^\vee$ be elements such that $\beta(\chi_0, \vec{v}_0)$ equals 1. Denote

$$T_0 : V \to V,\ T_0(\vec{v}) = \vec{v} - \chi_0(\vec{v}) \cdot \vec{v}_0.$$

Define left R-module homomorphisms

$$p_{N,0} : R \to V,\ p_{N,0}(A) = A(\vec{v}_0),$$

$$q_{N,0} : R \to R,\ q_{N,0}(B) = B \circ T_0.$$

Define right R-module homomorphisms

$$p_{0,M} : R \to V^\vee,\ p_{0,M}(A) = \chi_0 \circ A,$$

$$q_{0,M} : R \to R,\ q_{0,M}(B) = T_0 \circ B.$$

Prove that $p_{N,0}$ and $q_{N,0}$ give a presentation of N as a left R-module, and prove that $p_{0,M}$ and $q_{0,M}$ give a presentation of M as a right R-module. Conclude that $N \otimes_R M$ is the cokernel of the homomorphism of Abelian groups,

$$q_{0,M} \oplus q_{N,0} : R \oplus R \to R.$$

Finally, check that the following homomorphism of Abelian groups is a cokernel of $q_{0,M} \oplus q_{N,0}$,

$$\mathrm{ev}_0 : R \to k,\ \mathrm{ev}_0(A) = \chi_0(A(\vec{v}_0)).$$

Conclude that (k, β) is an R-tensor product of $V^\vee$ and V.

Problem 8 Let M' be a group. Let M be a nonempty set (typically, a particular element of M is specified, i.e., M is a pointed set). Let

$$\widetilde{q} : M' \times M \to M,$$

be a left action of K on M. Let M'' be a set, and let

$$p : M \to M'',$$

be a set map that is K-invariant, i.e., the following diagram of set maps commutes,

$$\begin{tikzcd} M' \times M \arrow[r, "\widetilde{q}"] \arrow[d, "\mathrm{pr}_2"'] & M \arrow[d, "p"] \\ M \arrow[r, "p"'] & M'' \end{tikzcd}.$$

(Note, in particular, that M'' is also nonempty, and the image of any specified point of M makes M'' into a pointed set such that p is a map of pointed sets.) There is an associated set map

$$(\widetilde{q}, p_2) : M' \times M \to M \times_{p,M'',p} M,\ (m', m) \mapsto (\widetilde{q}(m', m), m).$$

The datum $(M', M, M'', \widetilde{q}, p)$ is *left exact* if $(\widetilde{q}, p_2)$ is a bijection. The datum is *exact* if it is left exact and p is surjective.

(i) For a triple $(M', M, \widetilde{q})$ as above, prove that there exists an extension to a left exact datum if and only if the action $\widetilde{q}$ of M' on M is a free action. In this case, defining $p_0 : M \to M''_0$ to be the usual quotient of M by the M'-action, prove that the datum $(M', M, M''_0, \widetilde{q}, p_0)$ is a left exact extension that is initial in an appropriate sense, and this datum is exact. Conclude that an exact extension, if it exists, is unique up to unique isomorphism.

(ii) Assume now that M is a group, let $q : M' \to M$ be a group homomorphism, and let $\widetilde{q}$ be the action of M' on M induced by q. Prove that there is a left exact extension of $(M', M, \widetilde{q})$ if and only if q is injective, i.e., a monomorphism, and in this case $q(M')$ equals the fiber of $p^{-1}(p(e))$ containing the identity element e of M (we always consider a group as a pointed set using the identity element). Moreover, in this case, conclude that for the unique exact extension, there exists a unique left action of M on M'' such that p is a morphism of M-sets (where the action of M on itself is the left regular action).

(iii) Again assume that M is a group, with the identity as distinguished point. For a set map $p : M \to M''$, prove that there exists a group homomorphism $q : M' \to M$ such that $(M', M, M'', \widetilde{q}, p)$ is left exact if and only if both the fiber $p^{-1}(p(e))$ of p containing the identity e is a subgroup of M, and the induced map

$$(Q, \mathrm{pr}_2) : p^{-1}(p(e)) \times M \to M \times M,\ (m', m) \mapsto (m' \cdot m, m),$$

has image $M \times_{p,M'',p} M$ (equivalently, every (nonempty) fiber of p is a left $p^{-1}(p(e))$-coset). In this case, prove that for every $q : M' \to M$ with $(M', M, M'', \widetilde{q}, p)$ left exact, q is an isomorphism from M' to $p^{-1}(p(e))$.

(iv) Finally, assume that M', M, M'' are groups. Let $q : M' \to M$ and $p : M \to M''$ be group homomorphisms. Prove that $(M', M, M'', \widetilde{q}, p)$ is left exact if and only if q is an isomorphism to a normal subgroup $q(M')$ of M, and the kernel of p is $q(M')$. Prove that the datum is exact if and only if p is surjective, q is injective, and the image of q equals the kernel of p. Thus, in this case, the usual notions of left exact and exact agree with the notion above. However, the notion above makes sense even when M and M'' are merely (pointed) sets.

