MAT 401 FINAL PAPER

The final paper constitutes a significant portion of your grade (approximately 20%). It is due Thursday, December 7th. The paper should be approximately 10 pages long. The paper is to be expository in nature: you will learn about a topic of your choosing related to commutative algebra, varieties, or algorithms in algebra. Then you will write a paper in your own words explaining what you have learned, and what you find to be the important and interesting features of your topic. You are encouraged to include statements of important theorems and proofs of some of these, but your paper is not supposed to be a "textbook". It should be more like the notes of an interesting talk.

Below are a list of references and some potential topic areas. You are free to choose any topic from this list, any topic from one of the references, or a topic of your own choosing. To avoid overlap in papers (a little overlap is fine, but no two students should have exactly the same topic), and to make sure that the topic being attempted is attainable in the time remaining, please check with me regarding your choice of topic.

References:

Cox, David, Little, John, and O'Shea, Donal, *Using algebraic geometry.* Springer-Verlag, New York, 1998, Graduate Texts in Mathematics, 185.

Eisenbud, David, *Commutative algebra, with a view towards algebraic geometry.* Springer-Verlag, New York, 1995, Graduate Texts in Mathematics, 150.

Potential Topics:

1. Modules over a polynomial ring. Ideals in a polynomial ring are a special case of modules over a polynomial ring. This topic would consider the abstract theory of modules (the definition of modules, examples, simple operations on modules). It could also discuss the "module description problem" using generators and relations. It could also discuss how one can extend Groebner basis methods to modules to compute simple operations on modules (intersection of submodules, kernels and cokernels of maps between modules, preimage of a submodule under a module map, the annihilator ideal of a module, and tensor products of modules).

2. Blowing up. This topic requires some projective geometry. Given a variety V and a subvariety W, there is a process which produces a new (quasi-projective) variety $\widetilde{V}$ and a polynomial map $\widetilde{V} \to V$ with the properties that

(1) If E denotes the preimage of W in $\widetilde{V}$, then the ideal $\mathbb{I}_{\widetilde{V}}(E)$ is generated by a single equation. We call E the *exceptional divisor*.

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(2) The restricted polynomial map $\widetilde{V} - E \to V - W$ is an isomorphism of (quasi-projective) varieties.

This topic would discuss the definition/construction of the variety $\widetilde{V}$, often called the *blowing up* of V along W. It would give some examples, and it would discuss how one can use Groebner basis techniques to find equations for $\widetilde{V}$ and E in terms of equations for V and W.

3. Modules revisited. This requires some knowledge of topic 1. This topic would consider *graded modules* over a polynomial ring, which are to modules what homogeneous ideals (discussed in chapter 8, section 3) are to usual ideals. This topic could discuss the Hilbert function and Hilbert polynomial of a homogeneous ideal, and how one can use Groebner basis techniques to compute the Hilbert function and Hilbert polynomial of a homogeneous ideal.

4. Other monomial orders. This topic could discuss the special properties, advantages and disadvantages of different monomial orders. In particular, it might discuss the special properties of the graded reverse lexicographical order, and some computational aspects of different monomial orders.

5. (ADVANCED) Groebner bases and flat families. This is an advanced topic and is only suggested for someone who is willing to invest quite a bit of time. Learn what *flatness* means, learn what flat families of ideals/modules are, and learn the relationship between Groebner bases and flat families (essentially learn section 15.8 of Eisenbud's book).

6. Generic initial ideals. This is also advanced, but a little more approachable. What happens to the ideal of leading terms and Groebner basis when we change our monomial ordering by applying a linear change of variables to the variables in the polynomial ring? This question leads to the notion of generic initial ideal and Borel-fixed ideals. This material is discusses in section 15.9 of Eisenbud's book.

7. Primary decomposition. Recall that given a variety $V \subset k^n$, one can find a unique irreducible decomposition $V = V_1 \cup \cdots \cup V_r$. The algebraic translation of this is as follows: given a radical ideal $I \subset k[x_1, \ldots, x_n]$, one can write $I = P_1 \cap \cdots \cap P_r$ for some prime ideals P_i. What happens when we don't assume that I is radical. There is still a decomposition $I = Q_1 \cap \cdots \cap Q_r$ where the Q_i are *primary ideals*. Learn what primary ideals, learn the proof of primary decomposition for ideals, and learn to what extent primary decomposition is or is not unique.

