Name: ______________________________ **Problem 1:** ____________ /25

Problem 1(25 points) Consider the following matrices and column vectors,

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 4 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

(a)(15 points) Find necessary and sufficient conditions on the integers b_1, b_2 and b_3 such that there exist integers x_1, x_2, x_3, x_4 solving the linear system $AX = B$. Express your conditions as linear equations and linear congruences in the variables b_1, b_2, and b_3 (and only in these variables).

(b)(10 points) When (b_1, b_2, b_3) equals $(2, -8, 0)$, find the general solution $X \in \mathbb{Z}^4$ of the linear system $AX = B$.

(a) Beginning with the augmented matrix $\tilde{A} = \left[\begin{array}{c|c} A & Id_{3\times 3} \\ \hline Id_{4\times 4} & \end{array}\right]$, perform (integer coefficient) elementary row and column operations to find an equivalent matrix $\left[\begin{array}{c|c} D & U \\ \hline V & \end{array}\right]$ where D is in "block diagonal" form $D = \left[\begin{array}{c|c} d_1 \; \ddots \; d_r & 0 \\ \hline 0 & 0 \end{array}\right]$.

$$\tilde{A}_1 = \tilde{A} = \left[\begin{array}{cccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{array}\right]; \quad \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 3R_1 \end{array}, \quad \tilde{A}_2 = \left[\begin{array}{cccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 & -3 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{array}\right]; \quad \begin{array}{l} R_1' = R_1 + R_3 \\ R_2' = R_2 - 2R_3 \end{array},$$

$$\tilde{A}_3 = \left[\begin{array}{cccc|ccc} 1 & 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 & 1 & -2 \\ 0 & 0 & -2 & 0 & -3 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{array}\right]; \quad \begin{array}{l} R_2' = -R_3 \\ R_3' = R_2 \end{array}, \quad \tilde{A}_4 = \left[\begin{array}{cccc|ccc} 1 & 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 4 & 1 & -2 \\ \hline 1 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{array}\right]; \quad C_2' = C_2 - C_1, \quad \tilde{A}_5 = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 4 & 1 & -2 \\ \hline 1 & -1 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{array}\right];$$

$$\begin{array}{l} C_2' = C_3 \\ C_3' = C_2 \end{array}, \quad \tilde{A}_6 = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & -2 & 0 & 1 \\ 0 & 2 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 4 & 1 & -2 \\ \hline 1 & 0 & -1 & 0 & & & \\ 0 & 0 & 1 & 0 & & & \\ 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 0 & 1 & & & \end{array}\right].$$

So $D = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$, $U = \begin{bmatrix} -2 & 0 & 1 \\ 3 & 0 & -1 \\ 4 & 1 & -2 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Make the change of variables $\underline{X} = V\tilde{\underline{X}}$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} = \begin{bmatrix} \tilde{x}_1 - \tilde{x}_3 \\ \tilde{x}_3 \\ \tilde{x}_2 \\ \tilde{x}_4 \end{bmatrix}$, and

(over 2)

$\widetilde{B}=\mathcal{U}B$, $\begin{bmatrix}\widetilde{b}_1\\ \widetilde{b}_2\\ \widetilde{b}_3\end{bmatrix}=\begin{bmatrix}-2&0&1\\3&0&-1\\4&1&-2\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}-2b_1 \quad +b_3\\ 3b_1 \quad -b_3\\ 4b_1+b_2-2b_3\end{bmatrix}$. The original system $AX=B$ is equivalent to the new system $D\widetilde{X}=\widetilde{B}$, i.e.,

$$\begin{cases}1\cdot\widetilde{x}_1 = \widetilde{b}_1 = -2b_1 \quad + b_3\\ 2\widetilde{x}_2 = \widetilde{b}_2 = 3b_1 \quad - b_3\\ 0 = \widetilde{b}_3 = 4b_1+b_2-2b_3\end{cases}.$$

So the original system is consistent if and only if the new system is consistent, i.e., if and only if both

$$\begin{aligned}\widetilde{b}_2 &= \boxed{3b_1 - b_3 \equiv 0 \pmod 2\text{, and}}\\ \widetilde{b}_3 &= \boxed{4b_1+b_2-2b_3 = 0.}\end{aligned}$$

(b) When $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ equals $\begin{bmatrix}2\\-8\\0\end{bmatrix}$ then $\begin{bmatrix}\widetilde{b}_1\\ \widetilde{b}_2\\ \widetilde{b}_3\end{bmatrix}$ equals $\begin{bmatrix}-2(2) \quad +(0)\\ 3(2) \quad -(0)\\ 4(2)+(-8)-2(0)\end{bmatrix}=\begin{bmatrix}-4\\6\\0\end{bmatrix}$.