Problem 9 Let R be an associative, unital ring. Let M and N be left R-modules. Let $\mathrm{Hom}_{R-\mathrm{mod}}(M, N)$ be the set of homomorphisms of left R-modules.

(i) The zero homomorphism $0 : M \to N$ is a homomorphism of left R-modules, thus defines a distinguished element $0 \in \mathrm{Hom}_{R-\mathrm{mod}}(M, N)$. For every pair $f, g : M \to N$ of homomorphisms of left R-modules, check that both $f + g$ and $f - g$ are also homomorphisms of left R-modules. Thus, $\mathrm{Hom}_{R-\mathrm{mod}}(M, N)$ is a subgroup of the additive group of homomorphisms from M to N.

(ii) Let $\mu : M_1 \to M_2$ and $\nu : N_1 \to N_2$ be homomorphisms of left R-modules. Since composition of homomorphisms of left R-modules is a homomorphism of left R-modules, there are well-defined maps,

$$\mathrm{Hom}_{R-\mathrm{mod}}(M_i, \nu) : \mathrm{Hom}_{R-\mathrm{mod}}(M_i, N_1) \to \mathrm{Hom}_{R-\mathrm{mod}}(M_i, N_2),\ A \mapsto \nu \circ A,$$

$$\mathrm{Hom}_{R-\mathrm{mod}}(\mu, N_j) : \mathrm{Hom}_{R-\mathrm{mod}}(M_2, N_j) \to \mathrm{Hom}_{R-\mathrm{mod}}(M_1, N_j),\ B \mapsto B \circ \mu.$$

Prove that these are both homomorphisms of Abelian groups. Finally, check that the following two compositions are equal,

$$\mathrm{Hom}_{R-\mathrm{mod}}(M_2, N_1) \xrightarrow{\mathrm{Hom}_{R-\mathrm{mod}}(M_2,\nu)} \mathrm{Hom}_{R-\mathrm{mod}}(M_2, N_2) \xrightarrow{\mathrm{Hom}_{R-\mathrm{mod}}(\mu,N_2)} \mathrm{Hom}_{R-\mathrm{mod}}(M_1, N_2),$$

$$\mathrm{Hom}_{R-\mathrm{mod}}(M_2, N_1) \xrightarrow{\mathrm{Hom}_{R-\mathrm{mod}}(\mu,N_1)} \mathrm{Hom}_{R-\mathrm{mod}}(M_1, N_1) \xrightarrow{\mathrm{Hom}_{R-\mathrm{mod}}(M_1,\nu)} \mathrm{Hom}_{R-\mathrm{mod}}(M_1, N_2).$$

These two common compositions are usually denote by $\mathrm{Hom}_{R-\mathrm{mod}}(\mu, \nu)$.

(iii) Let Q, R and S be unital, associative rings. Let M be an $R-Q$-bimodule, and let N be an $R-S$-bimodule. For every $q \in Q$, right multiplication by q on M,

$$\mu_q : M \to M,\ m \mapsto m \cdot q,$$

is a homomorphism of left R-modules. For every $s \in S$, right multiplication by s on N,

$$\nu_s : N \to N,\ n \mapsto n \cdot s,$$

is a homomorphism of left R-modules. Thus, there are associated group homomorphisms,

$$\mathrm{Hom}_{R-\mathrm{mod}}(\mu_q, \mathrm{Id}_N) : \mathrm{Hom}_{R-\mathrm{mod}}(M, N) \to \mathrm{Hom}_{R-\mathrm{mod}}(M, N),$$

$$\mathrm{Hom}_{R-\mathrm{mod}}(\mathrm{Id}_M, \nu_s) : \mathrm{Hom}_{R-\mathrm{mod}}(M, N) \to \mathrm{Hom}_{R-\mathrm{mod}}(M, N).$$

Prove that these maps define a structure of $Q-S$-bimodule on $\mathrm{Hom}_{R-\mathrm{mod}}(M, N)$.

(iv) Repeat all of the above for right R-modules M and N and $\mathrm{Hom}_{\mathrm{mod}-R}(M, N)$.

Problem 10 Let Q, R, S, and T be unital, associative rings. Let M be an $R-Q$-bimodule, let N be an $R-S$-bimodule, and let P be an $S-T$-bimodule. Thus $\mathrm{Hom}_{R-\mathrm{mod}}(M, N)$ is a $Q-S$-bimodule, and $N \otimes_R P$ is an $S-T$-bimodule. Thus $\mathrm{Hom}_{R-\mathrm{mod}}(M, N) \otimes_S P$ and $\mathrm{Hom}_R(M, N \otimes_S P)$ are $Q-T$-bimodules. For a homomorphism of left R-modules,

$$f : M \to N,$$

and for an element $p \in P$, define $f \otimes p$ to be the set map

$$f \otimes p : M \to N \otimes_S P,\ m \mapsto \beta_{N,P}(f(m), p) = f(m) \otimes p.$$

(i) Prove that $f \otimes p$ is a homomorphism of left R-modules. Thus there is a well-defined set map,

$$\gamma_{M,N,P} : \mathrm{Hom}_{R-\mathrm{mod}}(M, N) \times P \to \mathrm{Hom}_{R-\mathrm{mod}}(M, N \otimes_S P).$$

(ii) Prove that $\gamma_{M,N,P}$ is S-bilinear. Thus, there is a unique group homomorphism,

$$u_{M,N,P} : \mathrm{Hom}_{R-\mathrm{mod}}(M, N) \otimes_S P \to \mathrm{Hom}_{R-\mathrm{mod}}(M, N \otimes_S P).$$

(iii) Prove that $u_{M,N,P}$ is a homomorphism of $Q - T$-bimodules.

