MAT 322 Problem Set 4

Homework Policy. Please read through all the problems. Please write up solutions of the required problems. Please also read and attempt the extra problems, but please do not write up those solutions for grading. I will be happy to discuss the extra problems during office hours.

Each student is encouraged to work on problem sets with other students, but each submitted problem set must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource (such as a webpage).

Required Problems.

Problem 1. Let V be a finite-dimensional vector space, say $\mathbb{R}^n$, and let $\langle \bullet, \bullet \rangle : V \times V \to \mathbb{R}$ be an inner product.

(a) Prove that $f(\mathbf{b}, \mathbf{c}) = \langle \mathbf{b}, \mathbf{c} \rangle$ is differentiable as a function from $V \times V$ to $\mathbb{R}$. For $(\mathbf{b}, \mathbf{c}) \in V \times V$, describe $Df_{\mathbf{b},\mathbf{c}} : V \times V \to \mathbb{R}$.

(b) Let U be an open subset of $\mathbb{R}^m$, and let $g : U \to V$ be a differentiable function. Denote by $h : U \to \mathbb{R}$ the composite function $h(\mathbf{a}) = f(g(\mathbf{a}), g(\mathbf{a})) = \langle g(\mathbf{a}), g(\mathbf{a}) \rangle$. For $\mathbf{a} \in U$, use the Chain Rule to describe $Dh_{\mathbf{a}}$ in terms of $Dg_{\mathbf{a}}$.

(c) Assume now that h is minimized at $\mathbf{a}$, and assume also that $Dg_{\mathbf{a}}$ is surjective. Prove that $g(\mathbf{a})$ equals 0.

Problem 2. Statement of the Picard-Lindelöf Theorem: For every open rectangle $S = (t_0 - a, t_0 + a) \times (y_0 - b, y_0 + b) \subset \mathbb{R}^2$, for every continuous $F : S \to \mathbb{R}$ that is C-Lipschitz in y, i.e., $|F(t, y) - F(t, \widetilde{y})| \leq C|y - \widetilde{y}|$ for all $(t, y), (t, \widetilde{y}) \in S$, there exists real ϵ with $0 < \epsilon < a$, and a unique differentiable function $u : (t_0 - \epsilon, t_0 + \epsilon) \to (y_0 - b, y_0 + b)$ such that $u(t_0)$ equals y_0 and $u'(t) = F(t, u(t))$ for every $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

There is a second proof of the Inverse Function Theorem that uses the Contraction Mapping Fixed Point Theorem, the same theorem used to prove the above Picard-Lindelöf Theorem. In this exercise, you will prove the $n = 1$ case of the Inverse Function Theorem directly from the Picard-Lindelöf Theorem. Thus, let $f : (y_0 - b, y_0 + b) \to (t_0 - a, t_0 + a)$ be a C^2 function such that $f(y_0) = t_0$ and $1/r < f'(y) < r$ for some real $r > 0$ and for all $y \in (y_0 - b, y_0 + b)$ (the hypothesis on r is our usual hypothesis that the derivative is nonzero, but note the strong hypothesis that f is C^2).

(a) Up to replacing b by a smaller positive number, prove that $F : S \to \mathbb{R}$ by $F(t, y) = 1/f'(y)$ is continuous and C-Lipschitz in y for some real C (this is where the C^2 hypothesis is useful).

(b) Use the Picard-Lindelöf Theorem to conclude that there exists real $\epsilon > 0$ and a unique differentiable function $u : (t_0 - \epsilon, t_0 + \epsilon) \to (y_0 - b, y_0 + b)$ such that $u(t_0)$ equals y_0 and $u'(t) = 1/f'(u(t))$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

(c) Use the Chain Rule to prove that the derivative of $f(u(t))$ equals 1. Equivalently, prove that the derivative of $f(u(t)) - t$ equals 0.

(d) Use the Mean Value Theorem to prove that $f(u(t)) - t$ equals 0 on $(t_0 - \epsilon, t_0 + \epsilon)$. In particular, conclude that $(t_0 - \epsilon, t_0 + \epsilon)$ is contained in the image of f. Repeating this argument for every $t_0 = f(y_0)$ in the image of f, it follows that f is an open mapping (this is the hard step in the textbook proof of the Inverse Function Theorem).

(e) As a matter of consistency, use the formula for $u'(t)$ and the hypothesis that f is C^2 to prove that also u is C^2.

Problem 3 Let $A \subset \mathbb{R}^2$ and $B \subset \mathbb{R}^2$ be open subsets. Let $F : A \to \mathbb{R}^3$, $G : B \to \mathbb{R}^3$ be C^1 functions. Let $\mathbf{a} \in A$ and $\mathbf{b} \in B$ be elements such that $F(\mathbf{a})$ equals $G(\mathbf{b})$, and such that the images of $DF_{\mathbf{a}}$ and $DG_{\mathbf{b}}$ are 2-dimensional subspace of $\mathbb{R}^3$ that have 1-dimensional intersection.

Prove that there exists an open subset $C \subset \mathbb{R}$, there exist C^1 functions $f : C \to A$, $g : C \to B$, and there exists an element $\mathbf{c} \in C$ such that $F \circ f$ equals $G \circ g$, such that $f(\mathbf{c}) = \mathbf{a}$ and $g(\mathbf{c}) = \mathbf{b}$, and such that $Df_{\mathbf{c}}$ and $Dg_{\mathbf{c}}$ each have rank 1. In this sense, the intersection in $\mathbb{R}^3$ at $F(\mathbf{a}) = G(\mathbf{b})$ of the transverse surfaces $F(A)$ and $G(B)$ is the curve $F(f(C)) = G(g(C))$ (compare Exercise 5, p. 79).

Problem 4(Problem 6, p. 79) Let $U \subset \mathbb{R}^m$ be an open subset. Let $F : U \to \mathbb{R}^n$ be a C^1 function. For every point $\mathbf{a} \in U$ such that $DF_{\mathbf{a}}$ has rank n, for every $\delta > 0$, prove that there exists $\epsilon > 0$ such that $F(B_\delta(\mathbf{a}))$ contains $B_\epsilon(F(\mathbf{a}))$.

Problem 5(Problem 3, p. 90) Let $F : [0,1] \times [0,1] \to \mathbb{R}$ be defined by $F(x, y) = 0$ if $x \neq y$, $F(x, y) = 1$ if $x = y$. Prove that F is integrable over $[0,1] \times [0,1]$.

Problem 6(Problem 5, p. 90) Let $F : [0,1] \to \mathbb{R}$ be defined by $F(p/q) = 1/q$ if p and q are positive integers with no common factor other than 1, and $F(x) = 0$ if x is irrational. Prove that F is integrable over $[0,1]$.

Extra Problems.

pp. 78—79, Exercises 1, 2, 3; p. 90, Exercises 1, 2, 4.