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Rationally Connected Varieties over $\mathbb{Q}_p^{nr}$

jt. w/ Brad Duesler

1. C_i fields
2. RC varieties / C_i fields
3. RC / $\mathbb{Q}_p^{nr}$

Def (Lang) A field k is C_i if every form $f \in k[x_1, \dots, x_n]$ w/ $d^i < n$ has a nontrivial solution.

Ex. $C_0 \Leftrightarrow$ alg. closed

C_1 also called "quasi-alg. closed" (E. Artin)

Example: $\mathbb{F}_q$ is C_1 (Chevalley-Warning)

- $K = k(\text{curve})$ is C_1 (Tsen's thm.)

- Lang. K = f. field of ~~r~~-dim'l variety over a C_i field is itself C_{i+r}.
- $\mathbb{Q}_p^{nr}$ is C_1 (Lang again).

$= \bigcup_{(p,n)=1} \mathbb{Q}_p(\mu_n)$.

- $\mathrm{Frac}(W(\overline{\mathbb{F}}_p))$ = fraction field of Witt vectors over $\overline{\mathbb{F}}_p$.

$[\mathbb{Q}_p^{nr} \subset \mathrm{Frac}(W(\overline{\mathbb{F}}_p))]$

Conjecture (Artin). $\mathbb{Q}_p$ is C_2

But Terjanian gave counterexamples over $\mathbb{Q}_2$ of degree 4 in 18 vars.

Ax-Kochen. $\mathbb{Q}_p$ is almost C_2.

Theorem. Fix an integer $d>0$. $\exists$ a finite set of primes S (depending on d) s.t. $\forall$ forms $f \in \mathbb{Q}_p[x_1, \dots, x_n]_d$ w/ $d^2 < n$, there is a nontrivial zero provided $p \notin S$.

② RC over a C_1-field.

Let X be a smooth, proj. variety $/ k$.

Fix k. TFAE (=defn of RC vars / $k = \bar{k}$ uncountable of char 0)

(1) $\forall\, x_1, x_2 \in X(k)$, $\exists\, f : \mathbb{P}^1 \to X$ non-const. s.t. $f(\mathbb{P}^1)$ contains x_1, x_2.

(2) $\forall$ finite set $S \subset X(k)$, $\exists\, f : \mathbb{P}^1 \to X$ s.t. $f(\mathbb{P}^1)$ contains S & w/ prescribed jet data at S.

(3) There exists a morphism $f : \mathbb{P}^1 \to X$ s.t. $f^* T_X$ is ample, $f^* T_X = \bigoplus_{i=1}^{\dim(X)} \mathcal{O}_{\mathbb{P}^1}(a_i)$ w/ all $a_i > 0$.

These are called "very free curves".

In positive char, (3) $\Rightarrow$ (1), (2) but not vice versa. (Examples due to Shioda & Katsura where (1) $\not\Rightarrow$ (3).) (Even older due to Zariski.)

__Def.__ A smooth, proj. variety X over k is __separably RC__ if $\bar{X}$ $(= X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k})$ contains a very free curve.

__Principle__ (Kollár). If X/k is sep. RC then $X(k) \neq \emptyset$ and has "lots" of pts if k is ~~sufficiently~~ nice.

Here, "nice" should mean C_1.

__Examples.__ k is a C_1-field, X is separa rationally conn'd, $X(k) \neq \emptyset$.

- $k = \mathbb{F}_q$ (Esnault).
- $k =$ fn'n field of a curve / alg. clos'd fld of char. 0 (Graber-Harris-Starr)
- " " char p (de Jong - Starr)

- Also known for all geom. rat'l surfaces over C_1-fields, (CT & Manin)

<u>Note</u>. "Lots" of points.

~~[illegible]~~. There exists a cubic surface $/\mathbb{F}_2$ with only 1 point.
Sw-D.

<u>Shuijing Li</u>. There are DP's of degree 2, resp. 1, over $\mathbb{F}_2$, resp. $\mathbb{F}_3$, w/ 1 point.

<u>Theorem</u> [Duesler-Knecht] Fix a numerical poly. P. There exists a finite set $e(P)$ of except pr's depends only on P s.t. $\forall X$ a sm. proj., sep. RC $X/\mathbb{Q}_p^{nr}$ w/ Hilb poly P, then $X(\mathbb{Q}_p^{nr}) \neq \phi$ when $p \notin e(P)$.

③ <u>RC / $\mathbb{Q}_p^{nr}$</u>. ~~[illegible]~~ Lang proved that a variety defined over $\mathbb{Q}_p^{nr}$ w/

a point in $\mathrm{Frac}(W(\overline{\mathbb{F}}_p))$ has a $\mathbb{Q}_p^{nr}$-pt.

