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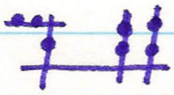
The Effective Cone of $\overline{M}_{0,n}$ and its Birational Contractions

joint work with Ana-Maria Castravet

Denote by $\overline{M}_{0,n}$ the moduli space of n-pointed stable rational curves. This contains as a dense, open subset $M_{0,n} := ((\mathbb{P}^1)^n - \text{diagonals}) / \operatorname{Aut}(\mathbb{P}^1)$.

Example stable map.  : At least 3 marked points or nodes on every irreducible component.

Effective divisors on $\overline{M}_{0,n}$. For every partition $\{1,\dots,n\} = A \sqcup B$ into sets with $\#A, \#B > 1$, the is a boundary divisor $\Delta_{A,B} = \overline{\{{}^{A}_{B}\!\!>\!\!<\}}$.

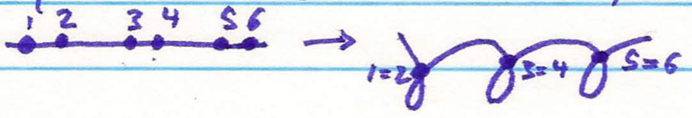
The Picard number of $\overline{M}_{0,n}$ is 2^{n-1} − lower order terms. There is a blow up model of $\overline{M}_{0,n}$. There are forgetful maps such as the one $\overline{M}_{0,n} \to \overline{M}_{0,n-1}$ forgetting the last marked point p_n. And there is a morphism $\psi: \overline{M}_{0,n} \to \mathbb{P}^{n-3}$; choose $n-1$ points in uniform position and consider rational curves of degree $n-3$ which contain these $n-1$ fixed points together with one variable point q in $\mathbb{P}^{n-3}$.

Example. $\psi: \overline{M}_{0,5} \to \mathbb{P}^2 \supset \{p_1, p_2, p_3, p_4\}$ gives the conics in $\mathbb{P}^2$ containing $p_1, \dots, p_4$ and gives a blow up model $\overline{M}_{0,5} = Bl_{p_1,\dots,p_4}(\mathbb{P}^2)$.

More generally $\overline{M}_{0,n}$ is the iterated blow up of $\mathbb{P}^{n-3}$ in the proper transforms of linear subspaces spanned by subsets of $p_1, \dots, p_{n-1}$.

One might guess from this that the effective cone of $\overline{M}_{0,n}$ is generated by boundary divisors.

<u>Keel-Vermeire Result</u>. For $\overline{M}_{0,6}$ the effective cone is not generated by boundary divisors alone.

Vermeire used the Kapranov model to show this. Keel had a different techique. There is a morphism $\overline{M}_{0,6} \to \overline{M}_3$ given by 

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Pull back the hyperelliptic divisor in $\overline{M}_3$ to get an effective divisor not in the cone spanned by boundary divisors.

One can continue this: $\overline{M}_3 \dashrightarrow \mathrm{Sym}^4(\mathbb{C}^3)/SL_3$ (the symmetric GIT quotient). This is a contracting birational transformation. So it is natural to expect the Keel-Vermeire divisor must be an extremal ray of the effective cone.

<u>Theorem</u> [Keel-McKernan] For the natural action of S_n on $\overline{M}_{0,n}$, $\mathrm{Eff}(\overline{M}_{0,n}/S_n)$ is generated by boundary divisors.

<u>Conjecture</u> [Castravet-Tevelev] The effective cone of $\overline{M}_{0,n}$ is generated by boundary divisors, Brill-Noether loci of hypergraph curves and pullbacks of these divisors by forgetful maps.

Missing transparency of hypergraph curves.

The number of these hypergraph curves grows quickly as n grows.

Definition. A hypergraph is a collection $\Gamma = \{\Gamma_1, \dots, \Gamma_d\}$ of subsets of $\{1, \dots, n\}$ such that

(1) for every $i = 1, \dots, d$, $\#\Gamma_i$ is ≥ 3,

(2) for every $j = 1, \dots, n$, the integer $v_j := \#\{i \mid v_j \in \Gamma_i\}$ is ≥ 2

(3) (Normalization) $\sum_{j=1}^{d} (\#\Gamma_j - 2)$ equals $n-2$

(4) (Convexity) For every nonempty subset $S \subset \{1, \dots, d\}$, $\sum_{j \in S} (\#\Gamma_j - 2)$ is $\leq \#(\bigcup_{j \in S} \Gamma_j) - 2$.

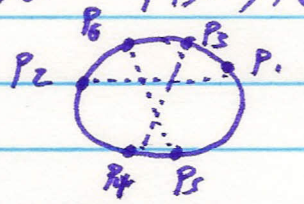
A hypergraph is a strong hypergraph if the convexity inequality is strict except when $S = \{1, \dots, d\}$ or when S is a singleton set.