Observe that $\begin{matrix}\widetilde{b}_2\equiv 0 \pmod 2\\ \widetilde{b}_3 = 0\end{matrix}$ so that the new system is consistent: $\begin{cases}1\widetilde{x}_1 = \widetilde{b}_1 = -4\\ 2\widetilde{x}_2 = \widetilde{b}_2 = 6\\ 0=\widetilde{b}_3=0\end{cases}$.

The general solution of the new system is $\begin{bmatrix}\widetilde{x}_1\\ \widetilde{x}_2\\ \widetilde{x}_3\\ \widetilde{x}_4\end{bmatrix}=\begin{bmatrix}-4\\3\\t_1\\t_2\end{bmatrix}$, where t_1 and t_2 are arbitrary integers.

Thus the general solution of the original system is given by $X=\mathcal{V}\widetilde{X}$,

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}\widetilde{x}_1-\widetilde{x}_3\\ \widetilde{x}_3\\ \widetilde{x}_2\\ \widetilde{x}_4\end{bmatrix}$, i.e., $\boxed{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-4-t_1\\t_1\\3\\t_2\end{bmatrix}\text{, where } t_1 \text{ and } t_2 \text{ are arbitrary integers.}}$

Name: ______________________________ **Problem 2:** ____________ /15

Problem 2(15 points) Consider the following integer, binary quadratic form

$$f(x,y) = 7x^2 - 4xy + y^2.$$

Find a 2×2, integer-valued matrix with determinant $+1$,

$$V = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

such that after the linear change of variables,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \alpha\tilde{x} + \beta\tilde{y} \\ \gamma\tilde{x} + \delta\tilde{y} \end{bmatrix},$$

the new binary quadratic form

$$g(\tilde{x},\tilde{y}) = f(x,y) = a\tilde{x}^2 + b\tilde{x}\tilde{y} + c\tilde{y}^2$$

is in reduced form, i.e., $|a| \le |c|$ and either $-|a| < b \le |a|$ if $|a| < |c|$ or $0 \le b \le |a|$ if $|a|$ equals $|c|$. Also give the binary quadratic form $g(\tilde{x},\tilde{y})$.

Step 1. Denote $f(x,y)$ by $f_1(x_1,y_1) = a_1x_1^2 + b_1x_1y_1 + c y_1^2 = 7x_1^2 - 4x_1y_1 + 1y_1^2$. Since we do not have $|a_1| \le |c_1|$, i.e., $7 \not\le 1$, we perform the admissible linear change of variables, $(x_1, y_1) = (y_2, -x_2)$. This gives an equivalent integer, binary quadratic form,

$f_2(x_2,y_2) = f_1(x_1,y_1) = 7(y_2)^2 - 4(y_2)(-x_2) + 1(-x_2)^2 = \underline{1x_2^2 + 4x_2y_2 + 7y_2^2} = a_2x_2^2 + b_2x_2y_2 + c_2y_2^2.$

Step 2. For the new form f_2, $|a_2| < |c_2|$ does hold, but $\cancel{-|a_2|}\ -|a_2| < b_2 \le |a_2|$ does not hold, $4 \not\le 1$. Thus apply the division algorithm to write

$b_2 = q(2|a_2|) + r$, $-|a_2| < r \le |a_2|$. Perform the admissible linear change of variables,

$4 = 2\cdot(2) + 0$, $(x_2, y_2) = (x_3 - q y_3, y_3) = (x_3 - 2y_3, y_3)$. This gives an equivalent form,

$f_3(x_3,y_3) = f_2(x_2,y_2) = 1(x_3 - 2y_3)^2 + 4(x_3 - 2y_3)y_3 + 7y_3^2 = \underline{1x_3^2 + 0x_3y_3 + 3y_3^2} = a_3x_3^2 + b_3x_3y_3 + c_3y_3^2.$

This form is reduced: $|a_3| < |c_3|$ and $-|a_3| < b \le |a_3|$. Thus we set $(\tilde{x},\tilde{y}) = (x_3, y_3)$

$1 < 3$ $\quad$ $-1 < 0 \le 1$

with admissible change of variables $(x,y) = (\tilde{y}, -\tilde{x} + 2\tilde{y})$, i.e. $\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}}$, giving the equivalent, reduced form $\boxed{g(\tilde{x},\tilde{y}) = 1\tilde{x}^2 + 0\tilde{x}\tilde{y} + 3\tilde{y}^2}$.