(iv) In the special case that S equals R, and N equals R as a left-right R-module, $u_{M,R,P}$ is the homomorphism of $Q - T$-bimodules,

$$\mathrm{Hom}_{R-\mathrm{mod}}(M, R) \otimes_R P \to \mathrm{Hom}_{R-\mathrm{mod}}(M, P).$$

Prove that the image of this homomorphism is contained in the subgroup of $\mathrm{Hom}_{R-\mathrm{mod}}(M, P)$ of homomorphisms of left R-modules $M \to P$ that have finitely generated image. Even more particularly, if M equals P as left R-modules, then Id_P is in the image of $u_{P,R,P}$ only if P is a finitely generated left R-module. Thus, $u_{M,R,P}$ is **not** always an isomorphism (unlike many of the natural transformations that arise from manipulating functors).

(v) Continuing the previous part, let S equal R, let N equal R, and let M equal P. Prove that the homomorphism of $Q - T$-bimodules,

$$\mathrm{Hom}_{R-\mathrm{mod}}(P, R) \otimes_R P \to \mathrm{Hom}_{R-\mathrm{mod}}(P, P),$$

is an isomorphism if and only if P is isomorphic as a left R-module to a direct summand of $R^{\oplus n}$ for some integer $n \geq 0$, i.e., P is a finitely generated, projective left R-module.

(vi) Repeat the above for right R-modules.

Problem 11 A left R-module N is **flat** as a left R-module if for every monomorphism of right R-modules,

$$\mu : M' \to M,$$

the induced homomorphism of Abelian groups,

$$\mu \otimes \mathrm{Id}_N : M' \otimes_R N \to M \otimes_R N,$$

is a monomorphism. The definition of flat for a right R-module is similar.

(i) For every free R-module, $i : \Sigma \to F_\Sigma$, prove that F_Σ is flat as both a left R-module and as a right R-module. Conclude that every projective left R-module, resp. right R-module, is flat as left R-module, resp. right R-module.

(ii) Let R be $\mathbb{Z}$, let n be a nonzero, noninvertible integer, and let N be $\mathbb{Z}[1/n] = \mathbb{Z}[x]/\langle nx - 1\rangle$. Let M_n be a $\mathbb{Z}$-module such that multiplication by n is injective on M_n. For $M_n[x] = M_n \otimes_{\mathbb{Z}} \mathbb{Z}[x]$, which contains M_n as the subset of elements of degree 0, for every nonzero element of degree $d \geq 0$,

$$f(x) = m_d x^d + \cdots + m_e x^e + \cdots + m_1 x + m_0,\ m_d \neq 0,$$

prove that $(nx-1)f(x)$ is nonzero of degree $d+1>0$. In particular, conclude that the intersection $M_n \cap (nx-1)M_n[x]$ in $M_n[x]$ is $\{0\}$. Conclude that the composition,

$$M_n \hookrightarrow M_n[x] \to M_n[x]/(nx-1)M_n[x],$$

is injective.

(iii) With notations as above, let M be a $\mathbb{Z}$-module, denote by $M[n^\infty]$ the submodule of M of all elements m such that there exists an integer $e \geq 0$ with $n^e \cdot m = 0$. Denote by M_n the quotient $M/M[n^\infty]$. Prove that multiplication by n is injective on M_n. Prove that $M[n^\infty] \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]$ is the zero module. Conclude that the natural homomorphism,

$$M \otimes_{\mathbb{Z}} \mathbb{Z}[1/n] \to M_n \otimes_{\mathbb{Z}} \mathbb{Z}[1/n],$$

is an isomorphisms. In particular, conclude that the kernel of the natural homomorphism

$$M \to M \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]$$

equals $M[n^\infty]$.

(iv) For every submodule M' of M, prove that $M' \cap M[n^\infty]$ equals $M'[n^\infty]$. Conclude that the natural homomorphism,

$$M' \otimes_{\mathbb{Z}} \mathbb{Z}[1/n] \to M \otimes_{\mathbb{Z}} \mathbb{Z}[1/n],$$

is a monomorphism. Therefore $\mathbb{Z}[1/n]$ is a flat $\mathbb{Z}$-module.

(v) For every free $\mathbb{Z}$-module F_Σ, for every nonzero element f of F_σ, prove that there exists an integer $e > 0$ such that f is not in the image of the homomorphism of multiplication by n^e. Conclude that $\mathbb{Z}[1/n]$ admits only the zero homomorphism to F_Σ. Therefore $\mathbb{Z}[1/n]$ is a flat $\mathbb{Z}$-module that is not projective.