A local–global principle for weak approximation for varieties over function fields

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This is joint work with Jason Starr.

Setup: B smooth projective curve over $\mathbb{C}$. $\pi\colon X \to B$ a family of projective varieties with general fibre rationally connected.

Fix $b_1, \ldots, b_k \in B$, and formal sections $\hat{t}_i$ over $\operatorname{Spec}(\hat{\mathcal{O}}_{b_i})$.

Q: Is it possible to find a section s of π agreeing with each $\hat{t}_i$ to order $\geq N$?

We say that weak approximation holds for π and $b_1, \ldots, b_k$ if we can find s for any $\hat{t_1}, \ldots, \hat{t}_k$. We say that weak approximation holds for π if it holds for all $b_1, \ldots, b_k$.

Why WA? Number theory: $X/\mathbb{Q}$ a projective variety. Then $X(\mathbb{Q}) = X(\mathbb{Z})$. We have reduction maps $X(\mathbb{Z}) \to X(\mathbb{F}_p), X(\hat{\mathbb{Z}}_p), X(\mathbb{R})$, etc. Number theoretic question: can we reverse these arrows?

Rational connectedness. Suppose Y is a projective variety. Say Y is *rationally chain connected* if given any two points p, q there exists a chain of rational curves connecting p and q. Say Y is *rationally connected* if given any 2 points (equivalently, any finite set of points) then there is a single rational curve containing all of them. Equivalently (if Y is smooth) there is a single map $f\colon \mathbb{P}^1 \to Y$ such that $f^*T_Y = \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 1$.

Theorem 0.1. *(Kollár–Miyaoka–Mori) If Y is smooth, RCC $\Rightarrow$ RC.*

Why RC? Suppose $\pi\colon X \to B$ satisfies WA.

Fix $b_0 \in B$ such that fibre is smooth and $p, q \in X_{b_0}$.

Pick points $b_1, b_2, \ldots$ in the base converging to b_0 and points $q_i \in X_{b_i}$ converging to q.

Look for sections passing through p and q_i, $i = 1, 2, \ldots$. In the limit, the section has to split, so there exists a chain of rational curves connecting p and q. (Actually, we need to take uncountably many points in the base, and pass to a subsequence such that the class of the section is fixed.) So WA implies that the general fibre is RC.

Theorem 0.2. *(Graber–Harris–Starr) General fibre is RC $\Rightarrow$ there are lots of sections.*

Conjecture 0.3. (Hassett–Tschinkel) General fibre RC $\Rightarrow$ WA holds.

Some results and the method of Hassett–Tschinkel. Suppose $\pi\colon X \to B$ is as in the setup and $b_1, \ldots, b_k \in B$.

Theorem 0.4. *WA holds at $b_1, \ldots, b_k$ if the fibres b_i are (a) smooth, or (b) cubic surfaces with at most RDP's.*

Also known if fibres are del Pezzo surfaces of degree ≥ 4, of degree ≥ 2 with at worst ?? singularities (A. Knecht), or del Pezzo surfaces with quotient singularitiews (C. Xu).

Reduction step: By blowing up (repeatedly) on $X_{b_i} \subset X$ we can translate problem of "matching to order N" to "matching to order 0".

Method of HT. Start with some section s (we know it exists by GHS). Connect $S \cap X_b$ with q by a creatively chosen chain of rational curves. Add other general rational curves in general fibres such that the resulting curve can be smoothed to pass through the correct point.

Q: Are there only local obstructions to WA? For example, if $\pi\colon X \to B$, $\pi'\colon X' \to B'$, $X_{\hat{b}_0} \simeq X'_{\hat{b}'_0}$, does WA at $b_0 \in B$ imply WA at $b'_0 \in B'$? Results:

Definition 0.5. Say *the method of HT holds for $X_{\hat{b}}$* if given any two sections $\hat{t}$ (target) and $\hat{s}$ (starting section) it is possible to add rational curves in fibre X_b to $\hat{s}$ and smooth to $\hat{t}$.

Theorem 0.6. *(A) If method of HT holds for $X_{\hat{b}}$ then WA holds at all $b' \in B'$ such that $X'_{\hat{b}'} \simeq X_{\hat{b}}$.*

Theorem 0.7. *(B) There is a purely local criterion in terms of the Laurent fibre $X_{\hat{K}} = X_{\hat{b}} \otimes_{\mathcal{O}_{\hat{b}}} \hat{K}$, $\hat{K} = \mathrm{Frac}(\mathcal{O}_{\hat{b}})$ that guarantees WA, namely, $X_{\hat{K}}(\hat{K})$ is "R-connected" in the sense of Manin.*

How to prove Theorem A: Look for mystery moduli space MMS parametrising intersection data of curves in X with X_{b_0}, satisfying

(1) Given family of curves, get map to moduli space of restriction data, in particular, have map

$$\mathrm{Hilb}_s \to \mathrm{MMS}$$

where Hilb_s is the Hilbert scheme of "section-like" curves in X.

(2) Given $[C] \in \text{MMS}$ it's possible to attach rational tails in fibres away from b_0 to get a curve C' such that the map $\text{Hilb}_s \to \text{MMS}$ is smooth at C'.

Given this, prove A: First blowup so that agreeing with t is the same as passing through a point.

(a) Start with some section s and restrict to get $\hat{s}$ over $\mathcal{O}_{\hat{b}}$.

(b) Use HT method: add rational curves to $\hat{s}$ and smooth to $\hat{t}$.

(c) Add same rational curves to s.

(d) Add more tails away from b_0 to get C' such that the map to MMS is smooth at C'.

(e) Follow deformation to get section.

What is MMS? The moduli space of pseudo ideal sheaves on X_b. A pseudo ideal sheaf is a finitely presented quasi coherent sheaf $\mathcal{F}$ together with a map $u\colon \mathcal{F} \to \mathcal{O}_{X_b}$ such that the map

$$\mathcal{F} \otimes \mathcal{F} \to \mathcal{F}, \quad f_1 \otimes f_2 \mapsto u(f_1)f_2 - f_1 u(f_2)$$

is identically zero. We have a map

$$\text{Hilb}_s \to \text{MMS}, \quad I_C \mapsto (I_C \otimes \mathcal{O}_{X_b} \to \mathcal{O}_{X_b}).$$