Name: ______________________ Problem 3: __________ /30

Problem 3(30 points) Consider the integer, binary quadratic forms

$$f(x,y) = ax^2 + bxy + cy^2$$

which are positive definite and which have discriminant $b^2 - 4ac$ equal to -12.

(a)(10 points) Find all such forms $f(x,y)$ which are reduced. In particular, give the class number $H(-12)$.

(b)(20 points) For all odd primes p different from 3, find a necessary and sufficient condition that p is properly represented by a quadratic form f as above (with discriminant equal to -12). Write your condition in terms of p being congruent to a list of residues modulo a fixed integer (using the Chinese Remainder Theorem if necessary to combine congruences modulo relatively prime integers).

(a). First of all, $b^2 \equiv -12 \equiv 0 \pmod 4$ so that b is even. Also, if a is even, then $-4ac \equiv 0 \pmod 8$ so that $b^2 \equiv -12 \equiv 4 \pmod 8 \Rightarrow b \equiv 2 \pmod 4$.

If $f(x,y)$ is reduced then $12 = 4ac - b^2 \geqslant 4ac - a^2 \geqslant 4aa - a^2 = 3a^2$.
(since $|b| \leq |a|$) (since $0 < a \leq c$)

So $a^2 \leq 4$ and $0 < a \Rightarrow a$ equals 1 or 2.

Case 1. $a=1$. Since $-a < b \leq a$ gives $-1 < b \leq 1$ and since b is even, b equals 0. Thus $12 = 4ac - b^2 = 4(1)c - (0)^2 = 4c$. Thus c equals 3. So if a equals 1, then $f(x,y) = 1 \cdot x^2 + 0xy + 3y^2$.

Case 2. $a=2$. Since a is even, we have $b \equiv 2 \pmod 4$. Also $-a < b \leq a$, i.e. $-2 < b \leq 2$. Thus b equals 2. So $12 = 4ac - b^2 = 4(2)c - (2)^2 = 8c - 4$. And then c equals 2. So if a equals 2 then $f(x,y) = 2x^2 + 2xy + 2y^2 = 2(x^2+xy+y^2)$.

So the reduced, positive definite forms of discriminant -12 are

$$\boxed{f_1(x,y) = x^2 + 3y^2 \quad \text{and} \quad f_2(x,y) = 2(x^2+xy+y^2).}$$

Note $f_1(1,0) = 1$, but always $f_2(x,y)$ is even. So these are inequivalent. Therefore the class number is $\boxed{H(-12) = 2.}$

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(b) By one of our theorems, for an odd prime p and a discriminant d which is a square modulo 4, there exists an integer, binary quadratic form with discriminant d representing p if and only if d is a square modulo p. (N.B. Since p is positive and $d=-12$ is negative, such a form is necessarily positive definite.)

Since $p\neq 2$ and $p\neq 3$, -12 is not divisible by p, i.e, $-12\not\equiv 0 \pmod p$.

Therefore p is represented if and only if $\left(\frac{-12}{p}\right)$ equals $+1$.

Since $-12=-1\cdot 2^2\cdot 3$, we have $\left(\frac{-12}{p}\right)=\left(\frac{-1}{p}\right)\left(\frac{2^2}{p}\right)\left(\frac{3}{p}\right)=\left(\frac{-1}{p}\right)(+1)\left(\frac{3}{p}\right)$.

By quadratic reciprocity, $\left(\frac{3}{p}\right)\left(\frac{p}{3}\right)=(-1)^{\frac{3-1}{2}\cdot\frac{p-1}{2}}=(-1)^{1\cdot\frac{p-1}{2}}=\left(\frac{-1}{p}\right)$.