A thm. of Ax-Kochen allows to compare $W(\overline{\mathbb{F}}_q)$ to $\overline{\mathbb{F}}_p((t))$ via ultraproducts.
To deal with $\overline{\mathbb{F}}_p((t))$ use ~~the~~ a generalizat. of GHS dJ$_{et}$ to find pts over $\overline{\mathbb{F}}_p((t))$.

<u>Theorem</u>. Let X be ~~a~~ smooth, proj & sep. RC / $k((t))$ w/ $k = \overline{k}$. Then $X(k((t)))$ is nonempty.

<u>Q</u>. How to go b/w $W(\overline{\mathbb{F}}_p)$ & $\overline{\mathbb{F}}_p((t))$?
<u>A</u>. Ultraproducts.

<u>Def</u>. Let S be a set. Let $\Sigma \subset \mathcal{P}(S)$ be a coll. of <u>non-empty</u> subsets

Then Σ is a <u>filter</u> if

(1) $S_1, S_2 \in \Sigma \Rightarrow S_1 \cap S_2$ is in Σ

(2) If S_1 is in Σ & $S_1 \subset S_2$, then also S_2 is in Σ.

The filter is <u>principal</u> if $\exists x \in S$ s.t. $\forall S$ in Σ, x is S_0. Otherwise the filter is <u>nonprincipal</u>. A filter is an <u>ultrafilter</u> if $S_1 \notin \Sigma \Rightarrow S - S_1$ is in Σ. (equiv. ultrafilter is a filter which is maximal).

The existence uses Zorn's Lemma.

<u>Lemma</u>. For every infinite subset $S_0 \subset S$ there exists a non-principal ultrafilter containing S_0.

Proof. Let Σ be the "Frechet filter": the subsets of S whose intersection w/ S_0 has finite complement in S_0. This is a non-principal filter. Now use Zorn's lemma to prove there is a maxl filter containing Σ. This is ~~an~~ a non-princ. ultrafilter. $\square$

Def. Given a non-principal ultrafilter Σ on S & a collection of rings $\{R_i\}_{i \in S}$, the ring $\prod_{i \in S} R_i / \Sigma$ where $a \sim b$ if their cpts agree on a set of indices contain in Σ. This ring is an ultraproduct.

Lemma. If the rings R_i are all fields, then the ultraproduct is a field.

Example. $S = \{1, 2, 3\}$, $R_1 = \mathbb{Z}/2\mathbb{Z}$, $R_2 = \mathbb{Z}/3\mathbb{Z}$ & $R_3 = \mathbb{Z}/5\mathbb{Z}$. Let $\mathcal{E}$ be $\{\{2\}, \{1,2\}, \{2,3\}, \{1,3\}\}$. Equiv. classes give ultraproduct is $R_2 = \mathbb{Z}/3\mathbb{Z}$.

Lemma. If $\mathcal{E}$ is principal, then ultraproduct $= R_n$ for some n.

Lemma. Let S be the set of prime integers. Let $\{K_p\}_{p \in S}$ be a set of fields s.t. $\mathrm{char}(K_p) = p$. Then for every non-principal $\mathcal{E}$, the ultraproduct has characteristic 0.

Proof of main thm. Let $U_P \subset \mathrm{Hilb}_{\mathbb{Z}}(\mathbb{P}^n)$ be the open subset par. smooth, sep. RC varieties w/ fixed Hilb. poly. Then $\mathcal{X} \to U_P$, the univ. family, is smooth & proj. w/ geom. fibers SRC of Hilb. poly. P. Now both $\mathcal{X}$ & U_P are finite type.

So reduce to the case where U is affine. There is a formula F in 1^{st} order logic in the language of rings s.t. F is true in $R \iff \mathcal{X}(R) \to U_p(R)$ is surj.

<u>Theorem</u> [Łoś] The formula F is true over the ultraproduct $\iff$ the indices i for which F is true in R_i is a member of $\mathcal{E}$.

Let $e(P)$ = the set of primes for which $\mathcal{X}(W(\mathbb{F}_p)) \to U_p(W(\mathbb{F}_p))$ is not surj.

By way of contradiction, assume $e(P)$ is infinite. By the lemma, $\exists$ an ultrafilter $\mathcal{E}$. Let $A = \prod_{p \in S} \mathbb{F}_p((t))/\mathcal{E}$ &

$B = \prod_{p \in S} W(\mathbb{F}_p)/\mathcal{E}$.

Ax & Kochen proved that $A \cong B$. Now there is a certain