MAT 311 Spring 2011 Midterm 2

Name: ________________________________ SB ID number: ______________________

Problem 1: __________ /25

Problem 2: __________ /15

Problem 3: __________ /30

Problem 4: __________ /30

Total: __________ /100

Instructions: *Please write your name at the top of every page of the exam.* This exam is closed book, closed notes, calculators are not allowed, and all cellphones and other electronic devices must be turned off for the duration of the exam. Electronic language translators may be approved by the proctor for ESL students, but the student must identify himself or herself to the proctor so that the translator may be approved prior to using the translator.

You will have approximately 80 minutes for this exam. The point value of each problem is written next to the problem – use your time wisely. **Please show all work, unless instructed otherwise.** Partial credit will be given only for work shown.

You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., raise your hand.

Name: ______________________________ **Problem 1**: ____________ /25

Problem 1(25 points) Consider the following matrices and column vectors,

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 4 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

(a)(15 points) Find necessary and sufficient conditions on the integers b_1, b_2 and b_3 such that there exist integers x_1, x_2, x_3, x_4 solving the linear system $AX = B$. Express your conditions as linear equations and linear congruences in the variables b_1, b_2, and b_3 (and only in these variables).

(b)(10 points) When (b_1, b_2, b_3) equals $(2, -8, 0)$, find the general solution $X \in \mathbb{Z}^4$ of the linear system $AX = B$.

Name: ______________________________ **Problem 2**: ______________ /15

Problem 2(15 points) Consider the following integer, binary quadratic form

$$f(x, y) = 7x^2 - 4xy + y^2.$$

Find a 2×2, integer-valued matrix with determinant $+1$,

$$V = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

such that after the linear change of variables,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \alpha\tilde{x} + \beta\tilde{y} \\ \gamma\tilde{x} + \delta\tilde{y} \end{bmatrix},$$

the new binary quadratic form

$$g(\tilde{x}, \tilde{y}) = f(x, y) = a\tilde{x}^2 + b\tilde{x}\tilde{y} + c\tilde{y}^2$$

is in reduced form, i.e., $|a| \leq |c|$ and either $-|a| < b \leq |a|$ if $|a| < |c|$ or $0 \leq b \leq |a|$ if $|a|$ equals $|c|$. Also give the binary quadratic form $g(\tilde{x}, \tilde{y})$.

Name: ____________________________ **Problem 3**: ______________ /30

Problem 3(30 points) Consider the integer, binary quadratic forms

$$f(x, y) = ax^2 + bxy + cy^2$$

which are positive definite and which have discriminant $b^2 - 4ac$ equal to -12.

(a)(10 points) Find all such forms $f(x, y)$ which are reduced. In particular, give the class number $H(-12)$.

(b)(20 points) For all odd primes p different from 3, find a necessary and sufficient condition that p is properly represented by a quadratic form f as above (with discriminant equal to -12). Write your condition in terms of p being congruent to a list of residues modulo a fixed integer (using the Chinese Remainder Theorem if necessary to combine congruences modulo relatively prime integers).

Name: ______________________________ **Problem 4**: ____________ /30

Problem 4(30 points) Consider the following integer, ternary quadratic form

$$f(x,y,z) = -11x^2 + y^2 + yz + 2z^2.$$

(a)(20 points) Find an invertible, 3×3 matrix with rational entries,

$$V = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ 0 & c_{2,2} & c_{2,3} \\ 0 & 0 & c_{3,3} \end{bmatrix}$$

with column vectors $\vec{w}_1$, $\vec{w}_2$, $\vec{w}_3$, such that after the linear change of variables,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ 0 & c_{2,2} & c_{2,3} \\ 0 & 0 & c_{3,3} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} = \tilde{x}\vec{w}_1 + \tilde{y}\vec{w}_2 + \tilde{z}\vec{w}_3,$$

the new binary quadratic form $g(\tilde{x},\tilde{y},\tilde{z})$ is in "Legendre diagonal form", i.e.,

$$g(\tilde{x},\tilde{y},\tilde{z}) = f(x,y,z) = q(a\tilde{x}^2 + b\tilde{y}^2 + c\tilde{z}^2)$$

for a nonzero rational number q and for integers a, b, c such that abc is square-free. Also give the binary quadratic form $g(\tilde{x},\tilde{y},\tilde{z})$.

(b)(10 points) Using Legendre's theorem, or otherwise, determine whether or not $g(\tilde{x},\tilde{y},\tilde{z})$ has a nontrivial rational solution $(\tilde{x},\tilde{y},\tilde{z}) \neq (0,0,0)$.

Bonus problem.(10 bonus points) **Please only attempt this if you have already solved the rest of the exam.** Find integer coefficient, homogeneous, quadratic polynomials $\tilde{x}(u,v)$, $\tilde{y}(u,v)$, $\tilde{z}(u,v)$ of integer variables u and v such that $(\tilde{x}(u,v),\tilde{y}(u,v),\tilde{z}(u,v))$ is a solution of $g(\tilde{x},\tilde{y},\tilde{z})$ for every choice of $(u,v) \in \mathbb{Z}^2$ and such that every rational solution of g is a rational multiple of $(\tilde{x}(u,v),\tilde{y}(u,v),\tilde{z}(u,v))$ for some choice of integers u and v (not necessarily unique).