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MAT 322 Problem Set 3

Homework Policy. Please read through all the problems. Please write up solutions of the required problems. Please also read and attempt the extra problems, but please do not write up those solutions for grading. I will be happy to discuss the extra problems during office hours.

Each student is encouraged to work on problem sets with other students, but each submitted problem set must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource (such as a webpage).

Required Problems.

Problem 1. Let $\| \bullet \|_1$ be the ℓ_1 norm on $\mathbb{R}^m$,

$$\| (a_1, \dots, a_n) \|_1 = |a_1| + \cdots + |a_n|.$$

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function. Let $M > 0$ be a real number. Assume that for every $\mathbf{a} \in \mathbb{R}^m$ and for every $i = 1, \dots, m$, the directional derivative $f'(\mathbf{a}; \mathbf{e}_i)$ is defined and $|f'(\mathbf{a}; \mathbf{e}_i)| \leq M$. Prove that f is M-Lipschitz with respect to $\| \bullet \|_1$. In particular, conclude that f is continuous.

Problem 2. Denote by $f : \mathbb{R}^2 \to \mathbb{R}^2$ the function $f(r, \theta) = (r\cos(\theta), r\sin(\theta))$.

(a) Compute the total derivative matrix $df_{(r,\theta)}$ as a 2×2-matrix. Also compute the real-valued function $\det(df_{(r,\theta)})$.

(b) For each of $r = -1, 0, 2$, sketch the image curve $f(\{r\} \times \mathbb{R})$ with the direction of increasing θ indicated. For each of $\theta = 0, \pi/4, 3\pi/2, 7\pi/4$, sketch the image curve $f(\mathbb{R} \times \{\theta\})$ with the direction of increasing r indicated.

(c) Finally, for the following region U of $\mathbb{R}^2$, sketch the image $f(U)$ with careful attention to the boundary,

$$U = \{(r, \theta) \in \mathbb{R}^2 : -\pi/2 < \theta < \pi/2, 0 < r < 2\cos(\theta)\}.$$

Problem 3. Denote by $f : \mathbb{R}^2 \to \mathbb{R}^2$ the function

$$f(r, t) = \left(r \cdot \frac{1 - t^2}{1 + t^2}, r \cdot \frac{2t}{1 + t^2} \right).$$

(a) Compute the total derivative matrix $df_{(r,t)}$ as a 2×2-matrix. Also compute the real-valued function $\det(df_{(r,t)})$.

(b) For each of $r = -2, 0, 1$, sketch the image curve $f(\{r\} \times \mathbb{R})$ with the direction of increasing θ indicated. For each of $t = -1, 0, 1/2, 2$, sketch the image curve $f(\mathbb{R} \times \{t\})$ with the direction of increasing r indicated.

Problem 4. Let $U \subset \mathbb{R}$ be an open subset, let $g : U \to \mathbb{R}$ be a function in the class C^2, and let $a \in U$ be an element. Prove that for every real $\epsilon > 0$ there exists $\delta > 0$ such that for every real h with $|h| < \delta$, each of $g(a+h)$ and $g(a-h)$ is defined and the following inequality holds,

$$| \ [g(a+h) + g(a-h) - 2g(a)] - B(h) \ | \leq \epsilon \ | \ h \ |^2,$$

where $B(h)$ is the homogeneous, degree 2 polynomial $B(h) = g''(a)h^2$. **Not to be written up.** Does this suggest a way to define the "second derivative" of a multivariable function in the class C^2?

Problem 5. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as follows,

$$f(x_1, x_2, x_3) = (e^{x_1+x_2}, e^{x_1-x_2}, e^{-2x_1}x_3), \quad g(y_1, y_2, y_3) = (y_1 + y_2 + y_3, y_1y_2 + y_1y_3 + y_2y_3, y_1y_2y_3).$$

(a) Compute the 3×3 matrix $df_{(x_1,x_2,x_3)}$. Also compute $\det(df_{(x_1,x_2,x_3)})$, simplifying your formula as much as possible.

(b) Compute the 3×3 matrix $dg_{(y_1,y_2,y_3)}$. Check that the matrix is singular if $y_2 = y_1$, if $y_3 = y_1$, or if $y_3 = y_2$. Also compute $\det(dg_{(y_1,y_2,y_3)})$, simplifying your formula as much as possible; your answer should be divisible by each of $y_2 - y_1$, $y_3 - y_1$ and $y_2 - y_1$ because of the previous computation.

(c) Compute the matrix products $df_{(0,0,0)} \cdot dg_{(0,0,0)}$ and $dg_{(0,0,0)} \cdot df_{(0,0,0)}$. Which of these is the derivative at $(0,0,0)$ of $f(g(y_1, y_2, y_3))$? Please do **not** expand $f(g(y_1, y_2, y_3))$ to solve this problem.

Problem 6. For $\mathbf{a} \in \mathbb{R}^n$, denote the Euclidean norm,

$$\| \ \mathbf{a} \ \| = \sqrt{|a_1|^2 + \cdots + |a_n|^2}.$$

Denote by $B_1(0) \subset \mathbb{R}^n$ the open unit ball of radius 1 centered at the origin with respect to $\| \bullet \|$. Prove that the following function f is one-to-one, onto, and differentiable, yet the inverse function g is not differentiable at the origin,

$$f : B_1(0) \to B_1(0), \quad f(\mathbf{a}) = f(a_1, a_2, \ldots, a_n) = (\| \ \mathbf{a} \ \|^2 \ a_1, \| \ \mathbf{a} \ \|^2 \ a_2, \ldots, \| \ \mathbf{a} \ \|^2 \ a_n).$$

Problem 7. Let $U \subset \mathbb{R}^n$ be a nonempty open subset. Let $f : U \to \mathbb{R}^n$ be a C^1 function such that $df_{\mathbf{a}}$ is nonsingular for every $\mathbf{a} \in U$. Prove that $f(U)$ is an open subset of $\mathbb{R}^n$, even in those cases that f is not one-to-one.

Extra Problems. pp. 54-55, Exercises 2,8,10. p. 63, Exercises 2,4. pp. 70-71, Exercises 1,4.