Hence $\left(\frac{3}{p}\right)$ equals $\left(\frac{-1}{p}\right)\left(\frac{p}{3}\right)$. Therefore we have $\left(\frac{-12}{p}\right)=\left(\frac{-1}{p}\right)\left(\frac{3}{p}\right)=\left(\frac{-1}{p}\right)\left(\frac{-1}{p}\right)\left(\frac{p}{3}\right)$

$=(+1)\cdot\left(\frac{p}{3}\right)$.

And $\left(\frac{p}{3}\right)=\begin{cases}+1, & p\equiv +1 \pmod 3\\ -1, & p\equiv -1 \pmod 3\end{cases}$.

Therefore, for an odd prime $p\neq 3$, p is represented by a quadratic form f with discriminant -12 if and only if $\boxed{p\equiv +1 \pmod 3.}$

(N.B. Obviously an odd number cannot be represented by $f_2(x,y)$. Therefore an odd prime $p\neq 3$ which is represented by a quadratic form f with discriminant -12 is represented by $f_1(x,y)=x^2+3y^2$.)

Name: ______________________________ Problem 4: ______________ /30

Problem 4(30 points) Consider the following integer, ternary quadratic form

$$f(x,y,z) = -11x^2 + y^2 + yz + 2z^2.$$

(a)(20 points) Find an invertible, 3×3 matrix with rational entries,

$$V = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ 0 & c_{2,2} & c_{2,3} \\ 0 & 0 & c_{3,3} \end{bmatrix}$$

with column vectors $\vec{w}_1$, $\vec{w}_2$, $\vec{w}_3$, such that after the linear change of variables,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ 0 & c_{2,2} & c_{2,3} \\ 0 & 0 & c_{3,3} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = \tilde{x}\vec{w}_1 + \tilde{y}\vec{w}_2 + \tilde{z}\vec{w}_3,$$

the new binary quadratic form $g(\tilde{x},\tilde{y},\tilde{z})$ is in "Legendre diagonal form", i.e.,

$$g(\tilde{x},\tilde{y},\tilde{z}) = f(x,y,z) = q(a\tilde{x}^2 + b\tilde{y}^2 + c\tilde{z}^2)$$

for a nonzero rational number q and for integers a, b, c such that abc is square-free. Also give the binary quadratic form $g(\tilde{x},\tilde{y},\tilde{z})$.

(b)(10 points) Using Legendre's theorem, or otherwise, determine whether or not $g(\tilde{x},\tilde{y},\tilde{z})$ has a nontrivial rational solution $(\tilde{x},\tilde{y},\tilde{z}) \neq (0,0,0)$.

Bonus problem.(10 bonus points) **Please only attempt this if you have already solved the rest of the exam.** Find integer coefficient, homogeneous, quadratic polynomials $\tilde{x}(u,v)$, $\tilde{y}(u,v)$, $\tilde{z}(u,v)$ of integer variables u and v such that $(\tilde{x}(u,v), \tilde{y}(u,v), \tilde{z}(u,v))$ is a solution of $g(\tilde{x},\tilde{y},\tilde{z})$ for every choice of $(u,v) \in \mathbb{Z}^2$ and such that every rational solution of g is a rational multiple of $(\tilde{x}(u,v), \tilde{y}(u,v), \tilde{z}(u,v))$ for some choice of integers u and v (not necessarily unique).

(a) First of all, $2f(x,y,z) = -22x^2 + 2y^2 + 2yz + 4z^2$ is $\underline{X}^t Q \underline{X} = \underline{X} \cdot (Q\underline{X})$ where $\underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and where Q is the symmetric matrix $\begin{bmatrix} -22 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$.

Perform Gram-Schmidt, to find a new basis $\vec{w}_1 = \begin{bmatrix} c_{11} \\ 0 \\ 0 \end{bmatrix}$, $\vec{w}_2 = \begin{bmatrix} c_{12} \\ c_{22} \\ 0 \end{bmatrix}$, $\vec{w}_3 = \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix}$ with $\vec{w}_i \cdot (Q\vec{w}_j) = 0$ for $i \neq j$.

Step 1. $\vec{w}_1 = \vec{v}_1 = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$.

$Q\vec{w}_1 = \begin{bmatrix} -22 \\ 0 \\ 0 \end{bmatrix}$, $\vec{w}_1 \cdot (Q\vec{w}_1) = -22$.

