Character formulas

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This is a partially expository talk based on joint work with Bill Graham.

1 Riemann–Roch Philosophy (1950's)

X smooth projective variety over $\mathbb{C}$.

Grothendieck: The K-group $K(X)$ is a ring-valued contravariant functor, as is the cohomology $H(X)$. The Chern character

$$\mathrm{ch}\colon K(X) \to H(X)$$

is a contravariant functor. It is defined by

$$\mathrm{ch}(L) = \exp(c_1(L)) = 1 + c_1(L) + \frac{1}{2}c_1(L)^2 + \cdots$$

for L a line bundle on X.

K is covariant by Rf_*. H is also covariant by similar reasoning: For any X we have a map

$$H^*(X) \to H_*(X), \quad c \mapsto c \cap [X],$$

an isomorphism for X smooth.

But ch is *not* covariant.

Grothendieck–Riemann–Roch: $\mathrm{ch}(\cdot)\,\mathrm{td}(T_X)$ *is* covariant.

2 Equivariant case

Donovan (1970's): T cyclic group acting on X (later: T will be an algebraic torus). He defined "Chern trace"

$$\mathrm{ct}\colon K_T(X) \to H(X^T)$$

a contravariant functor and "Todd trace" $\mathrm{Tt}(T_X) \in H(X^T)$, and proved:

Theorem 2.1. $\mathrm{ct}(\cdot)\,\mathrm{Tt}(T_X)$ *is covariant.*

A more natural approach is to factor ct as

$$K_T(X) \to H_T(X) \to H(X^T)$$

where the first arrow is the equivariant Chern character and the second is the restriction map. Then we can replace the restriction map $\mathrm{res}(\cdot)$ by

$$\mathrm{res}(\cdot)\frac{1}{c_{\mathrm{top}}(N)}$$

where $N = N_{X^T}X$ is the normal bundle of $X^T \subset X$ ("localisation") to obtain covariance.

Quart observed that alternatively one can factor ct as

$$K_T(X) \to K_T(X^T) = K(X^T) \otimes R(T) \to H(X^T)$$

where the first arrow is the restriction map res and the second is the Chern character. Here $R(T)$ is the representation ring $\mathbb{Z}[M]$ of T and we take $H(X^T)$ with coefficients in $R(T)$. For the Chern character we use GRR to obtain covariance. For the restriction map, we prove

Theorem 2.2. $\mathrm{res}(\cdot)\frac{1}{\lambda_{-1}(\mathcal{N})}$ *is covariant, where*

$$\lambda_{-1}(\mathcal{N}) = \sum (-1)^i \wedge^i \mathcal{N}$$

and $\mathcal{N} = (N_{X^T}X)^\vee$, *with values in* $K(X^T) \otimes S^{-1}R(T)$, *where* S *is the multiplicative set generated by* $1 - \chi^u$ *for* $u \neq 0$.

Proof. (1) X^T is smooth.

(2) $\lambda_{-1}\mathcal{N}$ is invertible in $S^{-1}K_T(X^T)$.

(3) $i\colon X^T \subset X$, $i^*i_*\colon K_T(X^T) \to K_T(X^T)$ is multiplication by $\lambda_{-1}(\mathcal{N})$. (Use deformation to the normal bundle and Koszul complex to compute pushforward.)

(4) (Iversen–Nielsen in torus case) $\mathrm{res} = i^*\colon K_T(X) \to K_T(X^T)$ is isomorphism after localising at S.

The theorem follows: $i^*(\cdot)\frac{1}{\lambda_{-1}(\mathcal{N})}$ is the inverse of i_*, and i_* is covariant! □

If $x \in X^T$ is isolated, $\lambda_{-1}\mathcal{N} = \prod(1 - \chi^{u_i})$, u_i the characters of the T-action on the cotangent space $T^*_x X$.

Corollary 2.3. *E equivariant vector bundle on X. X^T finite (T torus). Then*

$$\sum(-1)^i H^i(X,E) = \sum_{x\in X^T} \frac{E|_x}{e(x)}$$

in $S^{-1}\mathbb{Z}[M]$.

Singular case: RR (Baum-F-MacPherson) Find a $K_0 \to H_*$ to "go with"' ch: $K^0 \to H^*$. Here res: $K_T^0 X \to K_T^0 X^T$, $K_0^T X \to K_0^T X^T$ is $(i_*)^{-1}$.

Corollary 2.4. *X complete, X^T finite, have $r(x) \in S^{-1}\mathbb{Z}[M]$ so that for all E*

$$\sum(-1)^i H^i(X,E) = \sum E|_x \cdot r(x).$$

For $x \in X$ smooth $r(x) = 1/e(x)$.

3 Weyl character formula

$X = G/B$. A character $\lambda \in M$ defines a line bundle L_λ. We have $L_\lambda|_{x(w)} = \chi^w(\lambda)$.

The weights of T on $T^*_{x(w)}X$ are $w(\alpha)$, $\alpha \in R^+$.

Corollary 3.1. *(WCF)*

$$\sum(-1)^i H^i(X,L_\lambda) = \sum_{w\in W} \frac{\chi^{w(\lambda)}}{\prod_{\alpha\in R^+}(1-\chi^{w(\alpha)})}.$$

If $\lambda \in C^-$ then $H^i = 0$ for $i > 0$, and H^0 is irreducible representation with highest weight $\omega_0(\lambda)$.

4 Brion's formula

$X = X(\Delta)$ complete toric variety. $X^T = \{x(\sigma), \sigma \text{ maximal cone }\}$. $e(x(\sigma)) = \prod(1-\chi^{u_i})$.

If $x(\sigma) \in X$ is smooth, $u_1, \ldots, u_n$ generators of $\sigma^\vee$ (a basis of M), then

$$r(x(\sigma)) = \frac{1}{e(x(\sigma))} = \sum_{u\in\sigma^\vee\cap M} \chi^u \in \mathbb{Z}[[M]].$$

Definition 4.1. P in $\mathbb{Z}[[M]]$ is in $S^{-1}\mathbb{Z}[M]$ if there exists $u_i \neq 0$ such that $\prod(1-\chi^{u_i}) \cdot P = A \in \mathbb{Z}[M]$.

For any cone σ, $r_\sigma = \sum_{u \in \sigma^\vee \cap M} \chi^u$ is in $S^{-1}\mathbb{Z}[M]$ and is additive (it is zero on lower dimensional cones). In fact $r_\sigma = r(x(\sigma))$ of the singular RR story. (Consider $X(\tilde{\Delta}) \to X(\Delta)$ toric resolution, use rationality of toric singularities, etc.) So, for D an equivariant divisor, given by $\{u(\sigma)\}$,

$$\sum (-1)^i H^i(X, \mathcal{O}(D)) = \sum \chi^{u(\sigma)} \cdot r_\sigma.$$

P lattice polytope, dimension n in $M_\mathbb{R}$. Let $\Delta, X(\Delta), D$ be associated fan, toric variety, and equivariant divisor class. The vertices $u(\sigma)$ of P correspond to maximal cones $\sigma \in \Delta$. Let $C(u(\sigma), P)$ denote the cone with vertex $u(\sigma)$ generated by P (a translate of $\sigma^\vee$).

We have $H^i(X, \mathcal{O}(D)) = 0$ for $i > 0$, and $H^0(X, \mathcal{O}(D)) = \sum_{u \in P \cap M} \chi^u \in \mathbb{Z}[M]$.

Theorem 4.2. *(Brion)*

$$\sum_{u \in P \cap M} \chi^u = \sum_\sigma \left(\sum_{u \in C(u(\sigma), P)} \chi^u \right).$$