MAT 322 Problem Set 10

Homework Policy. Please read through all the problems. Please write up solutions of the required problems. Please also read and attempt the extra problems, but please do not write up those solutions for grading. I will be happy to discuss the extra problems during office hours.

Each student is encouraged to work on problem sets with other students, but each submitted problem set must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource (such as a webpage).

Required Problems.

Problem 1.(Problem 6, p. 244 and Problem 5, p. 273) Let W be a vector space with basis $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_m)$ and dual ordered basis $\mathcal{B}^\vee = (\phi_1, \dots, \phi_m)$ for $\mathcal{L}^1(W)$; $\phi_i(\mathbf{b}_j) = 1$ if $i = j$, resp. 0 if $i \neq j$. Let V be a vector space with basis $\mathcal{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ and dual ordered basis $\mathcal{A}^\vee = (\chi_1, \dots, \chi_n)$ for $\mathcal{L}^1(V)$. Let $T : W \to V$ be an $\mathbb{R}$-linear transformation with matrix representative $[T]_{\mathcal{B},\mathcal{A}} = [c_{i,j}]$, i.e., $T(\mathbf{b}_j) = \sum_i c_{i,j}\mathbf{a}_i$. There is a transpose transformation, $T^* : \mathcal{L}^1(V) \to \mathcal{L}^1(W)$, $T^*(f)(\mathbf{w}) = f(T(\mathbf{w}))$. The matrix representative is $[T^*]_{\mathcal{A}^\vee,\mathcal{B}^\vee} = [T]^\dagger_{\mathcal{B},\mathcal{A}} = [c_{j,i}]$, i.e., $T^*(\chi_i) = \sum_j c_{i,j}\phi_j$.

(a) Let $p \geq 1$ be an integer. Recall our notation for the associated basis of $\mathcal{L}^p(W)$, resp. $\mathcal{L}^p(V)$, $\phi_J = \phi_{j_1} \otimes \cdots \otimes \phi_{j_p}$, resp. $\chi_I = \chi_{i_1} \otimes \cdots \otimes \chi_{i_p}$, for every ordered p-tuple of integers $I = (i_1, \dots, i_p)$, resp. $J = (j_1, \dots, j_p)$, such that for every $k = 1, \dots, p$, $1 \leq i_k \leq n$, resp $1 \leq j_k \leq m$. Denote by $T^* : \mathcal{L}^p(V) \to \mathcal{L}^p(W)$ the pullback $\mathbb{R}$-linear transformation, $T^*(f)(\mathbf{w}_1, \dots, \mathbf{w}_p) = f(T(\mathbf{w}_1), \dots, T(\mathbf{w}_p))$. For every I and J as above, define $c_{I,J} \in \mathbb{R}$ to be the unique real number such that

$$T^*(\chi_I) = \sum_J c_{I,J}\phi_J.$$

Compute a formula for $c_{I,J}$ as a polynomial in the entries $c_{i,j}$.

(b) Repeat the previous part for the pullback $T^* : \mathcal{A}^p(V) \to \mathcal{A}^p(W)$ with repsect to the bases $\psi_J = \phi_{j_1} \wedge \cdots \wedge \phi_{j_p}$, resp. $\xi_I = \chi_{i_1} \wedge \cdots \wedge \chi_{i_p}$, where $1 \leq i_1 < \cdots < i_p \leq n$, resp. $1 \leq j_1 < \cdots < j_p \leq m$.

(c) Let $U \subset \mathbb{R}^m$ be an open subset, and let $F : U \to \mathbb{R}^n$ be a differentiable map. Let $(x_1, \dots, x_m)$ denote the standard coordinate system on $\mathbb{R}^m$, let $(y_1, \dots, y_n)$ denote the standard coordinate system on $\mathbb{R}^n$, and denote the components of F by $F(x_1, \dots, x_m) = (F_1(x_1, \dots, x_m), \dots, F_n(x_1, \dots, x_m))$. The pullback map $F^* : \mathcal{L}^1(\mathcal{T}(\mathbb{R}^n)) \to \mathcal{L}^1(\mathcal{T}(U))$ is given by $F^*(dy_i) = \sum_j \partial F_i/\partial x_j dx_j$. For the unique scalar-valued functions $c_{I,J} : U \to \mathbb{R}$ such that

$$F^*(dy_{i_1} \otimes \cdots \otimes dy_{i_p}) = \sum_J c_{I,J} \cdot dx_{j_1} \otimes \cdots \otimes dx_{j_p},$$

find a formula for $c_{I,J}$ as a polynomial in the partial derivatives $\partial F_i/\partial x_j$. Find a similar formula for the pullback of the differential form $F^*(dy_{i_1} \wedge \cdots \wedge dy_{i_p})$.

Problem 2(Glueing Tensor Fields / Differential Forms) Let $M \subset \mathbb{R}^m$ be a k-dimensional submanifold of class C^r, $r \geq 1$. Let $\omega \in \mathcal{L}^p(\mathcal{T}(M))$, resp. $\omega \in \Omega^p(M)$, be a p-tensor field on M of class C^r, resp. a differential form on M of class C^r. For every open subset $U \subset \mathbb{R}^k$, and for every function of class C^r, $\alpha : U \to \mathbb{R}^m$ that is a diffeomorphism to a (relatively) open subset of M, i.e., for every C^r coordinate patch on M, denote by $\omega_{(U,\alpha)}$ the pullback $\alpha^*\omega$ as a p-tensor fields on U, resp. as a differential p-form on U, of class C^r.