Step 2. $\vec{v}_2 \cdot Q\vec{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -22 \\ 0 \\ 0 \end{bmatrix} = 0$.

So $\vec{w}_2 = \vec{v}_2 = \boxed{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$.

$Q\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $\vec{w}_2 \cdot (Q\vec{w}_2) = 2$

Step 3. $\vec{v}_3 \cdot Q\vec{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -22 \\ 0 \\ 0 \end{bmatrix} = 0$ and

$\vec{v}_3 \cdot Q\vec{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 1$.

So $\vec{w}_3 = 2(\vec{v}_3 - \frac{1}{2}\vec{w}_2) = 2\vec{v}_3 - \vec{w}_2 = \boxed{\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}}$.

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$Q \cdot \vec{w_3} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$ (Bond, James Bond :) , $\vec{w_3} \cdot Q\vec{w_3} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} = \underline{\underline{14}}$.

So with respect to the change of basis matrix $V = [\vec{w_1} \vdots \vec{w_2} \vdots \vec{w_3}] = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}$,

Name: ______________________ Problem 4, continued

and the change of variables $\underline{X} = Q\tilde{\underline{X}}$, i.e. $\underline{X} = \tilde{x}\vec{w_1} + \tilde{y}\vec{w_2} + \tilde{z}\vec{w_3}$,

$2g(\tilde{x},\tilde{y},\tilde{z}) = 2f(x,y,z)$ equals $-\underline{\underline{22}}\,\tilde{x}^2 + \underline{\underline{2}}\,\tilde{y}^2 + \underline{\underline{14}}\,\tilde{z}^2$, i.e,

$\boxed{g(\tilde{x},\tilde{y},\tilde{z}) = -11\tilde{x}^2 + 1\tilde{y}^2 + 7\tilde{z}^2.}$

(b) Since $(a,b,c) = (-11, 1, 7)$, $(-bc, -ac, -ab)$ equals $(-7, 77, 11)$.

Notice that at least one of these is positive. Also,

$-bc \pmod{|a|}$ is $-7 \equiv 4 = 2^2 \pmod{11}$,

$-ac \pmod{|b|}$ is $77 \equiv 0 = 0^2 \pmod{1}$,

$-ab \pmod{|c|}$ is $11 \equiv 4 = 2^2 \pmod{7}$.

Thus, by Legendre's theorem, there does exist a nontrivial zero.

In fact, it is not too hard to find one: $\boxed{\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}$.

(Bonus). There is no nontrivial solution with $\tilde{x} = 0$, since -7 is not a square.

So, up to scaling, assume $\tilde{x}$ equals 1. Then write $(\tilde{x},\tilde{y},\tilde{z}) = (1, 2+t, 1+tm)$ for $t, m \in \mathbb{Q}$ (note that if $t=0$ then, up to a sign change, $\tilde{z} = 1$ so that $tm = 0$, i.e. $m \neq \infty$).

Substituting into $g(\tilde{x},\tilde{y},\tilde{z})$ gives $-11 + 1(4 + 4t + t^2) + 7(1 + 2tm + t^2m^2)$

$= t[2(2+7m) + t(1+7m^2)]$. So the solutions $\neq (1,2,1), (1,2,-1)$ w/ $\tilde{x} = 1$ are

$t = \dfrac{-2(2+7m)}{1+7m^2} \longrightarrow (\tilde{x},\tilde{y},\tilde{z}) = \left(1, \dfrac{2(-1-7m+7m^2)}{1+7m^2}, \dfrac{1-4m-7m^2}{1+7m^2}\right)$. Scale through

by $1+7m^2$ so that $(\tilde{x},\tilde{y},\tilde{z}) \overset{\text{propor.}}{\sim} (1+7m^2, 2(-1-7m+7m^2), 1-4m-7m^2)$.

Finally set $m = \dfrac{v}{u}$ and scale by u^2 to get

$(\tilde{x},\tilde{y},\tilde{z}) \overset{\text{propor.}}{\sim} \boxed{(u^2+7v^2, 2(-u^2-7uv+7v^2), u^2-4uv-7v^2)}$.

Notice this is proportional to $(1,2,-1)$ when $(u,v) = (0,1)$, it is proportional to $(1,-2,1)$ when $(u,v) = (1,0)$, and to $(1,2,1)$ when $(u,v) = (7,-2)$.

So this "parameterization" does account for all rational solutions.