MAT 322 Problem Set 9

Homework Policy. Please read through all the problems. Please write up solutions of the required problems. Please also read and attempt the extra problems, but please do not write up those solutions for grading. I will be happy to discuss the extra problems during office hours.

Each student is encouraged to work on problem sets with other students, but each submitted problem set must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource (such as a webpage).

Required Problems.

Problem 1.(p. 217, Problem 3) This problem continues Problem 1 from Problem Set 8. Let r be a positive real number with $0 < r < 1$. Denote elements of $\mathbb{R}^2$ with coordinates (θ, ϕ). Denote elements of $\mathbb{R}^3$ with coordinates (x, y, z). Define the function β by,

$$\beta : \mathbb{R}^2 \to \mathbb{R}^3,\ \beta(\theta, \phi) = (\cos(\phi), \sin(\phi), 0).$$

Define the function α by,

$$\alpha : \mathbb{R}^2 \to \mathbb{R}^3,\ \alpha(\theta, \phi) = (1 + r\cos(\theta)) \cdot \beta(\theta, \phi) + (0, 0, r\sin(\theta)),$$

i.e., scale $\beta(\theta, \phi)$ by the scalar $1 + r\cos(\theta)$ and then translate by $r\sin(\theta)$ in the direction of the z-axis.

(a) Expand the x-coordinate, y-coordinate, and z-coordinate of $\alpha(\theta, \phi)$.

(b) Check that $\beta(\theta, \phi + 2\pi)$ equals $\beta(\theta, \phi)$ for every θ and ϕ. For every interval $(a, a + 2\pi]$, for every fixed θ, check that the restriction of $\beta(\theta, -)$ to $\phi \in (a, a + 2\pi]$ is a continuous bijection from $(a, a + 2\pi]$ to the closed unit circle S in the xy-plane.

(c) Check that $f \circ \alpha$ equals $f \circ \beta$.

(d) Check that $g \circ \alpha$ is constant, with value equal to r^2. Check that the image of α is precisely the embedded, C^∞, 2-dimensional manifold $M = g^{-1}(\{r^2\})$. Since $g^{-1}(\{r^2\})$ is a closed bounded subset of $\mathbb{R}^3$, by the Heine-Borel Theorem, M is compact.

(e) Check that the directional derivative $\partial\alpha/\partial\phi$, resp. $\partial\alpha/\partial\theta$, equals $(1 + r\cos(\theta))\widetilde{\mathbf{w}}$, resp. $-r\mathbf{t}$, where $\mathbf{t}$ and $\widetilde{\mathbf{w}}$ are the orthonormal basis for the kernel of $D_{(x,y,z)}g$ at the point $(x, y, z) = \alpha(\theta, \phi)$.

In particular, the directional derivatives are orthogonal, and their norms are $1 + r\cos(\theta)$, resp. r. Use this to compute $\text{vol}_{\mathbb{R}^3,2}(D_{(\theta,\phi)}\alpha)$.

(f) Conclude that the restriction of α to the open subset $U = (0, 2\pi) \times (0, 2\pi) \in \mathbb{R}^2$ is a chart for M. Also, the image $\alpha(U)$ a relatively open subset of M whose complement, $Z = \alpha([0, 2\pi] \times \{0\}) \cup \alpha(\{0\} \times [0, 2\pi])$, is a measure zero set.

(g) Compute the 2-volume of M in $\mathbb{R}^3$ as $\int_{(U,\alpha)} 1$. What do you conclude about $\text{vol}_{\mathbb{R}^3,2}(M)/r$ and $|\text{vol}_{\mathbb{R}^3,2}(D_{(\theta,\phi)}\alpha)|/r$ as r goes to zero? Is your answer consistent with your intuition about the surface area of a torus?

Problem 2. For the following $(3 + 2) \times 3$ matrix A, compute $|\text{vol}_{\mathbb{R}^5,3}(A)|^2$. Then use this computation to extrapolate a formula for the volume element of a manifold given as the image of the graph $h : U \to U \times \mathbb{R}^2$ of a C^1-function $g : U \to \mathbb{R}^2$ for $U \subset \mathbb{R}^3$ an open set. Please write your answer as $\int_U I$ for some explicit integrand I.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Problem 3.(p. 209, Problem 5). The following problem proves that the subsets $\mathbf{O}_n(\mathbb{R})$ and $\mathbf{SO}_n(\mathbb{R})$ of $\mathbf{Mat}_{n\times n}(\mathbb{R})$ are embedded, C^∞-submanifolds of dimension $n(n-1)/2$. For $n = 1$, this is trivial. For $n = 2$, you checked this explicitly in Problem 1 of Problem Set 7. For this problem, it is fine to solve the problem only for $n = 3$.