8. Normalization. Given a variety $V \subset k^n$, one can find another variety $\widetilde{V}$ and a polynomial map $f : \widetilde{V} \to V$ which is birational and such that $\widetilde{V}$ is *normal*. This means that if $\frac{p}{q} \in K(\widetilde{V})$ is a rational function on $\widetilde{V}$ which satisfies a monic polynomial

$$x^r + a_1 x^{r-1} + \cdots + a_r \tag{1}$$

for some elements $a_1, \ldots, a_r \in k[\widetilde{V}]$, then in fact $\frac{p}{q}$ equals a polynomial $p' \in k[\widetilde{V}]$. Learn what the normalization is, how to compute it in some simple cases, and algorithms in some other cases. The general algorithm is complicated, but particularly nice are the normalizations of varieties cut out by monomial ideals, so perhaps you can concentrate on these varieties.

9. Free resolutions. This might be called "modules revisited II". A module can be described by generators and relations. But better than this is to give a *free resolution* of the module. Learn what this is, and learn the Hilbert syzygy theorem that explains why such a thing always exists. If you want more, relate the Hilbert polynomial of the module to the terms in the free resolution.

10. Invariants. Suppose that a variety V has a group of symmetries G. For example, you might consider the symmetries of k^2 given by $(x, y) \mapsto (y, x)$. What is the quotient of V by the action of G? What are the G-invariant polynomials on V. Is the quotient an affine variety? Is the ring of G-invariant polynomials a finitely generated ring? If it is, can one find the generators and relations of this ring (i.e. can one find an algorithm for determining the generators and relations)?

11. Zero-dimensional Gorenstein ideals. Gorenstein rings are ubiquitous (to paraphrase Hyman Bass). And zero-dimensional Gorenstein ideals are even simple to understand! Learn what zero-dimensional Gorenstein ideals are. Give some examples. There are many problems related to zero-dimensional Gorenstein ideals. One such problem which was a topic for a summer program last year is the following: what is the maximum of the minimal number of generators for a zero-dimensional Gorenstein ideal as the ideal varies among all zero-dimensional Gorenstein ideals with fixed socle degree. This is a paper that might be a good starting point for a UROP project.

12. Desingularization of plane curves. Suppose that $V \subset k^2$ is a 1-dimensional variety, i.e. a *plane curve*. Then there exists an algebraic variety $\widetilde{V}$ and a polynomial map $f : \widetilde{V} \to V$ which is birational and such that $\widetilde{V}$ is a nonsingular variety. What is this desingularization? How do we know it exists? Can we compute it? What does this topic have to do with topics 2 and 8?

13. Rational normal curves. Recall the twisted cubic curve $V \subset k^3$ is the subvariety with the parametrization (t, t^2, t^3). This variety has degree 3, the minimal possible degree for an irreducible 1-dimensional variety whose span is all of k^3. The generalization of the twisted cubic curve to k^n is know as the rational normal curve. Learn what rational normal curves are, parametrizations for rational normal curves, implicit forms, and special properties of these curves (e.g. that they are varieties of minimal degree as above).

14. Determinantal varieties. Suppose the one has an $m \times n$ matrix whose entries are variables $x_1, \ldots, x_{mn}$. For each integer $r = 1, \ldots, n$ (assuming $m \geq n$) one can consider the subvariety $D_r \subset k^{mn}$ whose ideal is generated by the $r \times r$ minors of this matrix (i.e. the determinants of all $r \times r$ submatrices obtained by choosing r columns and r rows form the matrix). Such a variety is called a determinantal variety. Learn about such varieties, what their properties are, their dimension, where they are singular, how to understand their tangent spaces, etc.

15. Groebner bases over $\mathbb{Z}$. Learn what can and can't be done with Groebner basis techniques for the ring $\mathbb{Z}[x_1, \ldots, x_n]$.

16. Factoring polynomials. Learn about algorithms for factoring polynomials over the integers, and learn Berlekamp's algorithm for factoring polynomials. This isn't as closely related to the material in the textbook as the topic above, but it should be alot of fun.

17. Non-reduced ring. The coordinate ring of a variety $V \subset k^n$ is always of the form $k[x_1, \ldots, x_n]/I$ where I is a radical ideal. What happens when I isn't radical? How should one think of the quotient ring? Is there a geometric object associated to such a ring which is V in the special case that I is radical? What is an affine scheme?

18. Prime spectrum. Learn about the prime spectrum of a ring $k[x_1, \ldots, x_n]/I$ and learn about the Zariski topology on the prime spectrum.

Of course you are also free (and encouraged) to pursue a topic of your own choosing not from the list above.