A planar realization of a hypergraph is a subset $\{p_1, \dots, p_n\}$ of $\mathbb{P}^2$ such that $\{p_i\}_{i \in S}$ is collinear if & only if S is a subset of some Γ_j.

Definition. A hypergraph divisor D_Γ in $\overline{M}_{0,n}$ is the closure of the locus obtained by choosing a planar realization of Γ and then projecting the vertices $p_1, \dots, p_n$ from general points in $\mathbb{P}^2$?

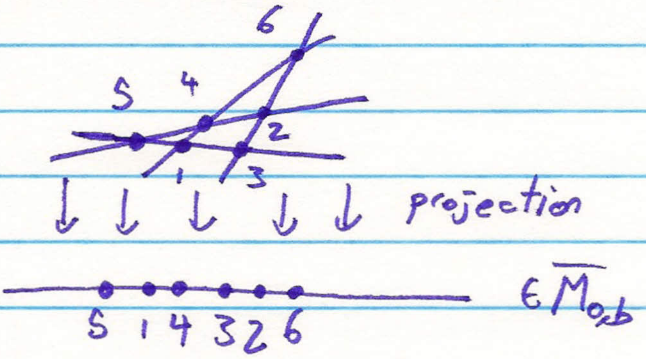
Even for $n=6$ this is a bit different from our earlier description.

Old description. For $p_1, \dots, p_6$ in $\mathbb{P}^1$, consider $\mathbb{P}^1$ as a conic in $\mathbb{P}^2$



Then $(\mathbb{P}^1, p_1, \dots, p_6)$ is hyperelliptic if & only if $\overline{p_1p_2}, \overline{p_3p_4}, \overline{p_5p_6}$ are concurrent.

New description.



Dolgachev pointed out that Joubert's 1856 paper proves that the old description is equivalent to the new description (these are the Segre cubic & Igusa quartic models for $M_{0,6}$).

Theorem 1 [Castravet-Tevelev] If Γ is any strong hypergraph then Γ admits a planar realization, D_Γ is an irreducible divisor, and D_Γ generates an extremal ray of $\mathrm{Eff}(\overline{M}_{0,n})$. This extremal ray is cut out by explicit linear inequalities.

The proof uses Brill-Noether loci. By way of motivation let C be a general curve of genus g. For every $[L]$ in $\mathrm{Pic}^{g+1}(C)$, $h^0(L)$ is ≥ 2. By the Brill-Noether theorem the locus $W^2_{g+1} = \{[L] \in \mathrm{Pic}^{g+1}(C) \mid h^0(L) \geq 3\}$ is ~~a divisor~~ of codimension $(r+1)(g-d+r) = 3 \cdot 1 = 3$.

Example. For $g=3$, W^2_4 equals $\{[\omega_C]\} \subset \mathrm{Pic}^4(C)$.

Consider the morphism $G^1_{g+1}(C) \xrightarrow{f} Pic^{g+1}(C)$ where $G^1_{g+1}(C) := \{([L], V) \mid [L] \in Pic^{g+1}(C),\ V \subset H^0(C, L),\ \dim V = 2\}$. Then $f^{-1}(W^2_{g+1})$ is an exceptional divisor in $G^1_{g+1}(C)$

This is related to the effective cone of G^1_{g+1} investigated by Bauer-Szemberg.

Definition. A hypergraph curve is a curve obtained by a pushout of a disjoint union $\coprod_{i=1} \mathbb{P}^1$ / gluing the same indices.

There may be moduli of these. Denote by M_Γ the moduli space. And denote by $\Sigma \to M_\Gamma$ the universal hypergraph curve. Form the relative Picard scheme,

$$Pic^{(1,\dots,1)}(\Sigma) \longrightarrow M_\Gamma .$$

degree 1 on every component

This is a torsor for an algebraic torus over M_Γ of dimension $g = n-3-\dim M_\Gamma$. Then a planar realization determines $M_{0,n} \to Pic(\Sigma)$: for a planar realization projecting to a point of $M_{0,n}$, consider the restriction of $\mathcal{O}_{\mathbb{P}^2}(1)$ to the planar curve. There is a compactification of $M_{0,n} \to Pic(\Sigma)$. And the locus where $h^0(f^*\mathcal{O}_{\mathbb{P}^2}(1))$ is >2 gives a divisor.

Theorem 2 [Castravet-Tevelev] The rational transformation $\overline{M}_{0,n} \dashrightarrow \overline{Pic(\Sigma)}$ is a birational contraction. And D_Γ is the unique component of the exceptional locus which intersects the interior, $M_{0,n}$.