Identify the $\mathbb{R}$-vector space $\mathbf{Mat}_{n\times n}(\mathbb{R})$ with $\mathbb{R}^{n^2}$ by identifying the elementary matrix $E_{(i,j)}$ with the basis vector $\mathbf{e}_{(i-1)n+j}$. Denote by $\mathbf{Sym}_n(\mathbb{R}) \subset \mathbf{Mat}_{n\times n}(\mathbb{R})$, resp. $\mathbf{Skew}_n(\mathbb{R}) \subset \mathbf{Mat}_{n\times n}(\mathbb{R})$, the subset of all $n \times n$ matrices A such that $A^\dagger$ equals A, resp. $A^\dagger$ equals $-A$. In terms of the Euclidean inner product, this is equivalent to the condition that for every $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,

$$\langle A\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, A\mathbf{w}\rangle, \quad \text{resp. } \langle A\mathbf{v}, \mathbf{w}\rangle = -\langle \mathbf{v}, A\mathbf{w}\rangle.$$

Define the following function,

$$\widetilde{S} : \mathbf{Mat}_{n\times n}(\mathbb{R}) \to \mathbf{Mat}_{n\times n}(\mathbb{R}),\ \widetilde{S}(B) = B^\dagger B.$$

Thus $\widetilde{S}(B)$ is the unique $n \times n$ matrix such that for every $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,

$$\langle B\mathbf{v}, B\mathbf{w}\rangle = \langle \widetilde{S}(B)\mathbf{v}, \mathbf{w}\rangle.$$

(a) Prove that $\mathbf{Sym}_n(\mathbb{R}) \subset \mathbf{Mat}_{n\times n}(\mathbb{R})$, resp. $\mathbf{Skew}_n(\mathbb{R}) \subset \mathbf{Mat}_{n\times n}(\mathbb{R})$, is an $\mathbb{R}$-vector subspace of dimension $n(n+1)/2$, resp. $n(n-1)/2$, with basis $\{(E_{i,j} + E_{j,i})/\sqrt{2} : 1 \leq i \lneq j \leq n\} \cup \{E_{i,i} : 1 \leq i \leq n\}$, resp. with basis $\{(E_{i,j} - E_{j,i})/\sqrt{2} : 1 \leq i \lneq j \leq n\}$. (These bases are orthonormal for the

Hilbert-Schmidt inner product.) Prove that the subspaces $\mathbf{Sym}_n(\mathbb{R})$ and $\mathbf{Skew}_n(\mathbb{R})$ give a direct sum decomposition of $\mathbf{Mat}_{n\times n}(\mathbb{R})$. (In fact, these subspaces are orthogonal for the Hilbert-Schmidt inner product.)

(b) Prove that the image of $\widetilde{S}$ is contained in the $\mathbb{R}$-vector subspace $\mathbf{Sym}_n(\mathbb{R})$. Thus, restricting the coimage / target of $\widetilde{S}$ defines a function,

$$S : \mathbf{Mat}_{n\times n}(\mathbb{R}) \to \mathbf{Sym}_n(\mathbb{R}),\ \ S(B) = B^\dagger B.$$

Check, moreover, that the entries of $S(B)$ are quadratic polynomials in the entries of B, so that S is of class C^∞.

(c) For every $B \in \mathbf{Mat}_{n\times n}(\mathbb{R})$, compute the total derivative $\mathbb{R}$-linear transformation,

$$D_B S : \mathbf{Mat}_{n\times n}(\mathbb{R}) \to \mathbf{Sym}_{n\times n}(\mathbb{R}),$$

of S at B. In other words, find the linear transformation such that for every $\epsilon > 0$, there exists $\delta > 0$ so that for every $A \in \mathbf{Mat}_{n\times n}(\mathbb{R})$ with $\|A\| < \delta$, also

$$\|(B + A)^\dagger(B + A) - B^\dagger B - D_B S(A)\| \leq \epsilon\|A\|.$$

(**Hint.** You are not asked here to prove the epsilon-delta definition (although it is not hard to do). Simply expand this expression and use that to guess the (correct) value of the derivative.

(d) By the definition of *orthogonal matrix*, B is in $\mathbf{O}_n(\mathbb{R})$ if and only if $S(B)$ equals $\mathrm{Id}_{n\times n}$. In this case, B is invertible with inverse $B^\dagger$. Thus, write $A = BC$ for unique $C = B^\dagger A$, and substitute in $A = BC$ to get a simplified expression for $D_B S(BC)$. Prove that your expression for $D_B S(BC)$ is independent of the particular element $B \in \mathbf{O}_n(\mathbb{R})$. In particular, conclude that the kernel $\mathrm{Ker}(D_B S)$ equals $B \cdot \mathrm{Ker}(D_{\mathrm{Id}_{n\times n}} S)$, so that the nullity of $D_B S$ equals the nullity of $D_{\mathrm{Id}_{n\times n}} S$ independent of B. Use the Rank-Nullity Theorem to conclude that also the rank of $D_B S$ equals the rank of $D_{\mathrm{Id}_{n\times n}} S$, independent of B.