(a) For every open subset $U' \subset U$, and for the restriction α' of α to U', check that $\omega_{(U',\alpha')}$ equals the restriction of $\omega_{(U,\alpha)}$ to U'. More generally, for every pair (U, α) and (U', α') of C^r coordinate patches, and for every function $\beta : U' \to U$ of class C^r such that α' equals $\alpha \circ \beta$ (e.g., inclusion of an open subset U' into the larger open set U), check that $\beta^*\omega_{(U,\alpha)}$ equals $\omega_{(U',\alpha')}$.

(b) For ω and η a pair of p-tensor fields on M, resp. differential p-forms on M, of class C^r, if $\omega_{(U,\alpha)}$ equals $\eta_{(U,\alpha)}$ for every C^r coordinate patch (U, α), prove that ω equals η.

(c) Let $((U_i, \alpha_i))_{i\in I}$ be a collection of C^r coordinate patches such that M equals the union over all i of the relatively open subset $\alpha_i(U_i)$. For every $i \in I$, let ω_i be a p-tensor field on U_i, resp. a differential p-form on U_i, of class C^r. Assume that for every $i, j \in I$, for every open subset $U_i' \subset U_i$, for every function of class C^r, $\beta : U_i' \to U_j$ with $\alpha_j \circ \beta = \alpha_i'$, the pullback $\beta^*\omega_j$ equals the restriction to U_i' of ω_i. Prove that there exists a unique p-tensor field ω on M, resp. differential p-form ω on M, such that for every $i \in I$, ω_i equals $\omega_{(U_i,\alpha_i)}$.

Problem 3 Let V be a finite dimensional $\mathbb{R}$-vector space, e.g., $\mathbb{R}^m$, and let $\langle \bullet, \bullet \rangle$ be an inner product on V, e.g., the Euclidean inner product on $\mathbb{R}^m$. For every linear transformation $H : V \to V$, define the adjoint $H^\vee : V \to V$ with respect to $\langle \bullet, \bullet \rangle$ by $\langle H^\vee(\mathbf{v}), \mathbf{w})\rangle = \langle \mathbf{v}, H(\mathbf{w})\rangle$ for every $\mathbf{v}, \mathbf{w} \in V$. Define $B_H : V \times V \to \mathbb{R}$ by $B(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, H(\mathbf{w})\rangle$.

(a) Prove that B_H is 2-multilinear, i.e., B_H is a 2-tensor.

(b) Prove that the rule $H \mapsto B_H$ defines an $\mathbb{R}$-linear isomorphism $B : \mathrm{Hom}_{\mathbb{R}}(V, V) \to \mathcal{L}^2(V)$.

(c) Prove that B_H is a symmetric tensor if and only if H is *self-adjoint*, i.e., $H^\vee$ equals H. Similarly, prove that B_H is an alternating tensor if and only if H is *skew-adjoint*, i.e., $H^\vee$ equals $-H$.

(d) For every linear transformation $T : V \to V$, prove that the pullback $T^*(B_H)$ equals $B_{T^\vee \circ H \circ T}$. In particular, for those H that are invertible, prove that $H^*(B_H)$ equals B_H if and only if H is orthogonal with respect to $\langle \bullet, \bullet \rangle$, i.e., $H^\vee \circ H$ equals Id_V.

Problem 4(approx. Problem 2, p. 260) On $\mathbb{R}^3$ with coordinates (x, y, z), consider the following differential 1-forms,

$$\omega = x \cdot dx + y \cdot dz - z \cdot dy, \quad \eta = yz \cdot dx + xy \cdot dy + xz \cdot dz.$$

Verify by direct computation that $d(d\eta)$ equals 0, and $d(\omega \wedge \eta)$ equals $(d\omega) \wedge \eta - \omega \wedge (d\eta)$.

Problem 5(approx. Problem 3, p. 273) This problem continues the previous problem. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be $\alpha(x, y, z) = (yz, xz, xy)$. Verify by direct computation that $\alpha^*(\omega) \wedge \alpha^*(\eta)$ equals $\alpha^*(\omega \wedge \eta)$, and $\alpha^*(d\omega)$ equals $d(\alpha^*\omega)$.

Problem 6 In each of the following cases, compute $d\omega$ on $\mathbb{R}^m$ and say whether it equals 0. If $d\omega$ equals 0, find a differential form (not unique) η such that $\omega = d\eta$.

(a) $\mathbb{R}^2$ with coordinates (x, y), $\omega = xdy - ydx$.

(b) $\mathbb{R}^2$ with coordinates (x, y), $\omega = xdy + ydx$.

(c) $\mathbb{R}^3$ with coordinates (x, y, z), $\omega = xdx + ydy + zdz$.

(d) $\mathbb{R}^3$ with coordinates (x, y, z), $\omega = xdy \wedge dz$.