PROBLEM SET 9

(1) Let $F$ be a field and let $A$ be an associative $F$-algebra with 1. Let $S$ be a subset of $A$ which commutes, i.e., for every pair $s, t$ in $S$, $st$ equals $ts$. Let $B$ be the smallest $F$-subalgebra of $B$ which contains $S$ and 1. Prove that $B$ commutes. Deduce the claim from the exercise in the middle of p. 9 of the notes on the spectral theorem.

(2) For the following linearly independent subset of $\mathbb{R}^3$, $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, find the orthonormal basis $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ satisfying the conditions of the Gram-Schmidt theorem.

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 3 \\ -8 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 4 \\ 25 \end{bmatrix}.$$  

(3) Let $n \geq 2$ be an integer, and let $a, b, c$ be integers. Define an $\mathbb{R}$-linear operator, $T_{a,b,c} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T_{a,b,c}(e_i) = \begin{cases} be_1 + ce_2, & i = 1, \\ ae_{i-1} + be_i + ce_{i+1}, & 2 \leq i \leq n - 1, \\ ae_{n-1} + be_n, & i = n \end{cases}$$

Prove that $T_{a,b,c}$ is normal if and only if $a^2 = c^2$. And when $a = c$ and $n = 2, 3$, diagonalize this matrix.

(4) Polar decomposition of normal operators. Let $V$ be a finite dimensional, complex Hermitian space and let $T$ be an invertible, normal operator on $T$. Prove that there exists a unique factorization $T = |T|U$ of $T$ into a product of commuting operators $|T|$ and $U$ on $V$ such that

(i) $|T|$ is a positive operator, i.e., $\langle |T|\vec{v}, \vec{v} \rangle$ is a positive real number for every nonzero $\vec{v}$ in $V$,

(ii) and $U$ is unitary.

Hint. For such a factorization, relate $T^*T$ and $|T|$. Use this to define $|T|$ and then prove the factor $(|T|)^{-1}T$ is unitary.