(e) Prove that the kernel of $D_{\mathrm{Id}_{n\times n}} S$ is precisely $\mathbf{Skew}_{n\times n}(\mathbb{R})$. Use (a) to conclude that the linear transformation,

$$D_{\mathrm{Id}_{n\times n}} S : \mathbf{Mat}_{n\times n}(\mathbb{R}) \to \mathbf{Sym}_n(\mathbb{R}),$$

is surjective, i.e., it has rank equal to $\dim_{\mathbb{R}}(\mathbf{Sym}_{n\times n}(\mathbb{R}))$. Use the Implicit Function Theorem to conclude that $\mathbf{O}_n(\mathbb{R})$ is an embedded, C^∞-submanifold of $\mathbf{Mat}_{n\times n}(\mathbb{R})$ of dimension $\dim_{\mathbb{R}}(\mathbf{Skew}_n(\mathbb{R}))$. Because $\mathbf{SO}_n(\mathbb{R})$ is the connected component that contains $\mathbf{Id}_{n\times n}$, conclude the same for $\mathbf{SO}_n(\mathbb{R})$. Finally, because it is the kernel of the derivative of the constraint function S that defines $\mathbf{SO}_n(\mathbb{R})$, the kernel of $D_{\mathrm{Id}_{n\times n}} S$ is often denoted by $\mathfrak{so}_n(\mathbb{R})$ (even though we have already given it another name).

(f) Let U and V be elements in $\mathbf{Skew}_n(\mathbb{R})$, so that $U^\dagger = -U$ and $V^\dagger = -V$. For the matrix $W = UV - VU$, compute $W^\dagger$ in terms of U and V. Is W in $\mathbf{Skew}_n(\mathbb{R})$? The $\mathbb{R}$-bilinear pairing $(U, V) \mapsto UV - VU$ is often called the *Lie bracket*.

Problem 4.(p. 209, Problem 3). Please first reread Problem 5 from p. 79. Let $U \subset \mathbb{R}^3$ be an open subset. Let $f, g : U \to \mathbb{R}$ be functions of class C^r for $r \geq 1$. Define $C = f^{-1}(\{0\}) \cap g^{-1}(\{0\})$. Find sufficient conditions for C to be a smooth curve without singularities (i.e., an embedded 1-dimension submanifold). Your conditions should involve the derivatives of f and g at points p in C. For C^r-functions $h(x)$, $k(x)$ defined on an interval $(a,b) \subset \mathbb{R}$, check that the functions $f(x,y,z) = y - h(x)$ and $g(x,y,z) = z - k(x)$ satisfy your conditions.

Problem 5.(p. 218, Problem 5). For every integer $n \geq 1$ and for every real $r > 0$, denote by $S^{n-1}(r) \subset \mathbb{R}^n$ the n-sphere of radius r centered at the origin, i.e., the set of all vectors $\mathbf{x}$ with $\|\mathbf{x}\|^2_{\text{Eucl}} = r^2$. Denote by $\overline{B}^n(r) \subset \mathbb{R}^n$ the closed unit ball of radius r, i.e., the set of all vectors $\mathbf{x}$ with $\|\mathbf{x}\|^2_{\text{Eucl}} = r^2$. Thus subset $\overline{B}^n(r)$ is an embedded, C^∞-manifold with boundary of dimension n, and the boundary, $S^{n-1}(r)$, is an embedded, C^∞-manifold without boundary of dimension $n-1$.

(a) Assume that $n \geq 3$. Follow the pattern in Example 2, p. 216 to express the volume of $S^{n-1}(r)$ in terms of the volume of $B^{n-2}(r)$.

(b) Prove that $\text{vol}_{\mathbb{R}^n,n-1}(S^{n-1}(r))$ equals $\frac{d}{dt}\text{vol}_{\mathbb{R}^n,n}\overline{B}^n(t)|_{t=r}$.

Problem 6.(p. 226, Problem 6). Let f and g be the following tensors on $\mathbb{R}^4$,

$$f(\mathbf{x},\mathbf{y},\mathbf{z}) = x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2,$$

$$g = \phi_{2,1} - \phi_{1,2}.$$

Express $h = f \otimes g$ as a linear combination of elementary tensors $\phi_{i_1,i_2,i_3,i_4,i_5}$, and then write out this tensor as a function $h(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v})$.